THE AMERICAN MATHEMATICAL MONTHLY

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THE MATHEMATICAL ASSOCIATION OF AMERICA

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

EDITED BY

H. E. SLAUGHT W. H. BUSSEY R. D. CARMICHAEL

WITH THE COÖPERATION OF

R. C. ARCHIBALD	R. P. BAKER	W. C. BRENKE
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D. A. ROTHROCK	C. S. SLICHTER	D. E. SMITH

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THE LOGICAL SKELETON OF ELEMENTARY DYNAMICS.[1]

By EDWARD V. HUNTINGTON, Harvard University.

The object of this article is to outline, in as compact and time-saving a form as possible, the logical structure of elementary dynamics. The definitions and theorems here stated are believed to contain all the theoretical equipment necessary for attacking problems in elementary dynamics, up to and including problems in the general motion of a system of rigid bodies in a plane.

Although the logical sequence may not always be the best teaching order, it is hoped that the article may be of service not only to teachers but also to students, especially in reviewing the subject.[2]

I. FUNDAMENTAL CONCEPTS.

Every logical science must begin with certain undefined concepts (in terms of which all the other concepts of the science are expressed), and certain unproved propositions (from which all the other propositions of the science are deduced).

Let us begin our inquiry, therefore, by asking what are the undefined concepts of dynamics. I take them to be these: (1) Space and time, with the derived concepts of velocity and acceleration; (2) forces, as suggested by the tension and compression in our own muscles; and (3) inert material bodies, on which our forces act. Let us briefly examine each of these preliminary notions.

Acceleration. The notions of time and space, as far as, and including,

[1] This article contains the substance of a paper read before the MATHEMATICAL ASSOCIATION OF AMERICA at its first summer meeting, September 1, 1916, under the title: The Teaching of Elementary Mechanics.

[2] Any teacher who is interested is requested to communicate with the Managing Editor of the MONTHLY, as arrangements may be made for supplying reprints. The article will be held in type for a limited period, until the possible demand for reprints has been ascertained.

the idea of velocity, may be assumed as fairly familiar to the student beginning mechanics; the idea of acceleration, however, is likely to be new, and no pains should be spared to make this concept vivid and real in the student's mind. In my own experience, a diagram showing a succession of snap shots of the moving object, taken at equal intervals of time against a stationary background, has proved very useful. Thus, in Fig. 1, as any boy can see, the body is moving

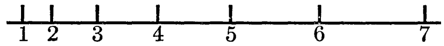

FIG. 1.

to the right with an increasing velocity; while in Fig. 2, the velocity first increases

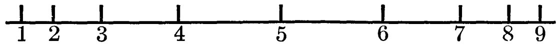

FIG. 2.

and then decreases again. All the four or five important types of accelerated motion can be illustrated in this way. But the matter is at best a difficult one, and any teacher who can devise a really successful method for making acceleration seem as definite and tangible a thing as velocity, would be doing us all a great service.

The units of measurement for acceleration, such as the railroad engineers' unit of a mile-per-hour per second, present no difficulty after the concept itself is clear.

The student should know enough calculus to understand how to get the velocity and the displacement from a known acceleration by integration. The following simple device will suffice for many common cases: If $dv/dt = f(t)$ or $f(x)$, *multiply through by dt or by dx;* in the latter case noting that $dx/dt = v$.

Force. The notion of force is also more or less familiar in the form of a push or a pull. We exert a force when we throw a stone or carry a load. Moreover, a simple instrument for measuring forces is also familiar, in the form of a spring balance. To find how hard I am pulling when I hold a kite string, I have merely to tie a spring balance to the end of the string and note how far the spring is stretched. Or, to find how much pressure is required to crack a nut, I have merely to replace the nut by a compression-balance between the jaws of the cracker, and note how far the spring is compressed. Of course, a spring balance, like every other physical instrument, is subject to numerous instrumental errors, and has to be somewhat idealized in order to serve our logical purpose. Nevertheless, a spring balance provides the best available concrete notion of a force.

The unit of force is perfectly arbitrary, being simply any agreed-upon amount of stretch in a standard spring; and a complete scale of multiples and submultiples of the unit is readily established by simply opposing one or more unmarked springs, in various combinations, against the standard, or unit, spring, and marking the positions reached by the pointer. (This process involves no knowledge of Hooke's Law; it assumes merely that the elastic properties of the spring do not vary with the time.)

By means of a portable, graduated spring balance thus constructed we are able theoretically to measure any force, anywhere, in terms of the arbitrarily chosen unit of force represented by our originally chosen standard spring. This simple, concrete notion of what a force *means* should not be complicated, at this stage, by any discussion of what a force *will do*, or by any inquiry as to how the ultimate unit of force can best be preserved for posterity. The important thing for the student to understand is that, theoretically at least, a force is always measurable by a spring balance, just as temperature is measurable by a thermometer, or time by a clock. Details of construction and standardization of the instrument should be postponed until later.

The following definitions will be important during the course:

(*a*) The *component, or resolved part, of a force* along a fixed axis is the magnitude of the force times the cosine of the angle which it makes with the axis.

(*b*) In a given plane, the *moment of a force* about a fixed point O in the plane is the magnitude of the force times the perpendicular distance from O to the line of action of the force. (The definition of the moment of a force about a skew axis in space need not be given until later in the course.)

As a matter of notation it is convenient to represent known forces by arrows with large closed heads, and unknown forces by arrows with large open heads; also to indicate the positive direction along a fixed axis by a small feathered arrow and the positive direction of rotation in the plane by a small, curved, feathered arrow.

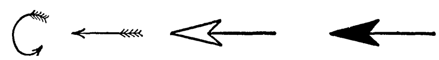

FIG. 3.

Matter. The last of our undefined concepts, namely the notion of inert material bodies, is also sufficiently familiar, provided we take it just as it comes, in its naïve, original form, and do not attempt to endow our material bodies with all sorts of mystical properties like "forces of inertia." A material body is passive. Force is the active agent, by which dead matter is buffeted about according to our will. All that we really need to know about matter, at the start, is that any given piece of matter, or any given collection of pieces, may be supposed to preserve its identity throughout any given discussion.

These, then, are our fundamental concepts: length and time, as measured by a meter stick and a clock; force, as measured by a spring balance; and inert lumps of matter, upon which our forces act.

Let us now turn to the fundamental principles, that is, the unproved propositions, of the theory.

II. Fundamental Principles of the Dynamics of a Single Particle.

To simplify the treatment, let us begin with the study of bodies which can be treated as particles. A particle is any material body which, for the particular purpose in hand, may be regarded as concentrated at a single point. Moreover,

let us assume for the present that all our observations are referred to a fixed frame of reference.

The fundamental question of dynamics is then the following: If a force gets hold of a free particle, and proceeds to act on it, what happens to the particle? The answer is contained in the first of our fundamental principles, namely:

1. **The principle of force and acceleration.** *A free particle, when acted on by a force, acquires an acceleration in the direction of the force; furthermore, if a given particle is acted on at different times by two forces F and F', and if a and a' are the corresponding accelerations, then*

$$F/F' = a/a';$$

that is, the accelerations are proportional to the forces.

This is what I like to describe as the fundamental equation of dynamics. It is best regarded as a scientific hypothesis, the truth of which has been abundantly verified by experiment. But its truth is not by any means obvious. For example, the immediate corollary that if *no force* is acting on a body, then there is *no change in the velocity* of the body, seems at first sight to contradict our commonest experience of motions which appear to die down of themselves. No pains should be spared to make clear by numerous concrete examples, including especially examples involving frictional forces, the full meaning of this equation. Each teacher will devise his own illustrations. All that I can do here is to point out that this principle of proportionality is really the kernel of the whole matter. However well a student may have done in statics, he cannot be said to have made even a fair start in dynamics until he has learned to associate force and acceleration inseparably in his mind.

Let us see what this principle enables us to do. We are on a desert island. We have our spring balance and our clock, and we are curious to discover the dynamical properties of each of the several inert lumps of matter that we find on the island. How do we proceed? Having selected any particular body for study, the essential thing is to ascertain *by experiment* (directly or indirectly) what acceleration would be produced in that body by some known force. Suppose, for example, that the given body proves to be such that a force of 10 lbs. gives it an acceleration of 2 ft./sec^2. Then the fundamental proportion enables us to state at once exactly what acceleration any other force would produce in that body.

Note on inertia. Since for any given body the ratio of the force to the acceleration produced is constant, the value of this ratio, F/a, is a characteristic of the body, which may be called its *inertia*. A body which requires a force of 10 lbs. to give it an acceleration of 2 ft./sec^2. has an inertia of 5 lbs. per (ft./sec^2.). The unit of inertia is thus a compound unit, namely a unit of force divided by a unit of acceleration; but as we shall see presently, this unit is one that seldom need be used in practice. The important thing to remember is that a separate experiment is required to determine the inertia of each individual body, and that when the inertia of any given particle is known, then the behavior of that

particle under any given force can be predicted by the aid of the fundamental equation.

Up to this point we have tacitly assumed that our particle is acted on by only one force at a time. Suppose now that the particle is acted on by several forces simultaneously.

To cover this case we state a second and a third fundamental principle, as follows:

2. **The principle of the vector addition of forces,** which may be analyzed into two parts, as follows:

(*a*) *Two forces acting in the same line on the same particle may be replaced by a single force which is their algebraic sum.*

This part (*a*) of principle 2 is all that we need, besides principle 1, for the dynamics of a single particle in one dimension.

(*b*) *Two forces, acting at right angles to each other on the same particle, may be replaced by a single force which is equal to the diagonal of the rectangle formed by the given forces.*

3. **The principle of the independence of two perpendicular forces.** *Suppose a particle P is acted on simultaneously by two perpendicular forces, F_x and F_y, one of which always remains parallel to the x axis, and the other always parallel to the y axis. Then the motion of the projection, P_x, of P on the x axis will be the same as if F_x were the only force acting; and the motion of the projection, P_y, of P on the y axis will be the same as if F_y were the only force acting.*

These principles are not wholly independent, since either 2(*b*) or 3 can be derived from the other by the aid of the principle of force and acceleration; on account of their simplicity, however, it seems preferable to assume them both.

From these principles it follows that if a particle P is acted on by any forces in the plane, the sum of the components of the forces along the x axis will determine the component of the motion along the x axis, while the sum of the components of the forces along the y axis will determine the component of the motion along the y axis. In other words, the general problem of the motion of a particle in a plane reduces to two simpler problems, each of which can be solved by the one-dimensional theory.

The extension of the method to the motion of a particle in three dimensions requires no new principles.

III. Weight and Falling Acceleration.

The theory, as so far developed, is entirely general, so general, in fact, as to appear rather abstract to beginners. It is desirable therefore to "come down to earth" as soon as possible, and see how the fundamental equation looks when applied to the simplest case of terrestrial force, namely, the force of gravity, or weight.

Definition. The *weight, W,* of a body, in a given locality, with respect to a given frame of reference, is best defined as the force required to support the body at rest with respect to that frame in the given locality.

Thus, if I plant a tripod in any locality on the earth (or on Mars), and suspend a kettle by a spring balance underneath the tripod, the reading of the spring balance gives immediately the local weight of the kettle with respect to the earth (or to Mars); all the measurements being made, of course, in terms of the standard unit of force.

Closely connected with the weight of the body is the "falling acceleration" of the body.

Definition. The *falling acceleration* of a body, in a given locality, with respect to a given frame of reference, is best defined as the acceleration with which the body, initially supported at rest in that locality, would begin to fall (in vacuo) if the supporting thread were broken—this acceleration being measured with respect to the given frame of reference.

I have stated these two definitions in a somewhat general form, so that they may apply without change to the case of a moving frame of reference.[1] We now have the following theorem, which holds true for either fixed or moving axes:

THEOREM. *If W is the weight of a given body, in a given locality, and g is the falling acceleration of that body in the same locality, then the ratio W/g is independent of the locality, and is a correct expression for the inertia of the body.*

The proof for the case of fixed axes follows immediately from principles 1 and 2*a*. The proof for the case of moving axes belongs later in the course.

By the aid of this theorem (for fixed axes), our *fundamental equations for the motion of a single particle in a plane* can be written in the useful form

$$F_x = (W/g)\ddot{x}, \qquad F_y = (W/g)\ddot{y},$$

where F_x is the sum of the components of the forces along the x axis,
F_y is the sum of the components of the forces along the y axis,
$\ddot{x}$, $\ddot{y}$ are the second derivatives of the x and y coördinates of the particle with respect to the time,
W is the weight of the particle in any locality,
and g is the falling acceleration of the particle in that locality.

The particular advantage of using the expression W/g for the inertia of a particle, instead of the general expression F'/a', is due to the following theorem:

THEOREM. *In any given locality, the falling accelerations of all bodies are equal.*

This theorem can be proved from general considerations; or, if preferred, it may be accepted as an empirical fact.

The locality in which $g = g_0 = 980.665$ cm./sec^2. $= 32.1740$ ft./sec^2. has been adopted by international agreement as the "standard locality," and the weight of the body in that locality is called the *standard weight* of the body. Since, now, the standard value of g is the same for all bodies, we see that *the inertia of a body is immediately determined when we know its standard weight.* For example, if the standard weight of a body is 3 lbs., we know that a force of 3 lbs. will give the body the standard acceleration g_0. In practice, therefore, we seldom

[1] First stated in this form in *Science*, July 30, 1915, page 161.

need refer to the inertia of a body by its original compound name; we speak simply of a *3 lb. body*, or a body whose standard weight is 3 lbs. By so doing we link up the abstract general theory with common commercial terminology, without any loss of scientific precision.

Any two bodies which balance each other on a beam balance have obviously the same local weight, and hence, by inference, the same inertia. It should be noted, however, that the process of balancing two bodies on a beam balance gives us no information about the inertia of either body by itself; it tells us merely that the inertia of one, whatever it may be, is the same as the inertia of the other.

Note on "mass" and "quantity of matter." A lump of matter whose standard weight is 3 lbs. is often spoken of as a *mass of three pounds;* or again, such a body is often said to have a mass of three pounds, or to *contain three pounds of matter*. Thus if we are told that a certain body "has a mass of 3 pounds," we are to understand that for that body W (in the standard locality) = 3 lbs. Similarly for the kilogram, etc.

IV. Accelerations along the Tangent and Normal.

In writing the differential equations of motion, the axes of reference may be chosen at pleasure, provided they remain fixed throughout the discussion. It is often convenient to take as these fixed axes the tangent and normal to the path of the particle at some fixed point, Q, of the path. Then, at the instant when the particle is passing through Q, its acceleration-components along the tangent and normal have the values

$$a_T = dv/dt, \qquad \text{and} \qquad a_N = v^2/r,$$

where v is the "path-velocity" of the particle at the instant in question, and r is the radius of curvature of the path at the point Q.

In case the path of the particle is a circle of radius r,

$$a_T = r\dot{\omega} \qquad \text{and} \qquad a_N = r\omega^2,$$

where $\omega = d\theta/dt$ is the angular velocity, in radians per unit of time, and $\dot{\omega} = d\omega/dt$ is the angular acceleration. All these expressions are of great importance.

The existence of the acceleration along the normal is apt to surprise and perplex the beginner, and no pains should be spared to make the proof convincing. One method is the following: Let Fig. 4 represent snap-shots of the

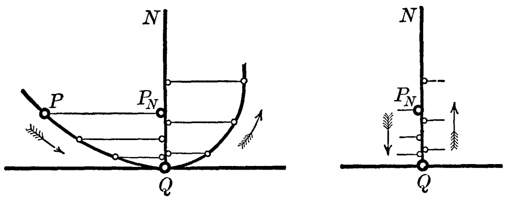

FIG. 4.

moving particle P, and let P_N be the projection of P on the normal axis. Then, by the acceleration-component of the original particle P along the normal axis, we mean simply the actual acceleration of the point P_N along that axis. Now if we draw a snap-shot picture of the point P_N, we see at once that P_N has a finite acceleration in the direction QN throughout the whole neighborhood of the instant at which it passes through Q. The existence of the normal acceleration being thus made evident, its magnitude can be readily computed by familiar methods of kinematics.

We mention also the expressions for acceleration along and perpendicular to the radius vector in polar coördinates:

$$a_r = \frac{d}{dt}\left(\frac{dr}{dt}\right) - r\omega^2, \qquad a_\theta = \frac{1}{r}\frac{d}{dt}(r^2\omega).$$

V. Fundamental Principles of the Dynamics of a System of Particles.

In the preceding sections, we have dealt only with the motion of a single particle. In dealing with a system of particles (and we shall suppose that every material body is composed of particles) it is important to distinguish between *the external and the internal forces* of the system. In regard to the external forces, that is, those which are impressed on the system from without, we should note that each force has its own definite point of application. In regard to the internal forces, we require a fourth and last fundamental principle, as follows:

4. **Principle of action and reaction.** *When two particles are in contact with each other, or attract or repel each other according to any law like that of gravitation or magnetism, the interaction between them may be represented by a pair of twin forces, equal in magnitude and opposite in direction—one of the twins acting on one particle and one on the other, along their joining line.*

In brief, the principle of action and reaction asserts that the internal forces of a system occur in pairs of twin forces.[1]

Two definitions are important at this point:

Definition. *Center of mass.* The centroid, or center of mass,[2] of a system of particles in a plane, is a point, $(\bar{x}, \bar{y})$, such that at every instant $\bar{x}$ is the weighted average of the x's of the several particles, and $\bar{y}$ the weighted average of the y's of the several particles; that is,

$$\bar{x} = \frac{x_1w_1 + x_2w_2 + \cdots + x_nw_n}{w_1 + w_2 + \cdots + w_n}$$

[1] If we think of the material universe as a whole, it must be admitted that all forces are internal forces, and hence that all forces occur in pairs; but in every practical problem, what we do first is to "isolate" some definite portion of the universe; that is, we pass a knife, as it were, around the boundary of that portion, dissecting each pair of twin forces that acts across the boundary into its two constituent forces, one of which we regard as an "external force" acting on our portion, the other of which we ignore. Whether a given force shall be called an external or an internal force depends entirely on what portion of the universe we are "isolating."

[2] The term "center of mass" is intended to suggest merely the center of the material of which the body is composed. A better term might be the "center of matter." The more common term "center of gravity" is apt to be misleading.

or in case of a continuous body,

$$\bar{x} = \frac{\int x\,dw}{\int dw},$$

(with similar expressions for $\bar{y}$), where $w_1, w_2, \cdots$ are the standard weights of the several particles. It is easily shown that the position of this point with respect to the material system is independent of the choice of axes.

Theorem on the motion of the center of mass. *If, in a plane, a system of particles, of total weight W, is acted on by any forces, then, at any instant,*

$$F_x = (W/g)\ddot{\bar{x}} \qquad \textit{and} \qquad F_y = (W/g)\ddot{\bar{y}},$$

where F_x is the sum of the components of the external forces along OX,
F_y is the sum of the components of the external forces along OY,
and $\ddot{\bar{x}}$, $\ddot{\bar{y}}$ are the acceleration components of the associated point which we have called the center of mass.

This theorem enables us to determine the motion of the center of mass of a system without any use of the internal forces.

The proof of the theorem consists simply in writing the differential equation of motion for each particle (taking account of both the external and the internal forces that act on that particle), and adding these equations; the internal forces will be found to cancel out by pairs.

The extension to three dimensions is obvious.

Definition. *Radius of gyration about a given axis.* Suppose a body is composed of particles whose distances from a given axis, fixed in the body, are r_1, r_2, $\cdots$, r_n. Then the radius of gyration of the body about that axis is a distance, k, such that the square of k is the weighted average of the squares of the r's of the several particles; that is,

$$k^2 = \frac{r_1^2 w_1 + r_2^2 w_2 + \cdots + r_n^2 w_n}{w_1 + w_2 + \cdots + w_n},$$

or, in case of a continuous body,

$$k^2 = \frac{\int r^2 dw}{\int dw}.$$

The following properties of the radius of gyration follow readily from the definition:

(*a*) If k is the radius of gyration about any axis, and $\bar{k}$ the radius of gyration about a parallel axis through the center of mass, then $k^2 = \bar{k}^2 + a^2$, where a is the distance between the axes.

(*b*) In case of a lamina in the x-y plane, if k is the radius of gyration about an axis perpendicular to the plane at the origin, then $k^2 = k_x^2 + k_y^2$, where k_x and k_y are the radii of gyration about the x and y axes, respectively.[1]

[1] The radius of gyration, k, seems to me a more tangible and useful quantity in dynamics than the so-called "moment of inertia." If you ask a student to estimate by eye the moment of

The student should know enough of the calculus to be able to find the center of mass and radius of gyration of all the simpler bodies by integration.

We are now in position to study the general problem of the motion of a rigid body in a plane.

First, we define the *angular acceleration*, $\ddot{\theta}$, or $\dot{\omega}$, of a rigid body in a plane as the second time-derivative of the angle θ which a line fixed in the body makes with a line fixed in the plane. Then we establish the following theorem:

Theorem of rotation, or **Theorem of moments about a point fixed in the body.** *If a rigid body, free to move in any manner in the plane, is acted on by any forces, and if Q is any suitably chosen point fixed in the body, then, at any instant,*

$$Fp = (W/g)k^2\dot{\omega},$$

where $\dot{\omega}$ is the angular acceleration of the body at the instant in question, Fp is the sum of the moments of the external forces about Q at that instant, and k is the radius of gyration of the body about Q. Here any point Q, fixed in the body, is a suitable point about which to take moments, provided either: (a) *Q is fixed in the plane (this is the case of rotation about a fixed axis); or* (b) *Q coincides with the center of mass; or* (c) *Q has a vector acceleration the direction of which passes through the center of mass.*

This theorem enables us to determine the angular acceleration of the body without any consideration of the internal forces.

The proof of the theorem for case (*a*) consists essentially in setting up a differential equation of motion for each particle separately, taking account of both internal and external forces, and choosing the axis along the tangent to the path in each case. Then, after multiplying each equation by a suitable factor and adding, we find that the internal forces will cancel out by pairs. The proof for the other cases follows without difficulty.[1]

inertia of a baseball bat about an axis through one end, he will be entirely at a loss to reply. He probably will not know even the name of the unit in which moment of inertia is measured. But if you ask him to estimate the radius of gyration of the bat about one end, he will see at once that something like 2 feet would be a plausible answer. The ability to give an approximate estimate of the numerical magnitude of the quantities with which one has to deal seems to me a very important consideration in favor of the use of radius of gyration instead of moment of inertia. Furthermore, the statement of theorem (*a*), above, is decidedly simpler in terms of radius of gyration than it would be in terms of moment of inertia. A similar remark applies to the following theorems:

(*c*) If a physical pendulum is swung about a fixed axis, then $k^2 = al$, where k is the radius of gyration of the pendulum about the axis, a = the distance from the axis to the center of mass, and l = the length of the isochronous simple pendulum.

(*d*) If a submerged vertical plane area serves as a dam, then $k^2 = \bar{x}p$, where k = the radius of gyration of the area about the water line, $\bar{x}$ = the distance from the water line to the center of mass of the area, and p = the distance from the water line to the center of pressure.

Neither of these theorems could be so simply stated in terms of moment of inertia.

Finally, the inappropriateness of the term "moment" should be noted; for moment of inertia is not the product of an inertia and a lever arm. If the concept is to be used in dynamics at all, the term "spin inertia," suggested by Professor W. S. Franklin to denote the quantity Wk^2, is recommended.

[1] For one method of proof, see a paper by the present writer in the AMERICAN MATHEMATICAL MONTHLY, Vol. 21, pp. 315–320, December, 1914.

An easy way to recall to mind the form of this important equation is the following: consider the motion of a single particle, attached to a fixed axis by a weightless arm, k, and acted on by a single force at right angles to the arm; then write the differential equation of motion along the tangent to the path, and note that $v = k\omega$.

These two theorems, the theorem of the center of mass and the theorem of rotation, bring us nearly to the end of our inquiry. For the principles and theorems here set forth, small as their number may seem, are sufficient for the solution of any elementary problem in the motion of a rigid body in two dimensions, as far as the dynamics of the problem is concerned. The only difficulties which remain are connected with the solution of the differential equations of motion; these difficulties are of a purely mathematical sort, and will not be considered here. Even for problems in three dimensions, no new dynamical principles are required, the only complications which arise being chiefly of a kinematical nature.

No discussion of dynamical principles would be complete, however, without mention, at least, of the useful theorem of work and kinetic energy, and the somewhat less important theorems of impulse and momentum (linear and angular). These theorems, while not essential for the theoretically complete solution of a problem, will often yield certain partial results of great value, with much less labor than the use of the general equations of motion would require.

VI. Work and Kinetic Energy.

Definition. *The work done by a force F, during a given interval* from t_1 to t_2, is defined as the quantity

$$U = \int_{x_1}^{x_2} F_x dx + \int_{y_1}^{y_2} F_y dy + \int_{z_1}^{z_2} F_z dz, \qquad \text{or} \qquad U = \int_{s_1}^{s_2} F_T ds,$$

where F_x, F_y, F_z are the components of F along the axes, and dx, dy, dz refer to the projections on the axes of the particle P on which F acts; or, where ds is the element of arc of the path traced by P, and F_T is the component of F along the tangent to the path.

In brief, *Work = Force × Distance.*

In applying this definition, it is important to note that every force must have a definite particle P as its point of application.

Definition. *The kinetic energy of a particle w, at a given instant,* is defined as the quantity

$$\text{K.E.} = (w/g)(v^2/2),$$

where v is the path velocity of the particle at that instant.

To recall these definitions to mind, for the case of a constant force, we have merely to note the result of integrating the fundamental equation $F = (W/g)dv/dt$ with respect to x; the left side gives work, the right side, change in kinetic energy.

Theorem of work and kinetic energy. *In any system of particles, the total* WORK DONE *by all the forces (both external and internal), during any interval, is equal to the total* GAIN IN KINETIC ENERGY *during that interval. But in the case of a rigid body, the work done by the internal forces is zero,* so that in applying the theorem to the case of a rigid body, only the external forces need be considered.

Note 1. In case of a rigid body rotating about a fixed axis (fixed in the body and fixed in space), the kinetic energy at any instant is $(W/g)k^2(\omega^2/2)$, where k is the radius of gyration of the body about that axis, and ω is the angular velocity of the body. In other words, if a rigid body is rotating about a fixed axis, its kinetic energy is the same as if all its material were concentrated at a distance k from that axis.

Note 2. In the case of a rigid body moving in any manner in a plane, the total kinetic energy, at any instant, is given by

$$(W/g)(\bar{v}^2/2) + (W/g)\bar{k}^2(\omega^2/2),$$

where $\bar{v}$ is the path velocity of the center of mass at the given instant,
ω is the angular velocity of the body at that instant,
and $\bar{k}$ is the radius of gyration of the body about the center of mass.

The first term of this expression is the *translational kinetic energy* (as if all the material were concentrated at the center of mass); the second term is the *rotational kinetic energy* (as if the center of mass were fixed).

VII. Impulse and Momentum.

Definition. The *impulse of a force F during a given interval* from t_1 to t_2, is defined as a vector quantity whose components along the axes of reference are

$$\int_{t_1}^{t_2} F_x\,dt, \qquad \int_{t_1}^{t_2} F_y\,dt, \qquad \int_{t_1}^{t_2} F_z\,dt,$$

where F_x, F_y, F_z are the components of F along the axes.

In brief, *Impulse = Force $\times$ Time.*

Since impulse (unlike work) is a vector quantity, we may speak of the component of an impulse along any fixed line, and of the moment of an impulse about any fixed axis.

Definition. The *momentum of a particle w at a given instant* is defined as a vector quantity whose components along the axes are

$$(w/g)v_x, \qquad (w/g)v_y, \qquad (w/g)v_z,$$

where v_x, v_y, v_z are the components of the path velocity v of the particle at that instant.

Since momentum (unlike kinetic energy) is a vector quantity, we may speak of the component of a momentum along any fixed line, and of the moment of a momentum about any fixed axis.

To recall these definitions to mind, for the case of a constant force, we have

only to note the result of integrating the fundamental equation $F = (W/g)dv/dt$ with respect to t; the left side gives impulse, the right side change in momentum.

Theorem of linear impulse and momentum. *In any system of particles, the total* IMPULSE *of all the external forces, in any given direction, during any interval, is equal to the total* GAIN IN MOMENTUM *in that direction during that interval.*

Note. In the case of any system of particles, whether rigidly connected or not, the total momentum in the x-direction, at any instant, is given by

$$(W/g)\bar{v}_x,$$

where $\bar{v}_x$ is the velocity component of the center of mass in that direction. In fact, the theorem of linear impulse and momentum is merely a corollary of the theorem on the motion of the center of mass.

Theorem of angular impulse and momentum, or **Theorem of moments about an axis fixed in space.** *In any system of particles, the total* MOMENT OF IMPULSE *of all the external forces about any fixed axis, during any interval, is equal to the total* GAIN IN MOMENT OF MOMENTUM *about that axis, during that interval.*

Note 1. In case of a rigid body rotating about a fixed axis (fixed in the body and fixed in space), the moment of momentum about that axis at any instant is $(W/g)k^2\omega$, where k is the radius of gyration of the body about that axis, and ω is the angular velocity of the body. In other words, if a rigid body is rotating about a fixed axis, its moment of momentum about that axis is the same as if all its material were concentrated at a distance k from the axis.

Note 2. In the case of a rigid body moving in any manner in a plane, the total moment of momentum about any point O fixed in the plane, at any instant, is

$$(W/g)p\bar{v} + (W/g)\bar{k}^2\omega,$$

where ω is the angular velocity of the body at the instant in question,
$p\bar{v}$ is the moment about O of the path velocity of the center of mass at that instant,
and $\bar{k}$ is the radius of gyration of the body about its center of mass.

The first term of this expression is the moment of the *translational momentum* of the body (as if all the material were concentrated at the center of mass); the second term is the moment of the *rotational momentum* of the body (as if the center of mass were fixed).

VIII. Note on the Fundamental Principles of Statics.

Problems in statics can always be regarded as special cases of problems in dynamics in which the velocity is zero, so that the theory of statics is completely covered by the theory of dynamics. If, however, one wishes to study statics by itself, before taking up the study of dynamics, it is interesting to note that the following four fundamental principles are sufficient as a logical foundation for statics.

A. Principle of the vector addition of forces [see 2, above].

B. Principle of the transmissibility of forces in a rigid body: Any force acting on a rigid body may be shifted at pleasure along its line of action.

By the aid of these principles A and B, any set of forces acting on a rigid body can be " boiled down " either to a single force, or else to a single couple (that is, a pair of equal forces acting in opposite directions along parallel lines). This force, or couple, is called the *resultant* of the given set of forces.

C. Principle of static equilibrium: Suppose a rigid body is acted on by a set of forces which " boil down," by principles A and B, to zero; then if the body was initially at rest (with respect to a fixed frame of reference), it will remain at rest (with respect to that frame).

D. Principle of action and reaction between two bodies [see 4, above].

The concepts of acceleration and inertia are not required in statics.

IX. Systems of Derived Units.

The units of measurement employed in mechanics divide themselves into (1) units for the fundamental quantities, force, length, and time; and (2) units for the derived quantities, such as velocity, acceleration, work, energy, etc.

The fundamental units may be chosen at pleasure. The derived units are defined in a systematic manner, in order to avoid much unnecessary labor in computation. For example, if the unit of length is the foot, the systematic unit of area is the square foot (ft^2.), not the acre.

The fundamental units of force, length, and time which are adopted as the basis of any system of units must of course be so defined as to be capable of exact reproduction at any future time; and these definitions, uninteresting as they properly are to the beginner, form an essential part of the logical skeleton of the science. Thus:

The *kilogram force* (1 kg.) is defined as the force required to support a certain carefully preserved lump of metal called the " international standard kilogram," in vacuo, in the standard locality. (Here the " standard locality," as fixed by the International Bureau of Weights and Measures, Paris, 1901, means any locality where $g = g_0 = 980.665$ cm./sec^2. $= 32.1740$ ft./sec^2., this being approximately the value of g at 45° latitude, sea level.) The *dyne* = 1/980665 kg.

The *pound force* (1 lb.) is defined as the force required to support another carefully preserved lump of metal, called the " standard pound avoirdupois," in vacuo, in the standard locality. (1 lb. = 0.4535924 $\cdots$ kg.)

The *meter length* (1 m.) is defined as the distance, at the temperature of melting ice, between two scratches on a certain carefully preserved metal bar, called the " international standard meter." The *foot* (1 ft.) is defined, in the United States, in terms of the meter, through the relation 1 meter = 39.37 inches (where 12 inches = 1 ft.). The *centimeter* (1 cm.) = 0.01 m.

The *second of time* is defined as 1/86400 of a mean solar day.

The three most important systems of units are the lb.-ft.-sec. system, the kg.-m.-sec. system, and the dyne-cm.-sec. system (the last being commonly called the " c.g.s." system). The names of the principal derived units in each of these three systems are shown in Table I, where the symbol / is to be read " per."

TABLE I.

PRACTICAL SYSTEMS OF DERIVED UNITS, AS ACTUALLY USED BY ENGINEERS AND PURE SCIENTISTS.

Dynamical Quantity.	Derivation.	Name of Unit in the Practical British System.	Name of Unit in the Practical Continental System.	Name of Unit in the Practical "C. G. S." System.	Dimensions in Terms of F, L, T.
FORCE	Fundamental	lb.	kg.	dyne	F
Length	Fundamental	ft.	m.	cm.	L
Time	Fundamental	sec.	sec.	sec.	T
Velocity	Length per unit time	ft./sec.	m./sec.	cm./sec.	L/T
Acceleration	Velocity per unit time	ft./sec^2.	m./sec^2.	cm./sec^2.	L/T^2
Pressure	Force per unit area	lb./ft^2.	kg./m^2.	dyne/cm^2.	F/L^2
Impulse, or Momentum	Force × time	lb.-sec.	kg.-sec.	dyne-sec.	FT
Work, or Kin. energy	Force × dist.	ft.-lb.	kg.-m.	dyne-cm.	FL
Power	Force × veloc.	ft.-lb./sec.	kg.-m./sec.	dyne-cm./sec.	FL/T
Inertia	Force per unit accel.	lb. per (ft./sec^2.) [= 1 "slug"]	kg. per (m./sec^2.) [= 1 "metric slug"]	dyne per (cm./sec^2.) [= 1 "Gram mass"]	$F/(L/T^2)$

All the units in this table, except the units of inertia, are in everyday practical use, the first two systems being preferred by engineers, the third by physicists.[1]

The units of inertia, as already explained, are seldom required, the inertias, or masses, of two bodies being most readily compared by means of their standard weights as determined by weighing on a beam balance. The commonly used units of mass are the *pound mass* and the *kilogram mass;* such systematic units as the "slug" (= 32.1740 pounds mass), and the "metric slug" (= 9.80665 kilograms mass) are never used in practice.

APPENDIX.

In spite of the simplicity and familiarity of the systems of units shown in Table I, most text-books make the attempt to impose upon the student's attention a different and unpractical group of systems, namely, systems in which the derived units are based on *mass*, length, and time, instead of on *force*, length, and time. That this attempt has not been successful outside of the text-books is sufficiently shown by Table II, where the names of the units systematically derived from mass, length, and time are given in a form in which they may be readily contrasted with the units of Table I. It will be seen that not one of the

[1] Other units, which do not belong to any system, are also in use, and are sometimes convenient for special purposes. For example, 1 atmosphere = 14.70 lb./in.^{2}; 1 horsepower = 550 ft.-lb./sec.; 1 joule = 10^7 dyne-cm.; 1 watt = 10^7 dyne-cm./sec.; etc. Every such non-systematic unit, if used at all, should be defined explicitly in terms of one of the systematic units of the same kind.

units of pressure, work, power, etc., named in Table II is in practical use, either in engineering or in pure science. It is earnestly to be hoped that all these superfluous units will eventually be dropped from the text-books.

TABLE II.

A USELESS COMPLICATION, STILL FOUND IN MANY TEXT-BOOKS.

Dynamical Quantity.	Derivation.	Name of Unit in the Scholastic British System.	Name of Unit in the Scholastic Continental System.	Name of Unit in the Scholastic "C. G. S." System.	Dimensions in Terms of M, L, T.
MASS	"Fundamental"	Lb.	Kg.	Gm.	M
Length	Fundamental	ft.	m.	cm.	L
Time	Fundamental	sec.	sec.	sec.	T
Velocity	Length per unit time	ft./sec.	m./sec.	cm./sec.	L/T
Acceleration	Velocity per unit time	ft./sec^2.	m./sec^2.	cm./sec^2.	L/T^2
Pressure	Mass-accel'n per unit area	Lb. per (sec^2.-ft.)	Kg. per (sec^2.-m.)	Gm. per (sec^2.-cm.)	$M/(T^2L)$
Momentum, cr Impulse	Mass × veloc.	Lb.-ft./sec.	Kg.-m./sec.	Gm.-cm./sec.	ML/T
Kin. energy, or Work	Mass × (veloc.)^2	Lb.-ft^2./sec^2.	Kg.-m^2./sec^2.	Gm.-cm^2./sec^2.	ML^2/T^2
Power	Mass-(veloc.)^2 per unit time	Lb.-ft^2./sec^3.	Kg.-m^2./sec^3.	Gm.-cm^2./sec^3.	ML^2/T^3
Force	Mass × accel'n	Lb.-ft./sec^2. = 1 "poundal"	Kg.-m./sec^2.	Gm.-cm./sec^2. = 1 "dyne"	ML/T^2

A comparison of these two tables—Table I simple and familiar, Table II strange and artificial—provides one of the best arguments in favor of the use of force rather than mass as the principal undefined concept of dynamics. The only reason why the text-books so insistently base their derived units on mass instead of on force is apparently that a standard lump of metal is easier to preserve in a museum than a standard spring balance. But this is no argument for the logical priority of mass over force. As a matter of fact, the fundamental unit of force is as easy to preserve as the fundamental unit of mass, though the method of doing so does not consist in simply storing away a spring balance. Every consideration of logical simplicity and practical convenience speaks in favor of Table I as against Table II.

Finally, it should be noted that it is not possible to take *both* force *and* mass as fundamental units, without giving up the whole idea of systematic derived units. Every attempt to treat the units of force and mass as equally fundamental proves to be really a definition of one of them in terms of the other, so that the necessity of choosing between Table I and Table II cannot in that way be escaped.[1]

[1] This matter is closely related to the question of the choice between $F/F' = a/a'$ and $F = ma$ as the fundamental equation of mechanics. On this question reference may be made to a recent controversy between the present writer and Professor L. M. Hoskins in *Science* (December 4, 1914; February 5, April 23, July 30, and September 10, 1915; March 3, June 30, September 8, and October 27, 1916).

SIMPLE HINTS ON PLOTTING GRAPHS IN ANALYTIC GEOMETRY.

By AUBREY KEMPNER, Urbana, Ill.

In courses in analytic geometry, much importance is generally attached to the problem of plotting curves from given equations. It is therefore surprising that in most textbooks on analytic geometry some very simple and effective hints on plotting should be omitted. The contents of the present note are well known; for §§ III, IV compare the article in the MONTHLY for November, 1916, on "Graphical constructions for a function of a function and for a function given by a pair of parametric equations," by Professor W. H. Roever and his reference to Professor E. H. Moore.

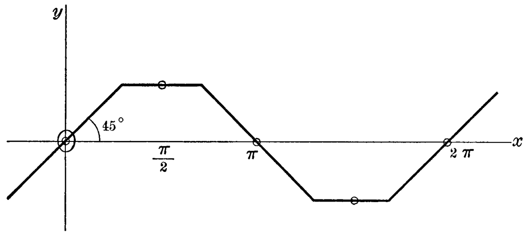

FIG. 1.

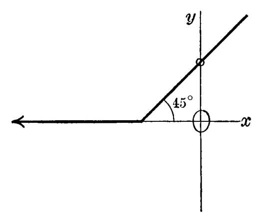

FIG. 3.

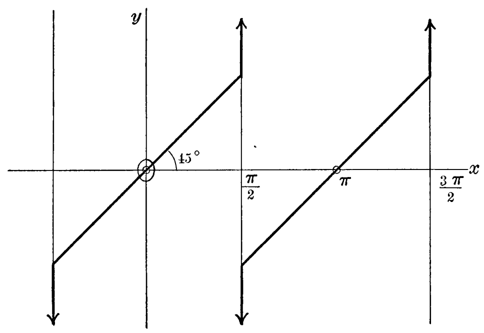

FIG. 2.

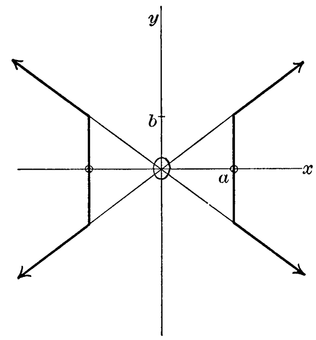

FIG. 4a.

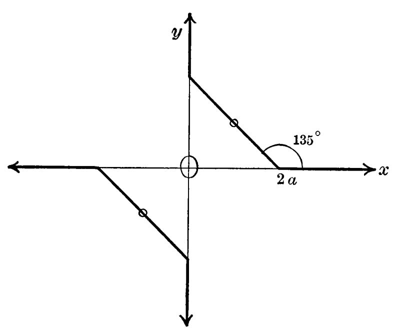

FIG. 4b.

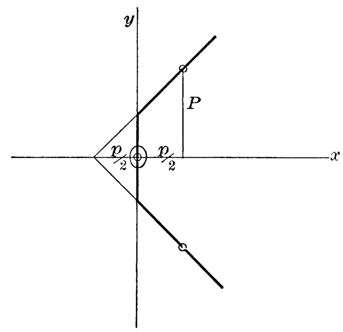

FIG 5.

I. In many cases it is very easy to construct a few tangents to the curve with the corresponding points of tangency, and thus to gain a skeleton of the curve. Besides being a help in the actual plotting, such a skeleton is a valuable aid to the memory. In Figs. 1–5 these skeletons are constructed for certain standard curves. The unit of measurement is supposed to be the same on both axes. Asymptotes are marked with an arrow-tip.

Figure 1: $y = \sin x$ (x in radians[1]), and similarly $y = \cos x$, and $y = \text{arc} \sin x$, $y = \text{arc} \cos x$.
Figure 2: $y = \tan x$, and similarly $y = \cot x$, $y = \text{arc} \tan x$, and $y = \text{arc} \cot x$.
Figure 3: $y = e^x$, and similarly $y = \log_e x$.
Figure 4a: The well-known figure for the hyperbola $x^2/a^2 - y^2/b^2 = 1$. (The corresponding figure for the ellipse consists simply of a rectangle, and is not here drawn.)
Figure 4b: The equilateral hyperbola $x \cdot y = a^2$.
Figure 5: The parabola $y^2 = 2px$.

II. To plot the graph of an equation of the type $y^2 = f(x)$ (or, similarly, of $x^2 = \varphi(y)$).

Rule: Construct the curve whose equation is $Y = f(x)$, using for Y-axis the original y-axis, and change the ordinates,[2] replacing Y by $y = \sqrt{Y}$.

The following relations between the curves $y^2 = f(x)$ and $Y = f(x)$ are evident:

(a) $y^2 = f(x)$ is symmetric with respect to x-axis.

(b) Corresponding to every loop of $Y = f(x)$ above the x-axis, $y^2 = f(x)$ has an oval.

(c) The parts of $Y = f(x)$ below the x-axis do not yield real points of $y^2 = f(x)$.

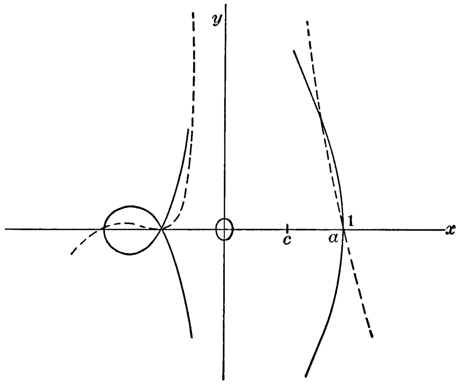

FIG. 6.

[1] The fraction $\frac{22}{7}$, as an approximate value for π, is exact to about 1/2000 of the value of π, and is therefore sufficiently accurate for all graphical purposes.

[2] This method of construction is really nothing but a particularly simple case of changing, along the Y-axis, from a uniform scale to a certain non-uniform scale. See, for example, C. RUNGE, *Graphical Methods*, 1912, § 6, for the general method.

(*d*) When $0 < Y < 1$, the curve $y^2 = f(x)$ lies above the curve $Y = f(x)$; and when $Y > 1$ it lies below.

(*e*) The curves $Y = f(x)$, $y^2 = f(x)$ cross each other in all points where $Y = 0$ and where $Y = 1$.

Curves of this type are, for example, the conic sections, the semi-cubical parabola, the Cissoid, the Strophoid, the Conchoid, the Versiera, when the equations are given in the forms commonly used.

Figure 6: The conchoid $y^2 = \frac{1}{x^2} \cdot (x + c)^2 \cdot (a^2 - x^2)$, where $a = 1$, $c = \frac{1}{2}$. The dotted curve is $Y = \frac{1}{x^2} \cdot (x + c)^2 \cdot (a^2 - x^2)$.

Figure 7: The parabola $y^2 = 2px$ (dotted line $Y = 2px$).

Figure 8: The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, or $y^2 = b^2 - \frac{b^2}{a^2} x^2 \left(\text{dotted curve } Y = b^2 - \frac{b^2}{a^2} x^2\right)$, and the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, or $y^2 = -b^2 + \frac{b^2}{a^2} x^2 \left(\text{dotted curve } Y = -b^2 + \frac{b^2}{a^2} x^2\right)$. The left half of the figure refers to the ellipse, the right half to the hyperbola. The two parabolas are congruent. $b = \frac{1}{2}\sqrt{2}$, $a = 1$.

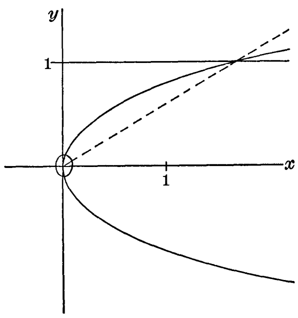

FIG. 7.

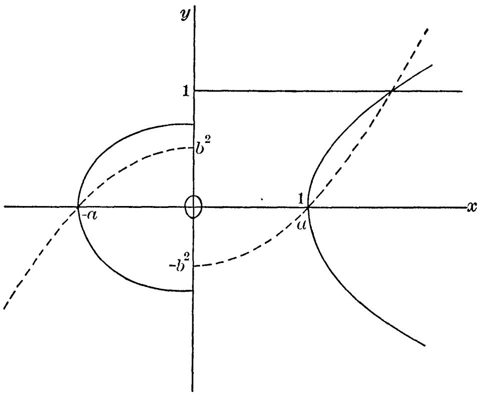

FIG. 8.

III. To construct the graph of $F(x, y) = 0$ when the equation is given in parametric form: $x = \varphi(t)$, $y = \psi(t)$.

Rule:[1] In a system of coördinates I, II (see Fig. 9) plot first $y = \psi(t)$, using I for the t-axis and II for the y-axis, and also $x = \varphi(t)$, using II for the t-axis and I for the x-axis. Next, draw the line of slope unity through the origin, assume any point on this line, draw through this point a line parallel to the axis II to its intersection (D in Fig. 9) with the curve $y = \psi(t)$, another line parallel to I to its intersection (E in Fig. 9) with the curve $x = \varphi(t)$, and complete the rec-

[1] Compare Professor Roever's article referred to above.

tangle. The fourth vertex (G in Fig. 9) is a point of $F(x, y) = 0$, with I for x-axis and II for y-axis.

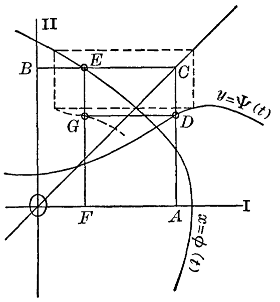

FIG. 9.

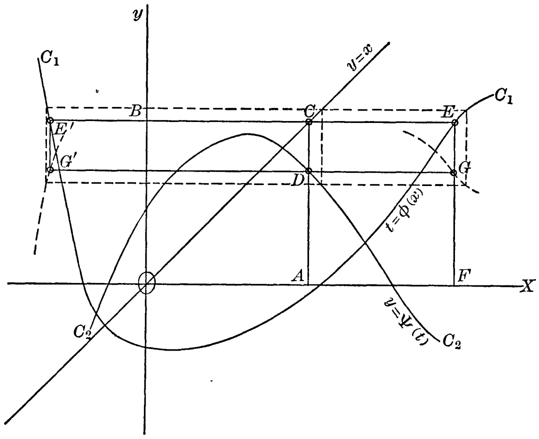

FIG. 10.

Proof: $AD = \psi(OA)$, $BE = \varphi(OB)$. (See Fig. 9.) Let $OA = OB = t$, then $BE = OF = \varphi(t) = x$, and $AD = FG = \psi(t) = y$.

IV. By interchanging x and t in $x = \varphi(t)$ we obtain

$$t = \varphi(x), \quad y = \psi(t) = \psi(\varphi(x))$$

and have thus the following rule for plotting the graph of a function of a function.

Rule: Plot in the $x - y$ system of coördinates the two curves (see Fig. 10)

C_1: $t = \varphi(x)$, using for t-axis the y-axis,

C_2: $y = \psi(t)$, using for t-axis the x-axis.

Draw the line $y = x$, assume any point C on it, draw through this point a line parallel to the y-axis to its intersection (D) with C_2, and another line parallel to the x-axis to its intersections (E and E' in Fig. 10) with C_1, and complete the rectangle. The fourth vertices (G and G' in Fig. 10) are points of the graph of $y = \psi(\varphi(x))$.[1]

Proof: $AD = \psi(OA)$, $OB = \varphi(BE)$. (See Fig. 10.) Let $OA = OB = t$, then $AD = FG = \psi(t)$, $t = \varphi(BE) = \varphi(OF)$, and $FG = \psi(\varphi(OF))$.

In order to reduce the number of cuts, no illustrative examples for Sections III, IV have been given. The reader is referred to the examples worked out in Professor Roever's paper.

From the rule of construction in the present paragraph one can derive in geometrical form the conditions which must be satisfied in order to have $\psi(\varphi(x)) \equiv \varphi(\psi(x))$.

By taking for φ and ψ the same function one finds a simple method for

[1] For another kind of graphical representation of a function of a function compare, for example, C. RUNGE, *Graphical Methods*, 1912, § 6.

plotting the iterated function $\phi_2(x) = \phi(\phi(x))$; repeating the process for $\phi_2(x)$ instead of for $\phi(x)$, $y = \phi_4(x) = \phi\phi\phi\phi(x)$, and in the same way $y = \phi_k(x)$, may be plotted, when k is any power of 2. When k is not a power of 2, the construction of this paragraph generally leads to complicated figures.

A COURSE IN GEOMETRY FOR COLLEGE JUNIORS AND SENIORS.[1]

By J. N. VAN DER VRIES, University of Kansas.

Courses in geometry for college juniors and seniors are usually given under the title "Modern Geometry." In many instances, the scope of the course is restricted by designating it as a course either in "Modern Synthetic Geometry" or in "Modern Analytic Geometry." An examination of the catalogs of a number of the leading American universities and colleges shows that there is no cleancut agreement as to the content of a course under any of these three titles. The term "Modern Geometry" seems to have grown up in our mathematical nomenclature for the purpose of distinguishing a college course in geometry under this name from the more elementary course of the secondary school, this latter course being the *ancient* geometry of the Greeks practically unchanged. This ancient geometry was, as Professor Cajori expresses it in his history, decidedly special. In fact, one of its principal characteristics was a complete lack of general principles and methods. Courses in modern geometry do not differ from the courses in elementary geometry so much in their content as in the methods by which they produce not only the well-known results of ancient geometry but also many results which could not be obtained by the equipment and methods of ancient geometry.

The college junior and senior ready to begin a course in modern geometry has had the usual amount of this elementary algebra and geometry, the latter for the larger part, if not entirely, the geometry of the plane. He will also have had the usual amount of college algebra, trigonometry, analytic geometry and calculus with some applications to mechanics and geometry. He may in some instances have had courses in unified mathematics both in his preparatory work and in his freshman year in college, but the various subjects will in most cases have been taught within well-defined boundary lines with no attempt at generalization or coördination. As a rule nowhere in his course has the student had much work tending to emphasize the homogeneity of all his work in *geometry* and the general principles which underlie it in its entirety. It is the belief of the writer that the chief aim of the course for college juniors and seniors should be to remedy the defects of the elementary geometry mentioned above by supplying the generalizing methods and principles towards which modern thought has contributed so greatly. A brief historical review of the introduction of these new concepts will not be out of place here.

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[1] Read before the Kansas Section of the Mathematical Association of America, March 18, 1916.

The introduction of infinitely great quantities into geometry by Kepler and the treatment of conics as projections of circles by Desargues and Pascal are among the first instances of the generalizing tendencies which characterize modern geometry. The announcement by Desargues and Pascal of the famous theorems named after them was the beginning of what is known as modern synthetic geometry. The introduction of the infinitely great into geometry by Kepler was closely coupled with the introduction of the infinitely small and led to the assumption by him of a powerful concept of modern geometry, namely, continuity. The middle of the seventeenth century witnessed the introduction into geometry by Descartes and others of the analytic method, another powerful agent in the new development. This new method retarded for a time the development of geometry on the purely synthetic side but more than compensated for this retardation by its generalizing tendencies and by the richness of its *suggestions* as to new concepts on the pure geometry side. The principle of duality was introduced from the geometric point of view by Gergonne early in the nineteenth century, was shortly afterwards applied to reciprocal polars by Poncelet, and was still further extended by Steiner. At about the same time Plücker introduced the principle of duality from the analytic point of view, coupling it with the idea of homogeneity. The introduction of the theory of imaginary points, lines and planes into projective geometry by Von Staudt followed soon after this. The concepts briefly outlined here form the principal instruments for the modern developments in geometry.

In many American universities the courses in modern geometry have been during the past two decades courses in advanced analytic geometry. The text followed was either Salmon's *Conic Sections*, Smith's *Conic Sections* or some other text mainly emphasizing the special properties of conic sections with some attention to homogeneous coördinates, abridged notation, etc., or else it was a text such as that of Miss Scott on *Modern Analytical Geometry*. During the past few years there has been a tendency evident in some institutions to swing the pendulum in the opposite direction and to make the course one in projective geometry from the synthetic point of view. But, as Professor Bussey points out in a strong plea for more synthetic work in geometry in No. 9, Vol. XX of THE AMERICAN MATHEMATICAL MONTHLY, a course in synthetic projective geometry does not commend itself to the prospective graduate student in mathematics at this stage of his development. For there is so much, even in geometry, that is of more importance to the student in his future work. The number of credit hours at the disposal of the student and the number and character of the students who are to take the course (and without them there will be no opportunity for the course) are matters that must also be considered in outlining a course, especially in the small college. In most of these institutions but one advanced course in geometry can be offered. The writer believes that this course should be both analytic and synthetic, emphasizing, if either, the synthetic side, as the analytic side has been in preponderance in the student's previous work. It should above all be an attempt to unify and coördinate the synthetic work of

the high school and the analytic work of the college as far as possible into a homogeneous whole. Such a course will be of benefit to the student whether he be a prospective high-school teacher of mathematics, a prospective research student in mathematics or one who is taking the course for its cultural value.

Thus, the concept of infinity and of parallel lines from the pure synthetic point of view will be greatly cleared up by an analytic discussion of the same, especially when homogeneous coördinates are used. The work in calculus now makes easy a comprehension of the idea of continuity and of limits in geometry which was not possible to the student of elementary geometry. The analytic treatment of imaginaries now makes possible a conception of the use and meaning of imaginary geometric elements. An illustration of this is the proof that a real straight line is the locus of the harmonic conjugates of a given point with respect to the two points in which a variable line through the point meets a conic, even when the variable line does not cut the conic in real points. The methods of the principle of duality when used synthetically appear much more valid to the student if they are reinforced by a thorough knowledge of the use of homogeneous coördinates and by a clear comprehension of the exact symmetry which arises in the analytic work. The beauty and the brevity of the synthetic work should, however, at all times be emphasized and the possibility of the avoidance of the analytic work, at times long and tedious, pointed out. The principle of central projection and the special cases arising from it can be studied simultaneously with work on linear transformations. When this is done a careful study may be made of the part played by the coefficients of the transformation. The geometry of each dimension above the first should be carefully built up on that of a lower dimension and coördinate systems in each dimension studied at the same time. It will then be possible to investigate the meaning and the properties of projective coördinates in each dimension and to point out the special systems, such as the metric Descartes system with its relationship to Euclidean geometry. The famous theorems of Desargues, Pascal, Brianchon, etc., should be carefully studied and the special theorems dependent on them investigated. Geometric addition and multiplication and their inverses ought to be explained and our ordinary algebraic operations shown as special cases. Quadrangular and other constructions may be illustrated by many examples. Throughout the course the idea of double ratio and its invariance under projection, the meaning of harmonics and their relation to involutions, the projective properties of conics, are all matters which should be emphasized.

Scarcity of time may make it necessary to make the treatment of some of the subjects mentioned in the previous paragraph very brief. The course is, however, believed by the writer to be such that the mind of the prospective teacher will be greatly enriched and that the prospective graduate student will have a first-class foundation for his future research work.

FIRST REGULAR MEETING OF THE MISSOURI SECTION.

Pursuant to the call from the committee appointed some time ago by Professor E. R. Hedrick, about forty-two men and women from universities, colleges, normal and high schools of Missouri, and some not connected with any educational institution but interested in mathematics, gathered in St. Louis at the Central High School on Saturday morning, Nov. 18, for the first regular meeting of The Missouri Section of The Mathematical Association of America. Professor C. A. Waldo called the meeting to order. Professor Hedrick was elected temporary Chairman and Mr. A. H. Huntington, temporary Secretary. Professor Hedrick gave a short report about the beginnings of this Section in the unorganized meeting one year ago at Washington University, and told of the subsequent successful organization of The Mathematical Association of America.

In accordance with the provision of the Constitution of the Association in regard to sections, the following members of the Association in attendance completed the formal organization of the Missouri Section:

L. D. Ames, University of Missouri, Columbia.
Charles Ammerman, McKinley High School, St. Louis.
C. J. Borgmeyer, St. Louis University, St. Louis.
Dorothy G. Calman, St. Louis.
Alan D. Campbell, Washington University, St. Louis.
Byron Cosby, State Normal School, Kirksville.
Otto Dunkel, Washington University, St. Louis.
Zoe Ferguson, Central High School, St. Joseph.
B. F. Finkel, Drury College, Springfield.
E. R. Hedrick, University of Missouri, Columbia.
A. H. Huntington, Central High School, St. Louis.
Byron Ingold, Christian University, Canton.
Louis Ingold, University of Missouri, Columbia.
Thos. W. Jackson, Fulton High School, Fulton.
G. H. Jamison, State Normal School, Kirksville.
Albert R. Nauer, Mechanical Engineer, St. Louis.
Paul R. Rider, Washington University, St. Louis.
W. H. Roever, Washington University, St. Louis.
Wallis G. Rowe, Smith Manual High School, St. Louis.
A. J. Schwartz, Cleveland High School, St. Louis.
J. I. Shannon, St. Louis University, St. Louis.
H. P. Stellwagen, Yeatman High School, St. Louis.
C. A. Waldo, Washington University, St. Louis.
Eula A. Weeks, Cleveland High School, St. Louis.
R. A. Wells, Park College, Parkville.
W. H. Zeigel, State Normal School, Kirksville.

A constitution for the Section was submitted and adopted; and the election of permanent officers for the coming year resulted as follows: President, Professor

C. A. Waldo, Washington University; Vice-President, Professor L. D. Ames, University of Missouri; Secretary-Treasurer, Mr. Albert H. Huntington, Central High School, St. Louis. After the election of officers, about three hours were given to the reading of papers, when adjournment was taken in a body to a near-by restaurant and the good fellowship of the meeting was continued.

Keen delight and satisfaction were expressed by all, the only criticism being that not enough time was given to the discussion of the papers. Here follows the program as presented.

ABSTRACTS OF PAPERS.

1. *A Course for Juniors in the School of Education.* By Professor L. D. AMES, University of Missouri.

Professor Ames described a course which for several years he has been giving in the University of Missouri for juniors in the school of education. Most of these students are preparing to teach in high schools, either in mathematics or in other departments. Many of them cannot take more than one or two years of mathematics beyond the calculus. It seems desirable to give them as broad a range of topics as is consistent with a clear grasp of the subject matter and the ability to apply it. This is done by putting emphasis on fundamentals, requiring the solution of carefully selected problems based on the fundamentals, but avoiding all unnecessary complications, requiring sound logic where logic is needed for clear understanding or vivid memory, but making frank use of assumptions where intuition and applications adequately serve these purposes. The amount of serious problem work or of thorough logic is not less than in the standard courses, but these are applied to fundamentals covering a large range. No undue effort is made towards fusion, but relations between topics are brought out where these are natural and useful. History is taught where it helps to an appreciation of subject matter, but the historical order is not followed. Topics are selected from all branches of mathematics as experience shows them to be workable. Algebra, geometry, and mechanics are represented. Perspective, graphical statics, foundations, non-euclidean geometry, a study of the number system, projective geometry, probabilities, finite groups, are typical topics. And withal it is believed that the impressions are not less vivid than in the standard courses.

2. *Graphical Solution of Spherical Triangles.* By Professor W. H. ROEVER, Washington University.

Professor Roever showed how to construct in the plane by means of circles and straight lines, the solutions of the six fundamental problems of spherical trigonometry. The statement of the rules was made so as to include all the possibilities which arise in the solution of the convex trihedral (spherical triangle of parts less than 180°). The rules are not always stated with sufficient generality, even in the standard foreign treatises on descriptive geometry.

3. *The Place of the Calculus in the Training of the High School Teacher.* By Professor BYRON COSBY, Kirksville Normal School.

Professor Cosby showed that parts of the calculus might well be given in place of parts of high-school algebra, being more economical of students' time and giving a better idea of the whole range of the mathematical field and of the continuity of the subject. Therefore the teacher should be equipped with the calculus.

4. *A Geometric Treatment of the Exponential Function.* By Dr. OTTO DUNKEL, Washington University.

In this paper the area under the hyperbola $y = 1/x$ was considered and the number e was defined as the abscissa of the point up to which the area is unity. From this definition the area up to any point was determined by elementary methods, using the principle of stretching (compressing) an area by stretching (compressing) all of its parallel chords in the same ratio. Having obtained an expression for the area, the derivative of e^x or of $\log_e x$ can be obtained from it rather simply. A method was then given for obtaining polynomial approximations to e^x, and from these closer approximations can be obtained for e than those given in the definition of e.

5. *Formulas for Approximate Integration.* By Professor BYRON INGOLD, Christian University.

A number of quite general formulas are known for approximating the value of a definite integral; for example, the well-known formulas of Gauss and the more recent formulas of Birkhoff. The formulas of Gauss, however, are inconvenient because of the fact that the division points of the interval of integration are in general irrational. It was the purpose of Professor Ingold's paper to obtain formulas comparable with the prismatoid formula in ease of application.

Among the formulas obtained may be mentioned:

$$\int_x^{x+h} f'(x)dx = \frac{h}{10}\left[f'(x) + 5f'\left(x + \frac{h}{3}\right) + 4f'\left(x + \frac{5h}{6}\right)\right],$$

obtained by inserting unknown coefficients and comparing with the expansion by Taylor's formula. This formula is usually better than the prismatoid formula for small values of h.

Other formulas deduced by methods analogous to the above are:

$$\int_x^{x+h} f'(x)dx = \frac{h}{4}\left[f'(x) + 3f'\left(x + \frac{2h}{3}\right)\right],$$

more accurate than the trapezoid formula in the third term of the expansion; and

$$\int_x^{x+h} f'(x)dx = hf'\left(x + \frac{h}{2}\right) + \frac{h^3}{24} f'''\left(x + \frac{h}{2}\right).$$

6. *Claims of Mathematics in the High School Course of Study.* By Mr. H. P. STELLWAGEN, Yeatman High School.

Mr. Stellwagen gave a résumé of the recent criticisms of the mathematics in the present course of study in the high schools of St. Louis, presented the general reasons, advanced by a committee of which he is chairman, for the retention of mathematics as a required subject, and invited discussion, in which as many took part as the time would permit.

7. *An Illustration of a Certain Necessary Condition in Minimizing a Definite Integral with a Discontinuous Integrand.* By Dr. PAUL R. RIDER, Washington University.

A curve C, which passes from a point P_0 to a curve D and thence to a point P_2 on the opposite side of D, minimizes the sum of the two integrals $I = \int F(x, y, x', y')dt$, $i = \int f(x, y, x', y')dt$. The first integral is to be taken along the arc from the point P_0 to the curve D, the second from D to P_2. Then at P_1, the point where the two arcs of the curve C meet the curve D, a certain relation must exist between the slope of the tangent to the curve D and the partial derivatives of F and f with respect to x' and y'. This paper illustrates the condition given in a problem that recently appeared in the MONTHLY (Vol. XXIII, page 125).

8. *On a Method of Sectioning Freshman and Sophomore Classes in Mathematics.* By Mr. ALAN CAMPBELL, Washington University.

This paper described a method in use in Washington University of dividing the freshman class in the schools of engineering and architecture into fast, medium, and slow sections. The sectioning is done on the basis of marks made by the students in tests covering high school mathematics, given during the first day or two of the college year. The sophomores are divided according to the grade of work they did in freshman year. This separation of good and poor students gives the better sections a chance to push forward much more rapidly, while the other sections move at a pace suited to the needs of the slower workers.

9. *The Equations and Models of a Large Group of Warped Surfaces.* By Professor C. A. WALDO, Washington University.

In this paper it was shown how to construct easily the models and the Cartesian equations of warped surfaces of two rectilinear directrices and one curvilinear (in particular rectilinear) directrix. These surfaces are special cases of surfaces called by Loria axoids and pseudoaxoids.

ALBERT H. HUNTINGTON,
Secretary.

THIRD REGULAR MEETING OF THE KANSAS SECTION.

The third regular meeting of the Kansas Section of THE MATHEMATICAL ASSOCIATION OF AMERICA was held in the Topeka High School, Topeka, on the morning of November 10, 1916.

Those present included the following 26 members of the ASSOCIATION:

C. H. Ashton, University of Kansas, Lawrence.
Lucy Dougherty, Kansas City, Kansas, High School.
Ottilia W. Dueker, Friends University, Wichita.
W. H. Garrett, Baker University, Baldwin.
W. A. Harshbarger, Washburn College, Topeka.
A. J. Hoare, Fairmount College, Wichita.
Emma Hyde, Kansas City, Kansas, High School.
H. E. Jordan, University of Kansas, Lawrence.
A. W. Larsen, University of Kansas, Lawrence.
S. Lefschetz, University of Kansas, Lawrence.
Theodore Lindquist, Kansas State Normal School, Emporia.
T. E. Mergendahl, College of Emporia, Emporia.
E. B. Miller, University of Kansas, Lawrence.
G. A. Miller, University of Illinois, Urbana, Ill.
U. G. Mitchell, University of Kansas, Lawrence.
Mary Newson, Washburn College, Topeka.
H. E. Porter, Kansas State Agricultural College, Manhattan.
B. L. Remick, Kansas State Agricultural College, Manhattan.
J. A. G. Shirk, Kansas State Manual Training Normal School, Pittsburg.
E. M. Stahl, Midland College, Atchison.
L. L. Steimley, University of Kansas, Lawrence.
E. B. Stouffer, University of Kansas, Lawrence.
W. T. Stratton, Kansas State Agricultural College, Manhattan.
J. N. Van der Vries, University of Kansas, Lawrence.
J. J. Wheeler, University of Kansas, Lawrence.
A. E. White, Kansas State Agricultural College, Manhattan.

Professor T. E. Mergendahl, of The College of Emporia, read a paper on "Algebra for students who enter college with only one unit of High School Algebra." The discussion of the paper was led by Miss Lucy Dougherty, of the Kansas City, Kansas, High School, and Professor E. M. Stahl, of Midland College. The following also took part in the discussion: Professors Van der Vries, Stouffer, Harshbarger, Ashton, Mitchell, Lefschetz, Remick and Professor G. A. Miller, of the University of Illinois.

The following officers were elected for the ensuing year:

J. A. G. SHIRK, State Manual Training Normal School, Chairman.
T. E. MERGENDAHL, College of Emporia, Vice-Chairman.
J. J. WHEELER, University of Kansas, Secretary-Treasurer.

The spring meeting will be held in Lawrence some time in March, 1917.

T. E. MERGENDAHL, *Secretary.*

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

William Oughtred, a Great Seventeenth-Century Teacher of Mathematics. By FLORIAN CAJORI. The Open Court Publishing Company, Chicago, 1916. vi + 100 pages. $1.00.

The story of William Oughtred is told by Professor Cajori in a delightful and entertaining manner. Oughtred's life covers the last quarter of the sixteenth century and the first half and part of the third quarter of the seventeenth century. This period in the history of mathematics is of particular significance because the wider dissemination and diffusion of mathematical knowledge during this time undoubtedly paved the way for the revolutionary developments of mathematics which took place in the seventeenth century. Mathematics is indeed a study which requires that its devotee withdraw himself from the world of affairs for reflection and meditation; however the great advances in this subject have been made at those times when large numbers of individuals were devoting themselves to the science. Such a time was this in which Oughtred lived. In England and in France there were many not connected with institutions of learning, men like Oughtred, like Napier, Vieta, Fermat and Descartes, who devoted themselves to the study of mathematics entirely because of an intellectual interest in the subject. Oughtred's devotion to the science was unbounded; not only did he instruct many in the "difficult poynts of the Art" gratis, but to certain specially gifted students he gave at the same time both bed and board.

The three principal works published by Oughtred are the *Clavis Mathematicae* of 1631, the *Trigonometria* of 1657, and *The Circles of Proportion* of 1632. All three works are of real importance in the development of elementary mathematics. The first work, which like the others is extremely concise, treats particularly of the numerical solution of equations; the *Trigonometria* and the *Clavis, The Key of the Mathematicks* as it was called in the English translation of 1647, are in advance of many contemporary mathematical works in the generous use of algebraic symbolism; *The Circles of Proportion* discusses the slide rule, of which, as has been proved by Professor Cajori, Oughtred was the inventor. Among his minor works several relate to the theory of mathematical instruments designed by Oughtred. These works enjoyed a wide popularity.

The two closing chapters of this little book are in many respects the most important, since they give a somewhat critical estimate of the Reverend Oughtred's influence upon mathematical progress and teaching and his ideas on the teaching of mathematics. Any teacher of mathematics and any "amateur" in mathematics of the type of Oughtred, if there be such in this practical day, will find great pleasure and intellectual profit in the reading of this work of Cajori's. I call attention to two mistakes. On page 83 the name of Erasmus O. Schreckenfuss is misspelled. On page 21 the statement is made that "the decimal point (or comma) was first used by the inventor of logarithms, John Napier, as early

as 1616 and 1617." But all the standard authorities on this subject agree that the point or comma was first used by the German Pitiscus in the 1612 edition of his trigonometry.[1] Napier has apparently no claim whatever to priority in the field, nor even to independent discovery for it is well established that he was familiar with the works of Pitiscus. The error is particularly unfortunate at the present time.

LOUIS C. KARPINSKI.

UNIVERSITY OF MICHIGAN.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

ALGEBRA.

473. Proposed by J. J. GINSBURG, Student, Cooper Union, New York.

Factor the expression $x^{30} + x^{25} + x^{20} + x^{15} + x^{10} + x^5 + 1$.

474. Proposed by A. A. BENNETT, University of Texas.

Show that the value of the infinite continued fraction, all of whose coefficients are unity, $1 + \frac{1}{1+}\frac{1}{1+}\cdots$, is $\frac{1}{2}(1 + \sqrt{5})$. Also find an explicit algebraic formula for the nth convergent.

GEOMETRY.

505. Proposed by O. S. ADAMS, Coast and Geodetic Survey, Washington, D. C.

Construct a triangle, having given the sum of two sides, the angle included by these sides, and the altitude from the given angle upon the third side.

506. Proposed by S. A. COREY, Albia, Iowa.

Given a pentagon, plane or gauche, whose sides a, b, c, d, e are represented by the vectors x, y, z, v and $(x + y + z + v)$, respectively; and a second pentagon whose sides a_1, b_1, c_1, d_1, e_1 are represented by the vectors r, s, t, u and $(r + s + t + u)$, respectively, where

$$r = c_1x - c_5c_2y - c_6c_3z + c_5c_6c_4v, \quad s = c_2x + c_1y - c_6c_4z - c_6c_3v, \quad t = c_1z + c_3x + c_5c_2v + c_5c_4y,$$

$$u = c_1v - c_4x - c_2z + c_3y, \qquad c_1, c_2, c_3, c_4, c_5 \text{ and } c_6 \text{ being ordinary scalars.}$$

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[1] As the introduction of the decimal point in 1616 and 1617 by Napier is claimed also in the recent Napier Tercentenary Volume it seemed to me worth while to verify the assertions which have been made in standard works on the subject, concerning the introduction of the decimal point by Pitiscus in 1612. I am enabled to do so through the courtesy of Professor David Eugene Smith and one of his students. In the *Canon Triangulorum emendatissimus et ad usum accommodatissimus, pertinens ad Trigonometriam Bartholomaei Pitisci, Grunbergensis Silesii* (Francofurti, Typis Nicolai Hoffmani, sumptibus Ionæ Rosæ, Anno M.DC.XII) Pitiscus uses for the sine of ten seconds the value 4.85 with the radius 100000; the value to 8 places is .00004848. Even more explicitly in the *Trigonometry* itself, also of 1612, Pitiscus uses a vertical bar and says: "Deinde pro latere AC nuper invento 13 | 00024 assumo 13 fractione scilicet 24/100000 neglecto quare ferme nullius fit momenti." It is thus evident that the 4.85 above which represents the decimal value was used with full consciousness of its significance by Pitiscus. I note also the use of the bar in the work of 1612, Bartholomaei Pitisi *Problematum Variorum* (Frankfurt). A further point worth noting is that Napier cites the earlier editions of Pitiscus, making it almost certain that between 1612 and 1616 he had copies of these works of 1612 in his hands. These works are in the New York Public Library.

Then prove the existence of the following relation between the sides of the two pentagons:

$$(c_1^2 + c_5c_2^2 + c_6c_3^2 + c_5c_6c_4^2)(x^2 + c_5y^2 + c_6z^2 + c_5c_6v^2) = r^2 + c_5s^2 + c_6t^2 + c_5c_6u^2.$$

507. Proposed by A. A. BENNETT, University of Texas.

With the use of the compasses alone construct a circle with area five times as great as that of a given circle. (This problem is said to be due to Napoleon I.)

CALCULUS.

421. Proposed by E. H. MOORE, The University of Chicago.

Given n continuous real-valued functions $\varphi_g(x)$ $(g = 1, 2, \cdots, n)$ of the real variable x on the interval (0 1) and set exp. $\int_0^1 \varphi_g(x)\varphi_h(x) = w_{gh}$ $(g, h = 1, 2, \cdots, n)$. Prove that the determinant $| w_{gh} |$ of the matrix (w_{gh}) is always $\gtreqq 0$ and that it is > 0 if no two of the functions $\varphi_1, \cdots, \varphi_n$ are identically equal on (0 1).

422. Proposed by O. S. ADAMS, Coast and Geodetic Survey, Washington, D. C.

Prove that

$$\int_0^1\int_0^1 f(xy)(1 - x)^{m-1}y^m(1 - y)^{n-1}dxdy = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m + n)}\int_0^1 f(z)(1 - z)^{m+n-1}dz,$$

$f(xy)$ being an arbitrary function of xy.

MECHANICS.

338. Proposed by J. B. REYNOLDS, Lehigh University.

A comet in a parabolic orbit crosses the earth's orbit (assumed circular) so that it remains a maximum time within it; find the comet's maximum velocity in miles per second and its time within the earth's orbit in years.

339. Proposed by C. N. SCHMALL, New York City.

A roll of cloth of very small uniform thickness a is coiled up tightly in the form of a circular cylinder of diameter d and is laid horizontally across a perfectly rough inclined plane so that its axis is parallel to the intersection of the plane with the horizontal. It is then allowed to unroll (without slipping) down the plane. Neglecting the motion of its center of gravity in the direction perpendicular to the plane, show that it will unroll entirely in the time

$$T = \frac{\pi}{4}\sqrt{\frac{6d^2}{ag \sin \phi}},$$

where ϕ is the inclination of the plane to the horizontal plane, and g the acceleration of gravity.

NUMBER THEORY.

257. Proposed by LOUIS O'SHAUGHNESSY, University of Pennsylvania.

Find a general expression for the number of positive integers from 1 to 10^t, inclusive, every one of which contains the figure 9 exactly r times $(0 \leqq r \leqq t)$.

258. Proposed by A. A. BENNETT, University of Texas.

Find a recursion formula in terms of binomial coefficients for a_n, where the a's are defined by the condition that the persymmetric determinants

$$\begin{vmatrix} a_0 & a_1 & a_2 & \cdot & \cdot \\ a_1 & a_2 & \cdot & \cdot & \cdot \\ a_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_{n-1} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_1 & a_2 & a_3 & \cdot & \cdot \\ a_2 & a_3 & \cdot & \cdot & \cdot \\ a_3 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_n \end{vmatrix}$$

are each equal to unity for every positive integer n.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

459. Proposed by C. N. SCHMALL, New York City.

By d'Alembert's test, or otherwise, show that in the infinite series

$$x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \cdots + \frac{n^nx^n}{n!} + \cdots$$

the upper limit of the interval of convergence is $1/e$, where e is the Napierian base, *i. e.*, $x < 1/e$ when the series is convergent. (Bromwich's *Infinite Series*, pp. 28, 33.)

SOLUTION BY HORACE OLSON, Chicago, Illinois.

Let r_n represent the ratio of the $(n+1)$th term of the given series to the nth term. Then $\lim\limits_{n \doteq \infty} r_n = \lim\limits_{n \doteq \infty} \left(1 + \frac{1}{n}\right)^n x = ex.$
Therefore, the series is convergent if $x < 1/e$, and divergent if $x > 1/e$.

If $x = 1/e$, $r_n = \left(1 + \frac{1}{n}\right)^n \cdot \frac{1}{e}$.

Let ρ_n be the ratio of the $(n+1)$th term of the harmonic series to the nth term; *i. e.*, $\rho_n = n/(n+1)$. Then $r_n/\rho_n = 1/e[1 + (1/n)]^{n+1}$.

$[1 + (1/n)]^{n+1}$ is a decreasing function of n, since its derivative

$$\left(1 + \frac{1}{n}\right)^{n+1} \left\{\log\left(1 + \frac{1}{n}\right) - \frac{1}{n}\right\} < 0.$$

But

$$\lim_{n \doteq \infty} \left(1 + \frac{1}{n}\right)^{n+1} = \lim_{n \doteq \infty} \left(1 + \frac{1}{n}\right) \cdot \lim_{n \doteq \infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Therefore r_n/ρ_n is always greater than unity.

Therefore the nth term of the given series is always greater than $1/e$ times the nth term of the harmonic series, and, since the latter is divergent, the former also is divergent.

Therefore, finally, the given series is convergent for $x < 1/e$ and divergent for $x \geqq 1/e$.

Also solved by E. W. WORTHINGTON, A. M. HARDING, G. W. HARTWELL, E. J. OGLESBY, and the PROPOSER.

460. Proposed by J. J. GINSBURG, Student, Cooper Union, New York.

Find the value of $\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}$ to infinity.

SOLUTION BY NATHAN ALTSHILLER, University of Oklahoma.

The number of operations increasing indefinitely, the above expression has no value, strictly speaking, unless the word "value" is taken in this connection to be the exact equivalent of the word "limit." The problem means: given a sequence of terms

$$u_1 = \sqrt{1}, \quad u_2 = \sqrt{1 + \sqrt{1}}, \quad u_3 = \sqrt{1 + \sqrt{1 + \sqrt{1}}}, \quad \cdots \quad u_n = \cdots$$

find the limit of u_n when n increases indefinitely, if such a limit exists.

The problem may be generalized by considering $\sqrt{a + \sqrt{a + \sqrt{a + \sqrt{a + \cdots}}}}$ to infinity, where a is any positive quantity.

We observe that the values of the u's increase with n so that (1) $u_{n+1} > u_n$. It is also easily seen that (2) $u_{n+1}^2 = u_n + a$.

From (1) and (2) it follows that $u_n^2 - u_n - a < 0$.

There is an upper limit to the values of u_n which satisfy this inequality, this being the larger of the roots of the equation:

$$u_n^2 - u_n - a = 0.$$

Thus u_n cannot exceed a given quantity independent of n, and on the other hand u_n constantly increases with n. Hence, u_n *has a limit*, according to the fundamental principle of the theory of limits. Let this limit be denoted by s. The value of s may be found by passing to the limit in (2).

$$\lim_{n\to\infty} u_{n+1}^2 = \lim_{n\to\infty} u_n + a, \quad \text{or} \quad s^2 = s + a,$$

hence $s = \dfrac{1 + \sqrt{1 + 4a}}{2}$, since s is necessarily positive.

The problem may be further generalized by considering the expression

$$\sqrt[p]{a + \sqrt[p]{a + \sqrt[p]{a + \sqrt[p]{a + \cdots}}}} \text{ to infinity}$$

where p is a positive integer, and a being any real quantity when p is odd, or any positive quantity, when p is even. The method of solution would be the same, in the main, as the one outlined above.

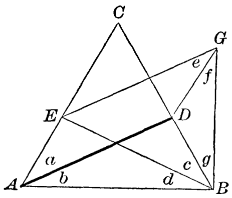
Also solved by W. P. Ransom, C. N. Schmall, A. M. Harding, E. E. Clark, Horace Olson, O. S. Adams, H. N. Carleton, Paul Capron, J. M. Stetson, N. P. Pandya, and the Proposer.

GEOMETRY.

487. Proposed by H. B. PHILLIPS, Massachusetts Institute of Technology.

If segments from the vertices A and B of a triangle to the opposite sides are of equal length and divide the angles A and B proportionally, the triangle is isosceles.

I. Solution by J. J. Ginsburg, Student, Cooper Union, New York City.



In the triangle ABC, we have by hypothesis $AD = BE$ and

(1) $$\frac{a}{b} = \frac{c}{d} = \frac{m}{n},$$

when a, b, c, d are the measures of the angles into which $\angle A$ and $\angle B$ are divided by AD and BE, and m/n is any given ratio.

We are to prove that $\angle A = \angle B$. Suppose that $\angle A \neq \angle B$, and suppose $\angle A > \angle B$. Then from (1)

(2) $$a > c \text{ and } b > d.$$

In the triangles ABD and AEB, $AD = BE$, $AB = AB$ and $b > d$. Hence

(3) $$BD > AE.$$

Draw $EG \parallel AD$ and $DG \parallel AE$, and draw BG. Then $EG = AD$ and $AD = BE$ by hypothesis. Hence $\triangle ABG$ is isosceles.

We have, then, $e + f = c + g$ and $e = a > c$. Hence,

(4) $$f < g \quad \text{and} \quad BD < DG < AE.$$

But (3) and (4) are contradictory. Hence the supposition that $\angle A > \angle B$ is impossible; Likewise $\angle A < \angle B$ is impossible. Hence $\angle A = \angle B$, and triangle ABC is isosceles.

II. Solution by Paul Capron, U. S. Naval Academy.

Denote the length of each segment by l, and the length of $\overline{AB}$ by c. Let the part of the angle A next to c be kA; then the part of the angle B next to c is kB. $[0 < k < 1.]$

Then

$$\frac{l}{\sin B} = \frac{c}{\sin(kA + B)}, \quad \text{and} \quad \frac{l}{\sin A} = \frac{c}{\sin(kB + A)}.$$

Hence,

$$\frac{\sin B}{\sin A} = \frac{\sin (kA + B)}{\sin (kB + A)}.$$

By composition and division,

$$\frac{\sin B - \sin A}{\sin B + \sin A} = \frac{\sin (kA + B) - \sin (kB + A)}{\sin (kA + B) + \sin (kB + A)};$$

whence,

$$\frac{\tan \frac{1}{2}(B - A)}{\tan \frac{1}{2}(B + A)} = \frac{\tan \frac{1-k}{2}(B - A)}{\tan \frac{1+k}{2}(B + A)}; \quad \text{or} \quad \frac{\tan \frac{1}{2}(B - A)}{\tan \frac{1-k}{2}(B - A)} = \frac{\tan \frac{1}{2}(B + A)}{\tan \frac{1+k}{2}(B + A)}$$

Since $0 < k < 1$ and the tangent is an increasing function, it follows that if B and A were unequal, the last relation would be untrue, for then the left-hand fraction would be greater than unity, and the right-hand fraction would be less than unity. Hence, $B = A$, and the triangle is isosceles.

488. Proposed by ROGER A. JOHNSON, Western Reserve University.

If triangles are constructed on a given base, having the radii of the incircle and circumcircle in a constant ratio, determine the locus of the vertex (necessarily the constant ratio is not greater than $\frac{1}{2}$).

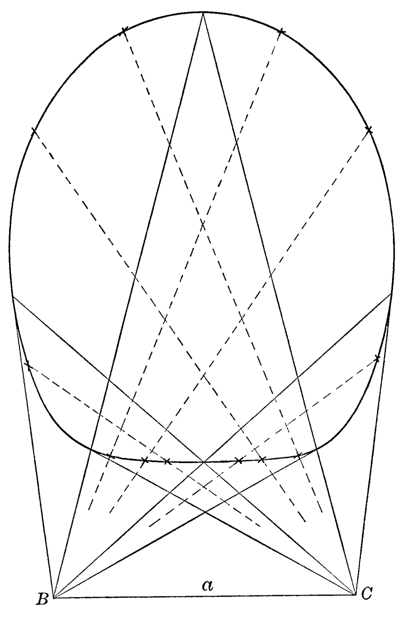
SOLUTION BY PAUL CAPRON, U. S. Naval Academy.

Let the constant ratio be $n = 2m$, the given base a, with its left end at B, its right end at C; and let the sides a, b, c of any one of the triangles be opposite the vertices A, B, C respectively. Then

$$n = 2m = = \frac{ac \sin B}{a + b + c} \div \frac{b}{2 \sin B} = \frac{2ac \sin^2 B}{b(a + b + c)}.$$

Let $c = r$, $B = \theta = 2\phi$; then $b^2 = a^2 + r^2 - 2ar \cos \theta = (a + r)^2 - 4ar \cos^2 \phi$. On substituting these values, we have, for the locus of A:

(1) $$m(a + r) \sqrt{(a + r)^2 - 4ar \cos^2 \phi} = 4ar \cos^2 \phi(m + \sin^2 \phi) - m(a + r)^2.$$



Rationalizing:

$$(2) \qquad r\cos^2\phi[(m^2+2m\sin^2\phi)(a+r)^2 - 4a\cos^2\phi(m+\sin^2\phi)^2 r] = 0.$$

As $A \doteq B$, $m \doteq 0$; except in this trivial case, $r = 0$ may be discarded. Solving (2) for $(r + a)$ we find,

$$(3) \qquad r + a = \frac{2a\cos^2\phi(m+\sin^2\phi)^2}{m^2+2m\sin^2\phi} \pm \frac{2a\sin\phi\cos\phi}{m^2+2m\sin^2\phi}(m+\sin^2\phi)\sqrt{\sin^2\phi - (m+\sin^2\phi)^2}.$$

Replacing m and ϕ by $\frac{1}{2}n$ and $\frac{1}{2}\theta$,

$$(4) \qquad r = a\cos\theta + \frac{a\sin\theta}{n^2+2n(1-\cos\theta)}[\sin\theta(1-\cos\theta) \pm (n+1-\cos\theta)\sqrt{1-2n-(\cos\theta-n)^2}],$$

or

$$(4)' \qquad r = a\cos\theta + \frac{a\sin^2\theta\operatorname{vers}\theta}{n^2+2n\operatorname{vers}\theta} \pm \frac{a\sin\theta(n+\operatorname{vers}\theta)}{n^2+2n\operatorname{vers}\theta}\sqrt{\sin^2\theta - 2n\operatorname{vers}\theta - n^2},$$

where $\operatorname{vers}\theta = 1 - \cos\theta$.

When $\theta = 0$, $r = a$, and when $\theta = \pi$, $r = -a$; this point C must be discarded from the locus, just as B was.

Aside from this, r is real when and only when

$$(5) \qquad n - \sqrt{1-2n} \leq \cos\theta \leq n + \sqrt{1-2n},$$

or

$$1 - n + \sqrt{1-2n} \geq \operatorname{vers}\theta \geq 1 - n - \sqrt{1-2n}.$$

(6) When θ has either of the limiting values given by (5), then $r = a$ and the third sides of these isosceles triangles are

$$a(1 \pm \sqrt{1-2n}).$$

Considerations of symmetry show that the high and low points of the curve occur where $r = a/2 \sec\theta$. Substituting this value in (2), we have:

$$m(m+1-\cos\theta)(2\cos\theta+1)^2 - \cos\theta(1+\cos\theta)(2m+1-\cos\theta)^2 = 0;$$

whence

$$(\cos^2\theta - \cos\theta + m)(\cos^2\theta - (m+1)) = 0,$$

and

$$(7) \qquad \cos\theta = \pm\tfrac{1}{2}\sqrt{4+2n},$$

or

$$(8) \qquad \cos\theta = \tfrac{1}{2}(1 \mp \sqrt{1-2n}).$$

The values given by (7) are without the limits set in (5); those given by (8) are within these limits. From (8),

$$(9) \qquad r = \frac{a}{2n}(1 \pm \sqrt{1-2n}),$$

the high and low points; or

$$(10) \qquad r = a(1 \mp \sqrt{1-2n}),$$

giving two other points.

The curve consists of two ovals, symmetrically situated with regard to $\theta = 0$, the common base of the triangles, and each oval is symmetrical with regard to $2r = a\sec\theta$, the mid-perpendicular to this base. In addition to the limiting tangents from B, given by (6), there is a symmetrically situated pair from C; these, however, are tangent at the points given by (10). Thus in each oval we can readily construct six points, with the tangent at each. These points determine the isosceles triangles satisfying the conditions of the problem. The six triangles form two sets of three similar triangles.

The accompanying figure shows one of the ovals, constructed for $n = 3/8$.

The values used are:

$\cos\theta$	$r = c$	
$-\frac{1}{8}$	a	$(b = \frac{3}{2}a)$
$\frac{1}{4}$	$\begin{cases} 2a \\ \frac{1}{2}a \end{cases}$	High Point $(b = 2a)$ $(b = a)$

$\frac{3}{8}$	$\frac{a}{39}(49 \pm 4\sqrt{55}) = 2.017a$ or $0.496a$	
$\frac{9}{16}$	$\frac{a}{384}(461 \pm 13\sqrt{385}) = 1.865a$ or $0.536a$	
$\frac{3}{4}$	$\frac{2}{3}a$	Low Point $(b = \frac{2}{3}a)$
	$\frac{3}{2}a$	$(b = a)$
$\frac{13}{16}$	$\frac{a}{128}(133 \pm \sqrt{1305}) = 1.221a$ or $0.757a$	
$\frac{7}{8}$	a	$(b = \frac{1}{2}a)$

CALCULUS.

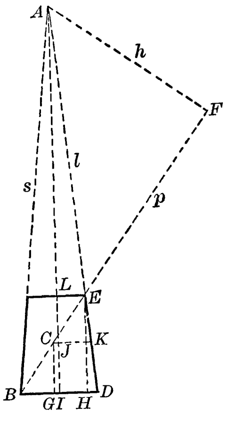
407. Proposed by PAUL CAPRON, Annapolis, Maryland.

A coffee pot in the form of a conical frustum, 10 inches high, with a lower base 8 inches in diameter and an upper base 6 inches in diameter, is held on a slant so that the lower base is barely covered by the coffee within, and the upper base is barely uncovered. How much coffee does the pot contain?

I. Solution by H. S. Uhler, Yale University.

The following solution may be of interest because it is based exclusively on theorems of elementary geometry. The diagram (drawn to scale) represents a plane section passing through the axis $\overline{IA}$ of the frustum $BDEL$ and the major axis $\overline{BE}$ of the elliptical free-surface of the coffee. The required volume will be gotten by subtracting the volume of the oblique cone ABE from that of the right cone ABD. We are given that $\overline{IL} = 10$ in., $\overline{ID} = 4$ in., and $\overline{LE} = 3$ in.

From the similar triangles AID and ALE, $\overline{AI} : \overline{ID} = \overline{AL} : \overline{LE}$ or $\overline{AI} : 4 = (\overline{AI} - 10) : 3$, hence $\overline{AI} = 40$ in. Hence, the volume of the right cone equals $(\frac{640}{3})\pi$ cu. in.



The altitude of the oblique cone may be obtained from the right triangles ABF and AEF, for, $s^2 = (2a + p)^2 + h^2$ and $l^2 = p^2 + h^2$, where a denotes the semi-major axis of the ellipse. That $a = 0.5\sqrt{149}$ may be seen at once from the right triangle BHE since $\overline{BH} = 7$ and $\overline{HE} = 10$. $s^2 = (\overline{BI})^2 + (\overline{AI})^2 = 1616$ and $l^2 = (\overline{LE})^2 + (\overline{LA})^2 = 909$. Elimination of p from the twoliteral equations gives $h = 240/\sqrt{149}$ in. It remains to find the semi-minor axis b of the ellipse. This axis will pass through C, the middle point of $\overline{BE}$, it will be perpendicular to the plane of the diagram, and it will be a chord of the circle whose radius is $\overline{JK}$ and whose plane is parallel to the bases of the given frustum. Since C bisects $\overline{BE}$ and $\overline{CK}$ is parallel to $\overline{BD}$, $\overline{BG} = 0.5\overline{BH} = 3.5$. $\overline{CJ} = \overline{BI} - \overline{BG} = 4 - 3.5 = 0.5$. $\overline{JK} = 0.5(\overline{ID} + \overline{LE}) = 3.5$. Since the minor axis of the ellipse constitutes the chord of a circle of radius 3.5 in. and is at a distance 0.5 in. from the center it follows that $b^2 = (3.5 - 0.5)(3.5 + 0.5) = 12$. Hence, $b = 2\sqrt{3}$ in. The area of the ellipse $= \pi ab = \pi\sqrt{3 \times 149}$;hence, the volume of the oblique cone equals

$$\tfrac{1}{3}(\pi\sqrt{3 \times 149})\left(\frac{240}{\sqrt{149}}\right) = 80\pi\sqrt{3} \text{ cu. in.}$$

Consequently, the volume of the coffee equals

$$\frac{80\pi}{3}(8 - 3\sqrt{3}) \text{ cu. in.} \doteq 234.894{,}585{,}349{,}6 \text{ cu. in.}$$

II. Solution by H. S. Uhler, Yale University.

Let us choose a set of rectangular coördinate axes such that the origin is at the center of the lower base of the frustum, the axis of y lies in the plane of the lower base with its negative segment passing through the lower end of the major axis of the elliptical free-surface of the coffee, and the axis of z is directed from the origin toward the apex of the completed cone. If the axes of y and z are drawn to the right and upward respectively, the axis of x will be out from the plane of the paper toward the reader.

We shall take as element of volume the frustum of an oblique pyramid bounded as follows: Two of the lateral faces are portions of the surfaces of right circular cones having the same basal plane and apex as the given cone, and with radii r and $r + dr$. The two remaining lateral faces lie in planes which contain the axis of the given cone (*i. e.*, OZ) and make angles θ and $\theta + d\theta$ with the plane YOZ. The upper base of this frustum is parallel to the lower base and passes through the point of intersection of the plane of the ellipse with the edge which is nearest to the axis OZ and to the plane YOZ. No finite error can arise from neglecting the little solid between the plane of the ellipse and the plane of the upper base of the chosen element of volume because it is an infinitesimal of the third order, whereas the element of volume just defined is of the second order.

The volume of the element may be looked upon as the difference between the volumes of two pyramids. The area of the base of the larger pyramid equals $rdrd\theta$. The altitude of this pyramid is identical with that of the given cone, and is obviously equal to 40 inches. Hence, the volume of the taller pyramid equals $40rdrd\theta/3$. If h denotes the altitude of the element of volume then the altitude of the shorter pyramid will be $40 - h$. Since the volumes of two similar pyramids are to each other as the cubes of their altitudes it follows that the volume of the lesser pyramid equals $(40 - h)^3 rdrd\theta/4800$. Consequently, the element of volume equals $40rdrd\theta/3 - (40 - h)^3 \cdot rdrd\theta/4800$.

h may be obtained from its definition as follows. The plane of the ellipse contains the line $y = -4$ and the point $(0, 3, 10)$ so that its equation is $-10y + 7z = 40$. The direction cosines, with reference to the axes of y and z respectively, of the edge drawn from $(r \sin\theta, r\cos\theta, 0)$ to $(0, 0, 40)$ are easily found to be in the ratio $-r\cos\theta : 40$, so that one equation of the edge is

$$-\frac{y}{r\cos\theta} = \frac{z - 40}{40}.$$

Elimination of y between this equation and that of the plane of the ellipse gives

$$z = h = \frac{40(4 + r\cos\theta)}{28 + r\cos\theta}.$$

Finally, the element of volume reduces to

$$\frac{40}{3} rdrd\theta - \frac{184{,}320 rdrd\theta}{(28 + r\cos\theta)^3}.$$

Volume of coffee $\equiv V = \frac{80}{3}\int_0^\pi d\theta \int_0^4 rdr - 368{,}640 \int_0^\pi d\theta \int_0^4 \frac{rdr}{(28 + r\cos\theta)^3}$. Making use of the fact that

$$\int \frac{xdx}{(a + bx)^3} = \frac{1}{b^2}\left[-\frac{1}{a + bx} + \frac{a}{2(a + bx)^2}\right]$$

we find

$$V = \frac{640\pi}{3} - 92{,}160 \int_0^\pi \left[-\frac{1}{7 + \cos\theta} + \frac{7}{2(7 + \cos\theta)^2} + \frac{1}{14}\right] \sec^2\theta d\theta$$

or

$$V = \frac{640\pi}{3} - \frac{46{,}080}{7}\int_0^\pi \frac{d\theta}{(7 + \cos\theta)^2}.$$

The tabular integrals

$$\int \frac{dx}{(a + b\cos x)^2} = \frac{1}{a^2 - b^2}\left[-\frac{b\sin x}{a + b\cos x} + a\int \frac{dx}{a + b\cos x}\right]$$

$$\int \frac{dx}{a + b\cos x} = \frac{-1}{\sqrt{a^2 - b^2}} \cdot \sin^{-1}\left[\frac{b + a\cos x}{a + b\cos x}\right]$$

lead to

$$\int_0^\pi \frac{d\theta}{(7+\cos\theta)^2} = \frac{7\sqrt{3}}{576}\pi$$

so that

$$V = \frac{80\pi}{3}(8 - 3\sqrt{3}) \text{ cu. in.}$$

$$V \doteq 234.894,585,349,6 \text{ cu. in.}$$

Also solved by G. A. Knapp, C. N. Schmall, George Paaswell, H. N. Carleton, O. S. Adams, and the Proposer.

408. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

The ellipse $(x^2/81) + (y^2/16) = 1$ is revolved around the y-axis. Find the area of the surface generated.

Solution by Clyde S. Atchison, Washington and Jefferson College.

From the given equation, we have

$$x = \frac{9}{4}\sqrt{16 - y^2}; \quad dx = -\frac{9ydy}{4\sqrt{16-y^2}}; \quad \text{and} \quad ds = \frac{1}{4}\sqrt{\frac{256 + 65y^2}{16 - y^2}} \cdot dy.$$

Then the area of the surface generated is

$$2\pi\int_{y=-4}^{y=4} x \cdot ds = 2\pi\int_{-4}^{4}\left(\frac{9}{4}\sqrt{16-y^2}\right)\left(\frac{1}{4}\sqrt{\frac{256+65y^2}{16-y^2}}\right)dy = \frac{9\pi}{8}\int_{-4}^{4}\sqrt{256+65y^2}\cdot dy$$

$$= \frac{9\pi\sqrt{65}}{16}\left[y\cdot\sqrt{y^2 + \frac{256}{65}} + \frac{256}{65}\log\left(y + \sqrt{y^2 + \frac{256}{65}}\right)\right]_{-4}^{4}$$

$$= \frac{9\pi\sqrt{65}}{16}\left\{\frac{144}{\sqrt{65}} + \frac{256}{65}\log\left(4 + \frac{36}{\sqrt{65}}\right) + \frac{144}{\sqrt{65}} - \frac{256}{65}\log\left(-4 + \frac{36}{\sqrt{65}}\right)\right\},$$

$$= 162\pi + \frac{144\pi}{\sqrt{65}}\log\frac{9+\sqrt{65}}{9-\sqrt{65}}.$$

Also solved by A. M. Harding, Nellie L. Ingals, Horace Olson, C. C. Yen, G. W. Hartwell, H. C. Feemster, J. A. Eckson, George Paaswell, O. S. Adams, and Paul Capron.

MECHANICS.

321. Proposed by E. J. MOULTON, Northwestern University.

The attraction, A, in any direction, due to a homogeneous sphere, on a particle at the center of the sphere, using the Newtonian law, is obviously zero. Find the error in the following method of computing A. Take cylindrical coördinates with origin at the center of the sphere; let the Z-axis extend in the direction of the attraction to be computed, and let r, θ be the polar coördinates used. Let δ be the density and R the radius of the sphere, and k the constant of gravitation. Then

$$A = \int_{z=-R}^{z=R}\int_{r=0}^{r=\sqrt{R^2-z^2}}\int_{\theta=0}^{\theta=2\pi}\frac{k\delta r dz d\theta dr}{[r^2+z^2]^{\frac{3}{2}}} \tag{1}$$

$$= 2\pi k\delta\int_{z=-R}^{z=R}\left[\frac{-z}{(r^2+z^2)^{\frac{1}{2}}}\right]_{r=0}^{z=\sqrt{R^2-z^2}} dz \tag{2}$$

$$= 2\pi k\delta\int_{-R}^{R}\left[\frac{-z}{R} + 1\right]dz \tag{3}$$

$$= 4\pi k\delta R. \tag{4}$$

Solution by Elijah Swift, University of Vermont.

In the integrand of (1) the radical is to be taken with the plus sign, as $[r^2 + z^2]^{\frac{1}{2}}$ is a length, namely, the radius vector from the origin to the element in question. In evaluating (2), then, we must take the radical with the plus sign. But when r is zero the value of the radical is $\pm z$, so that we must take $+ z$ when z is positive, $- z$ when z is negative. In obtaining (3), however, this value was taken as $+ z$ for all values of z. (3) should read

$$2\pi k\delta \left[\int_0^R \left[\frac{-z}{R} + 1 \right] dz + \int_{-R}^0 \left[\frac{-z}{R} - 1 \right] dz \right] = 0.$$

Also solved by Paul Capron, J. W. Clawson, George Paaswell, K. P. Williams, and the Proposer.

323. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Two equal bodies are placed on a rough inclined plane, being connected by a light inelastic string; if the coefficients of friction are respectively 1/3 and 1/4, show that they will both be on the point of motion when the inclination of the plane is $\sin^{-1}(7/25)$.

Solution by H. S. Uhler, Yale University.

In order to obtain the given inclination it is necessary to assume that the string lies in a vertical plane which is perpendicular to the intersection of the inclined plane with any horizontal plane, that it is taut, etc. From simple mechanical considerations it is evident that the body associated with the greater coefficient of friction must be higher up on the incline than its tandem body. Let m and T denote, respectively, the mass of either body and the tension of the string. By resolving the weight (mg) of each body parallel and perpendicular to the incline, and by employing the definition of the coefficient of friction, we find at once that the condition for being on the point of motion is expressed by the equations

$$T + mg \sin \alpha - \tfrac{1}{3} mg \cos \alpha = 0,$$

$$- T + mg \sin \alpha - \tfrac{1}{4} mg \cos \alpha = 0,$$

where α symbolizes the angle of elevation of the inclined plane. By first adding these equations and then dividing through by mg we get

$$2 \sin \alpha - (7/12) \cos \alpha = 0,$$

or

$$\alpha = \tan^{-1}(7/24) = \sin^{-1}(7/25).$$

Also solved by J. Rosenbaum, Horace Olson, H. C. Feemster, C. A. Nickle, G. Paaswell, W. C. Eells, J. A. Caparo, W. H. Thome, and A. G. Rau.

NUMBER THEORY.

211. (April, 1914.) Proposed by E. T. BELL, University of Washington.

If an odd perfect number exists, the total number of its divisors is a multiple of 2 but not of 4; or, what is the same thing, an odd perfect number must be of the form $p^{2a-1}n^2$ where p is a prime and a is odd.

Remark by Tracy A. Pierce, Harvard University.

Lucas in his *Théorie des Nombres* proved that an odd perfect number must be of the form $(4m + 1)^{4k+1}n^2$, where $4m + 1$ is a prime. See an article by Bourlet in *Nouv. Ann. de Math.*, 1896, pp. 297–312.

208. (March, 1914.) Proposed by E. T. BELL, University of Washington.

If an odd number be perfect it cannot be the sum of two squares.

Remarks by Tracy A. Pierce, Harvard University.

It is well known that every prime of the form $4m + 1$ is the sum of two squares. Using Lucas's form of an odd perfect number (see Problem 211 above) we see that

$$(4m + 1)^{4k+1}n^2 = (x^2 + y^2)[(4m + 1)^{2k}]^2 n^2 = (x^2 + y^2)N^2 = X^2 + Y^2,$$

contrary to the proposition as proposed.

229. (March, 1915.) Proposed by WALTER C. EELLS, Whitman College.

If p and q are integers and p is prime and positive, find the condition on q that the equation $p^x = qx$ shall have integral solutions, solve for x, and show that for a special value of p it has two solutions for a certain q, otherwise only one.

I. Solution by Frank Irwin, University of California.

Since p is prime, we must have $x = p^a$, $q = p^b$, where a and b are positive integers or zero (for a). Now $p^{a+b} = qx = p^{p^a}$, so that $b = p^a - a$, and q is necessarily of the form $p^{(p^a-a)}$. This condition is evidently also sufficient.

Given, then, such a q, the exponent $p^a - a$ may be determined; then a, which will give us x, is that number which must be added to this exponent to make it equal to the *next higher* power of p. For no power of p can lie between $p^a - a$ and p^a, since $p^a - a > p^{a-1}$, as may be readily proved, for instance by mathematical induction.

Of the cases that require special investigation, $a = 0$, and $p = 2$ with $a = 1$ or 2, the only one for which, given $p^a - a$, there is more than one solution for a, is the case

$$2^a - a = 1,$$

which has two solutions $a = 0, 1$. There are two solutions of our problem then for the case $p = 2$, $q = 2$, viz., $x = 1, 2$.

II. Solution by the Proposer.

Consider the two functions, $y = p^x$, $y = qx$. For $x = k$ (any integer), $y = p^k$. The slope of line $y = qx$, passing through (k, p^k), is $q = p^k/k$, which is integral if and only if

$$k = p^n \qquad (n = 0, 1, 2, 3, \cdots), \ i.\ e.,\ q = p^{(p^n-n)}.$$

(If $k < 0$, q is fractional since it is then $= 1/kp^k$).

Substituting this value of q in the given equation, it is easily seen that it is satisfied if and only if

$$x = p^n \qquad (n = 0, 1, 2, 3, \cdots).$$

Consider the exponent of p, namely $p^n - n$. We have

$$[p^n - n]_{n=0} = 1, \quad \text{and} \quad [p^n - n]_{n=1} = p - 1.$$

Then $x_1 = p^0$ and $x_2 = p^1$ will be solutions of the given equation if $1 = p - 1$, *i. e.*, if $p = 2$. From the graphs of the exponential function it is easily seen that $y = qx$ can have but one integral intersection if $p \neq 2$, $n_1 \neq 0$, $n_2 \neq 1$.

The equation having two solutions is $2^x = 2x$, of which $x_1 = 1$, $x_2 = 2$.

230. (April, 1915.) Proposed by E. B. ESCOTT, Ann Arbor, Michigan.

Find three numbers such that their sum, the sum of their squares, and the sum of their cubes, shall be a cube.

Note.—W. D. Cairns says this problem, which was proposed in *L'Intermediaire* in 1900, remains unsolved to date, even though it was reprinted in February, 1913.

Remarks by Artemas Martin, Washington, D. C.

The above problem was published in the *Mathematical Visitor*, Vol. I, No. 1 (Erie, Pa., March, 1877), page 6, as No. 9 in a list of "Unsolved Problems." So far as the writer at present knows that was the first publication of the problem and it still remains "unsolved."

233. (June, 1915.) Proposed by V. M. SPUNAR, Chicago, Illinois.

Solve in rational numbers $x^2 + y^2 = a^2$, $xy = m/n$, when m and n are integers and relatively prime to each other.

SOLUTION BY A. H. HOLMES, Brunswick, Maine.

Putting $x = (p^2 - q^2)/(2p \pm 2q + 1)$, $y = (2pq)/(2p \pm 2q + 1)$, we shall have

$$a = (p^2 + q^2)/(2p \pm 2q + 1), \qquad m = 2pq(p^2 - q^2),$$

and $n = (2p \pm 2q + 1)^2$ except in the cases where $p = q + 1$, $2q + 1$, or $5q + 1$, in which cases $n = 2p \pm 2q + 1$.

For smallest values put $q = 1$, $p = 2$, using the minus sign of the denominator. Then $x = 1$, $y = 4/3$, $a = 5/3$, $m = 4$, and $n = 3$.

Also solved by H. N. CARLETON.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence.

DISCUSSIONS.

I. RELATING TO AN INSTRUCTIVE PROBLEM IN ATTRACTION.

BY EDWIN BIDWELL WILSON, Massachusetts Institute of Technology.

In B. O. Pierce's *Newtonian Potential Function* (3d ed.) there is found on page 28 this exercise:

Show that the attraction at the focus of a segment of a paraboloid of revolution bounded by a plane perpendicular to the axis at a distance b from the vertex is of the form $4\pi\rho a \log(1 + b/a)$.

This problem is also found in other texts. The answer is very easy to get; it is also wrong. One way in which the answer may be found is this:

The attraction of a disc of mass M and radius r at a point on its axis at a distance c from the center is

$$\frac{2M}{r^2}\left(1 - \frac{c}{\sqrt{c^2 + r^2}}\right).$$

If the equation of the revolved parabola is $4ay = x^2$, the attraction of a segment at the focus is

$$\frac{2\pi\rho \cdot 4ay}{4ay}\left(1 - \frac{y - a}{\sqrt{(y - a)^2 + 4ay}}\right) dy,$$

and the total attraction is

$$A = 2\pi\rho \int_0^b \left(1 - \frac{y - a}{\sqrt{(y + a)^2}}\right) dy.$$

To the careless person this gives

$$A = 2\pi\rho \int_0^b \frac{2a\,dy}{y + a} = 4\pi\rho a \log(1 + b/a)$$

because $\sqrt{(y+a)^2}$ for him equals $y+a$. Indeed, since $y+a$ does not pass through zero, it might seem as though the root could be extracted in this way. In the numerator, however, there occurs the expression $y-a$ which does pass through zero when $y=a$ (*i. e.*, at the section through the focus) and this should be a danger signal.

There is another sign of trouble and a very fundamental one. We can not expect to get a single formula to express the attraction of the segment for the two cases $b<a$ and $b>a$, any more than we can expect to get a single formula for the attraction of a sphere on a point irrespective of whether the point is inside or outside of the sphere. The attraction is a continuous function of b; but the derivative of the force with respect to b is discontinuous when $b=a$, as is familiar to all who are acquainted with Poisson's Equation and its physical significance. The amount of the discontinuity is $4\pi\rho$.

The solution of this, and similar problems, is carried out in two steps. First when $b<a$, the attraction is toward the vertex of the paraboloid and equal to

$$A = 2\pi\rho\int_0^b\left(1-\frac{a-y}{a+y}\right)dy = 4\pi\rho a\left[\frac{b}{a}-\log\left(\frac{b}{a}+1\right)\right],$$

which, when $b=a$, gives the value $4\pi\rho a(1-\log 2)$. When $b>a$, the calculation of the attraction of the part of the segment beyond the focus gives

$$2\pi\rho\int_a^b\left(1-\frac{y-a}{y+a}\right)dy = 4\pi\rho a\left[\log\left(\frac{b}{a}+1\right)-\log 2\right]$$

and directed away from the vertex. The whole segment from 0 to b, therefore, has the attraction equal to

$$A = 4\pi\rho a\left[1-\log\left(\frac{b}{a}+1\right)\right], \qquad b>a,$$

toward the vertex. This value vanishes when $b=a(e-1)$ or when $b-a=(e-2)a=0.718a$. When b exceeds $a(e-1)$ the resultant attraction is away from the vertex.

II. Relating to a Problem in Minima.

By Dunham Jackson, Harvard University.

On page 339 of Osgood's Calculus (Ex. 9), the following problem is proposed:

"Find the point so situated that the sum of its distances from the three vertices of an acute-angled triangle is a minimum."

The answer is given:

"The lines joining the point with the vertices make angles of 120° with one another."

Then the note is added,

"For a complete discussion of the problem for any triangle see Goursat-Hedrick, *Mathematical Analysis*, Vol. 1, § 62."

The following simpler proof, which the writer has not seen in print, is so very brief that it may be worth while to give space to it, in spite of the inherent probability that it has already appeared somewhere.[1]

In the first place, let P be a point free to move along a straight line, and O a fixed point not on the line; let r be the length of OP and θ the angle which it makes with the given line; and let x be the distance of P from a fixed point of the given line. Then it is seen immediately, either geometrically or analytically, that

$$\frac{dr}{dx} = \cos\theta,$$

except for the question of the algebraic sign, which is readily determined in any given case.

Now let ABC be the given triangle, and O the point, assumed to exist somewhere, which makes the sum of the three distances a minimum. Let the angles BOC, COA, AOB, be denoted by θ_1, θ_2, θ_3, respectively. Since the sum of the distances is less for O than for any other point of the plane, it is in particular less for O than for any other point of the line OA. Let O' be a variable point of OA, and x its distance from O, measured positively toward A. Let r_1, r_2, r_3 stand for the distances $O'A$, $O'B$, $O'C$ respectively, and θ_1', θ_2', θ_3' for the angles $BO'C$, $CO'A$, $AO'B$. By the remark above,

$$\frac{dr_1}{dx} = -1, \qquad \frac{dr_2}{dx} = -\cos\theta_3', \qquad \frac{dr_3}{dx} = -\cos\theta_2';$$

as to the signs, it is clear that r_1 diminishes when x increases,[2] while r_2 and r_3 diminish or increase according as the respective angles are acute or obtuse. When O' coincides with O, so that θ_2' and θ_3' reduce to θ_2 and θ_3, the sum of the three distances is to be a minimum, and the sum of the derivatives must be zero. That is,

$$\cos\theta_2 + \cos\theta_3 + 1 = 0. \tag{1}$$

If we apply the same reasoning to the lines OB, OC, we find

$$\cos\theta_3 + \cos\theta_1 + 1 = 0, \tag{2}$$

$$\cos\theta_1 + \cos\theta_2 + 1 = 0. \tag{3}$$

By addition of these three equations and division by 2,

$$\cos\theta_1 + \cos\theta_2 + \cos\theta_3 + \frac{3}{2} = 0. \tag{4}$$

Subtracting (1), (2), and (3) in turn from (4), we obtain

$$\cos\theta_1 = -\frac{1}{2}, \qquad \cos\theta_2 = -\frac{1}{2}, \qquad \cos\theta_3 = -\frac{1}{2}.$$

This is equivalent to the statement to be proved.

[1] It is equivalent in substance to the proof given in de la Vallée Poussin's *Cours d'analyse infinitésimale*, Vol. 1, end of Chapter III; but it is slightly more elementary in form.

[2] If O' passes beyond the vertex A, of course r_1 increases with x; but it is not necessary to consider such points O'.

It will be noticed that in this demonstration no use has been made of the assumption that the angles of the triangle are acute. On the other hand, if an angle of the triangle is greater than 120°, it is clear that there is no point O such that the lines OA, OB, OC make angles of 120° with each other. The explanation of the paradox is, that it was tacitly assumed that O does not coincide with a vertex, and in the case of an angle greater than 120° the point which gives the minimum actually is the vertex of the obtuse angle.

NOTES AND NEWS.

Send communications to D. A. Rothrock, Indiana University, Bloomington, Ind.

The Royal Society recently granted the Sylvester Medal to Professor Gaston Darboux for his contributions to mathematical science.

Professor W. R. Smith, of the University of Minnesota, has gone to Dallas, Texas, as teacher of mathematics in the Terrill School for Boys.

Dr. Gillie A. Larew, who has been on leave during the past year, has returned to her position in Randolph-Macon Woman's College as adjunct professor of mathematics.

Dr. Isaac Sharpless, formerly instructor and professor of mathematics at Haverford College, and president since 1887, retires from active service at the end of the present college year.

The following appointments to instructorships in mathematics have recently been made: Mr. H. C. Clevenger, at the school of mines of the University of Montana; Mr. C. V. Reynolds, at Wesleyan University; and Mr. F. L. Smith, at Princeton University.

Promotions have been announced as follows: Dr. W. M. Smith, to a professorship of mathematics at Lafayette College; Assistant Professor R. M. Winger, to a professorship at the University of Oregon; Dr. J. K. Lamond, to a professorship at Pennsylvania College; Mr. H. W. Myers, to a professorship at Huron College; Dr. H. W. Roever, to an associate professorship of mathematics and Dr. G. O. James, to an associate professorship of mechanics and astronomy, at Washington University; Dr. R. L. Moore, to an assistant professorship at the University of Pennsylvania; Dr. W. E. Milne, to an assistant professorship at Bowdoin College; Dr. A. R. Crathorne, to an assistant professorship at the University of Illinois; Dr. J. J. Luck, to an adjunct professorship of mathematics at the University of Virginia; and Dr. W. V. Lovitt and Dr. T. E. Mason, to assistant professorships at Purdue University.

Professor H. B. Leonard, of the University of Oregon, becomes professor of mathematics at the University of Arizona, and assistant professor G. H. Cresse has been promoted to an associate professorship of mathematics.

The tenth regular meeting of the Southwestern Section of the American Mathematical Society was held at the University of Kansas, on December 2, 1916. Eleven papers were presented by representatives from the universities of Texas, Kansas, Nebraska, Missouri and Oklahoma, and from Washington University and Rice Institute. Visiting members of the Society were entertained by the mathematics department of the University of Kansas on Friday evening at a smoker and on Saturday noon at a luncheon.

Among the scientific papers in the tenth number of Vol. 2, *Proceedings of the National Academy of Sciences* are: "Point sets and Cremona groups," by A. A. Coble, of Johns Hopkins University; "Some problems of Diophantine approximation," by G. H. Hardy and J. E. Littlewood, of Trinity College, Cambridge and "Newton's method in general analysis," by A. A. Bennett, of Princeton University.

"The mathematics of common things" is the subject of an interesting paper in Vol. 16, No. 7, *School Science and Mathematics,* by Miss Harriet C. Glazier, of The Western College for Women, Oxford, Ohio. The writer of this paper enumerates a large number of illustrations of the fundamental notions of algebra and geometry as related to the facts of nature. Quantity, quality, conditions, functions, and geometric facts are considered in the paper.

A Mathematics Club has been organized at the University of Alabama, with D. Skolnik as chairman. Such clubs among the students of the colleges serve as a great stimulus to individual effort along special lines of study, and in many instances furnish a means of instructing students less advanced in mathematical knowledge. Last year the members of the Mathematical Society of the College of the City of New York volunteered a few hours each weeks for instructing freshmen who had been retarded in their mathematics. Such service is most valuable, enabling many students who otherwise drop out of college to gain normal standing. Mathematics clubs in high schools could very readily be organized to undertake similar volunteer service for those retarded in the high-school studies.

Mathematics clubs are maintained in a number of the Chicago high schools. Interesting programs are provided, and at times are made to include a union of several clubs. On November 27, a union program was given by the clubs of Bowen, Wendell Phillips, Harrison Technical, and Hyde Park high schools as follows: "The nine point circle," by J. Weiss, of Wendell Phillips; "An algebraic proof of the Pythagorean theorem," by E. A. Vorisek, of Harrison Technical; "Modern Geometry," by C. Bartlett, of Hyde Park; and "The sum of the angles of a triangle on the sphere and pseudosphere," by C. Peterson, of Bowen high school.

We record the death of a number of mathematicians: Professor F. Prym, of the University of Würzburg, December 15, 1915, at the age of seventy-five years;

Professor P. DUHEM, of the University of Bordeaux, at the age of sixty-seven; Dr. WILLIAM ESSON, Savilian professor of mathematics in the University of Oxford, at the age of eighty-eight; Professor S. B. MACLAREN, of University College, Reading, who died from wounds received in battle; Dr. HENRY GUNDER, formerly professor of mathematics at Findley College, Ohio, and later at Little Rock University, Arkansas, at the age of seventy-nine years; Dr. PERCIVAL LOWELL, director of the Lowell observatory, Flagstaff, Arizona, at the age of sixty-one years; Professor A. G. SMITH, head of the department of mathematics and astronomy at the University of Iowa, at the age of forty-eight years; Dr. W. C. ESTY, professor of mathematics and astronomy at Amherst College from 1862 to 1905 and since 1905 professor emeritus of mathematics; Mr. J. C. RAYWORTH, assistant professor of mathematics at Washington University, St. Louis, at the age of forty-two; and Professor CLEVELAND ABBE, of the United States Weather Bureau, at the age of seventy-eight.

The Association of Mathematics Teachers of New Jersey held its fifth regular meeting in the Central High School, Newark, N. J., on November 25, 1916. Professor C. O. GUNTHER, of Stevens Institute of Technology, is president, and Mr. A. S. HEGEMAN, of Central High School, Newark, is secretary. This organization of the teachers of mathematics of New Jersey has conducted a number of investigations along pedagogical and administrative lines, and at the present meeting received the "Final report of the committee on trigonometry courses," by the chairman of the committee, Professor C. O. GUNTHER. Secondary mathematics was provided for on the program by two papers: "The first-year high-school course in mathematics," by Mr. A. W. BELCHER, of the East Side High School, Newark; and "Teaching first year algebra," by Mr. WILLIAM STRADER, of Dickinson High School, Jersey City. A discussion on "Newton's analytical triangle" was given by Mr. W. D. REES, of Rutgers College, and "Some theorems on regular polygons described on the sides of a triangle," by Professor C. R. MACINNES, of Princeton University.

"The zero and principle of local value used by the Maya of Central America" is the subject of an interesting historical note by Professor FLORIAN CAJORI in *Science,* Nov. 17, 1916. Attention is called to the early use of a symbol for *zero* and the principle of local value of number symbols employed by the Maya probably dating back near the beginning of the Christian era. The Maya glyphs first deciphered by FÖRSTEMANN of Dresden, 1886, and independently by GOODMAN of California, relate for the most part to the calendar, to chronology, and to astronomy. The unit of this number system was 20, for which a special symbol, a half closed eye with a dot above, was used. Separate symbols of dots and bars represented the numbers from 1 to 19, each dot representing a unit, and each bar representing five units. Professor CAJORI gives as his source of information the recent Bulletin 57 of the Bureau of American Ethnology, Smithsonian Institution, Washington, D. C. This bulletin is an elementary presentation of the principles

of American archeology entitled "An introduction to the study of the Maya hieroglyphs," by S. G. MORLEY. Copies of this valuable publication may be obtained at a nominal cost from the superintendent of documents, Washington, D. C.

College-entrance mathematics receives a vigorous criticism from a writer in *School and Society*, October 21, 1916. The occasion for the article seems to have been the lowering of the passing grade for freshmen entering Harvard University under the examination in algebra conducted in June by the College Entrance Board. It seems that some 75 per cent. of the candidates fell below the passing grade of 60 per cent. in algebra, but in other subjects 75 per cent. passed. This situation the university authorities met by lowering the passing grade in algebra. The writer of the above-mentioned paper launches his discussion by saying: "Why should we longer annoy and worry our boys and girls with algebra (and geometry?) in their busy preparation for college? especially as these subjects are for the large majority as useless as they are plainly worrisome?" And again, "algebra is a time-wasteful fetish, an anachronism for the great majority of boys as well as for nearly all the girls." But what shall we have in the place of mathematics? Can we not as well say that ancient language, or the modern languages, or history, or science are *time-wasteful* for a large number of boys and girls? Few people use a foreign language after leaving school, and why do we modern mortals want to know anything of the history of the dead past? So far as college entrance examinations are concerned, these subjects may be made just as *unpassable* as algebra. It is interesting to note the recent memorandum (*School and Society*, Dec. 9, 1916) of the Teaching Committee of the Mathematical Association of Great Britain on what the pupil should acquire from a school course in mathematics. After concurring with the councils of the Classical, English, Geographical, Historical and Modern Language Associations in the view that any reorganization of the educational system should make adequate provision for both humanistic and scientific studies, should discourage premature specialization, and should insure some of the essentials of a liberal education to those preparing for the profession, the committee goes on to "submit that from a school course in mathematics the pupil should acquire: (1) an elementary knowledge of the properties of number and space; (2) a certain command of the methods by which such knowledge is reached, together with facility in applying mathematical knowledge to the problems of the laboratory and workshop; (3) valuable habits of precise thought and expression; (4) some understanding of the part played by mathematics in industry and practical arts, as an instrument of discovery in the sciences, and as a means of social organization and progress; (5) some of the appreciation of organized abstract thought as one of the highest and most fruitful forms of intellectual activity."

THE NEW YORK MEETING OF THE ASSOCIATION.

The second annual meeting of the Mathematical Association of America was held in New York City on Thursday, Friday, and Saturday, December 28–30, 1916. A full report of the meeting will appear in the February issue of the MONTHLY. There is opportunity now, as the present issue goes to press, for only one announcement, namely, concerning the election of officers.

Great interest had been manifested from the outset in the scheme of elections provided by the constitution. The result of the primaries would have put in nomination for the presidency the names of Professors Hedrick and Huntington, but upon Professor Hedrick's insistence that a precedent should be established from the beginning in favor of a single term for the president, the nominating committee acceded to his withdrawal and selected as candidates from the succeeding names on the primary ballot Professor E. V. Huntington and Professor Florian Cajori.

For the two vice-presidents, the four candidates indicated by the primaries were Professors Oswald Veblen, D. N. Lehmer, H. W. Tyler, and C. S. Slichter. For Secretary-Treasurer, the two candidates were Professors W. D. Cairns, and Dunham Jackson; and for the four members of the Council, the eight candidates were Professors R. E. Moritz, K. D. Swartzel, D. E. Smith, E. R. Hedrick, D. A. Rothrock, F. L. Griffin, Elizabeth B. Cowley, and Helen A. Merrill.

The total number of votes cast by mail (in time to be counted) and in person at the meeting amounted to 405, a most gratifying number in view of the fact that this was the first opportunity for the members to participate in a real election and to demonstrate that such matters need not be merely perfunctory.

The tellers appointed at the business meeting, Professors G. H. Ling, W. R. Ransom, and H. E. Hawkes, Chairman, performed their task with deliberation and care, thus prolonging the suspense to the point of acute expectancy both for the members present and especially for the presidential candidates, between whom the most friendly rivalry prevailed and concerning whom persistent rumors declared the contest to be very close.

The tellers finally reported the results as follows: For President, Professor Florian Cajori (winning over Professor Huntington by a vote of 203 to 202). For Vice-Presidents, Professors Oswald Veblen and D. N. Lehmer. For Secretary-Treasurer, Professor W. D. Cairns. For members of the Council to serve until January, 1920, Professors E. R. Hedrick, D. E. Smith, R. E. Moritz, and Helen A. Merrill.

Important Notice. Reprints of Professor Huntington's article in this issue will be furnished at cost (from five to ten cents per copy according to the number demanded) to all who apply before January 25, 1917. Please send orders to the MANAGING EDITOR.

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SECOND ANNUAL MEETING OF THE ASSOCIATION.

The second annual meeting of the Mathematical Association of America was held at Hamilton Hall, Columbia University, New York City, on Thursday, Friday and Saturday, December 28–30, 1916, in affiliation with the American Association for the Advancement of Science. There were 184 persons present at the various meetings, including 141 members of the Association as follows:

Joseph Allen, College of the City of New York, New York, N. Y.
R. B. Allen, Kenyon College, Gambier, Ohio.
R. C. Archibald, Brown University, Providence, R. I.
C. H. Ashton, University of Kansas, Lawrence, Kan.
Clara L. Bacon, Goucher College, Baltimore, Md.
Ida Barney, Smith College, Northampton, Mass.
L. A. Bauer, Carnegie Institution, Washington, D. C.
R. D. Beetle, Dartmouth College, Hanover, N. H.
C. A. Bergstresser, Boys' High School, Brooklyn, N. Y.
Harry Birchenough, N. Y. State College for Teachers, Albany, N. Y.
G. D. Birkhoff, Harvard University, Cambridge, Mass.
Vevia Blair, Graduate School, Columbia University, New York, N. Y.
Joseph Bowden, Adelphi College, Brooklyn, N. Y.
J. W. Bradshaw, University of Michigan, Ann Arbor, Mich.
E. W. Brown, Yale University, New Haven, Conn.
H. S. Brown, Hamilton College, Clinton, N. Y.
T. H. Brown, Brown University, Providence, R. I.
Daniel Buchanan, Queen's University, Kingston, Ontario, Can.
H. E. Buchanan, University of Tennessee, Knoxville, Tenn.
W. G. Bullard, Syracuse University, Syracuse, N. Y.
R. W. Burgess, Brown University, Providence, R. I.

W. D. Cairns, Oberlin College, Oberlin, Ohio.
Florian Cajori, Colorado College, Colorado Springs, Col.
J. A. Caparo, University of Notre Dame, Notre Dame, Ind.
F. E. Carr, Oberlin College, Oberlin, Ohio.
W. B. Carver, Cornell University, Ithaca, N. Y.
Mary E. Caster, Paterson, N. J.
J. W. Clawson, Ursinus College, Collegeville, Pa.
A. B. Coble, Johns Hopkins University, Baltimore, Md.
Abraham Cohen, Johns Hopkins University, Baltimore, Md.
E. C. Cook, College of the City of New York, New York, N. Y.
J. L. Coolidge, Harvard University, Cambridge, Mass.
Elizabeth B. Cowley, Vassar College, Poughkeepsie, N. Y.
Louise D. Cummings, Vassar College, Poughkeepsie, N. Y.
C. H. Currier, Brown University, Providence, R. I.
C. N. Dickinson, Hollins College, Hollins, Va.
C. E. Dimick, U. S. Coast Guard Academy, New London, Conn.
Eleanor C. Doak, Mount Holyoke College, South Hadley, Mass.
H. R. Dougherty, N. Y. Military Academy, Cornwall-on-Hudson, N. Y.
C. R. Duncan, Massachusetts Agricultural College, Amherst, Mass.
T. W. Edmondson, New York University, New York, N. Y.
L. P. Eisenhart, Princeton University, Princeton, N. J.
L. C. Emmons, Michigan Agricultural College, East Lansing, Mich.
T. C. Esty, Amherst College, Amherst, Mass.
G. W. Evans, Charlestown High School, Boston, Mass.
F. C. Ferry, Williams College, Williamstown, Mass.
H. B. Fine, Princeton University, Princeton, N. J.
C. A. Fischer, Columbia University, New York, N. Y.
T. S. Fiske, Columbia University, New York, N. Y.
J. D. Flynn, Trinity College, Hartford, Conn.
T. M. Focke, Case School of Applied Science, Cleveland, Ohio.
W. B. Ford, University of Michigan, Ann Arbor, Mich.
W. S. Franklin, South Bethlehem, Pa.
A. S. Gale, Rochester University, Rochester, N. Y.
W. V. N. Garretson, University of Michigan, Ann Arbor, Mich.
O. E. Glenn, University of Pennsylvania, Philadelphia, Pa.
Matilda Goertz, New York, N. Y.
T. E. Gravatt, Pennsylvania State College, State College, Pa.
G. M. Green, Harvard University, Cambridge, Mass.
C. C. Grove, Columbia University, New York, N. Y.
H. V. Gummere, Drexel Institute, Philadelphia, Pa.
M. W. Haskell, University of California, Berkeley, Cal.
H. E. Hawkes, Columbia University, New York, N. Y.
Olive C. Hazlett, Bryn Mawr College, Bryn Mawr, Pa.
E. R. Hedrick, University of Missouri, Columbia, Mo.

A. A. Himwich, Physician, New York, N. Y.
F. C. Hodgson, Publisher, New York City, N. Y.
F. J. Holder, University of Pittsburgh, Pittsburgh, Pa.
L. S. Hulburt, Johns Hopkins University, Baltimore, Md.
E. V. Huntington, Harvard University, Cambridge, Mass.
W. A. Hurwitz, Cornell University, Ithaca, N. Y.
Dunham Jackson, Harvard University, Cambridge, Mass.
G. H. Jamison, State Normal School, Kirksville, Mo.
S. A. Joffe, Assistant Actuary, New York, N. Y.
Edward Kasner, Columbia University, New York, N. Y.
Edward Kircher, Harvard University, Cambridge, Mass.
E. H. Koch, Jr., High School of Commerce, New York, N. Y.
K. W. Lamson, Graduate School, University of Chicago, Chicago, Ill.
Florence P. Lewis, Goucher College, Baltimore, Md.
G. H. Ling, University of Saskatchewan, Saskatoon, Can.
Joseph Lipka, Massachusetts Institute of Technology, Cambridge, Mass.
L. L. Locke, Brooklyn Training School for Teachers, Brooklyn, N. Y.
W. R. Longley, Yale University, New Haven, Conn.
Emilie N. Martin, Mt. Holyoke College, South Hadley, Mass.
James McClay, Columbia University, New York, N. Y.
Helen A. Merrill, Wellesley College, Wellesley, Mass.
Mansfield Merriman, Consulting Engineer, New York, N. Y.
E. J. Miles, Yale University, New Haven, Conn.
Bessie I. Miller, Rockford College, Rockford, Ill.
G. A. Miller, University of Illinois, Urbana, Ill.
J. A. Miller, Swarthmore College, Swarthmore, Pa.
J. S. Miller, Emory and Henry College, Emory, Va.
H. B. Mitchell, Columbia University, New York, N. Y.
F. M. Morgan, Dartmouth College, Hanover, N. H.
Frank Morley, Johns Hopkins University, Baltimore, Md.
Richard Morris, Rutgers College, New Brunswick, N. J.
B. L. Newkirk, University of Minnesota, Minneapolis, Minn.
E. J. Oglesby, College of William and Mary, Williamsburg, Va.
G. D. Olds, Amherst College, Amherst, Mass.
F. W. Owens, Cornell University, Ithaca, N. Y.
George Paaswell, Civil Engineer, New York, N. Y.
Alexander Pell, South Hadley, Mass.
Anna J. Pell, Mount Holyoke College, South Hadley, Mass.
A. D. Pitcher, Western Reserve University, Cleveland, O.
W. R. Ransom, Tufts College, Tufts College, Mass.
H. W. Reddick, Cooper Union, New York, N. Y.
Emma M. Requa, Hunter College, New York, N. Y.
R. G. D. Richardson, Brown University, Providence, R. I.
H. L. Rietz, University of Illinois, Urbana, Ill.

R. B. Robbins, Yale University, New Haven, Conn.
E. D. Roe, Jr., Syracuse University, Syracuse, N. Y.
R. E. Root, U. S. Naval Academy, Annapolis, Md.
D. A. Rothrock, Indiana University, Bloomington, Ind.
Mary E. Sinclair, Oberlin College, Oberlin, O.
H. E. Slaught, University of Chicago, Chicago, Ill.
Clara E. Smith, Wellesley College, Wellesley, Mass.
D. E. Smith, Columbia University, New York, N. Y.
P. F. Smith, Yale University, New Haven, Conn.
Sarah E. Smith, Mount Holyoke College, South Hadley, Mass.
W. M. Smith, Lafayette College, Easton, Pa.
Jessie Spearing, Graduate School, Columbia University, New York, N. Y.
J. M. Stetson, Western Reserve University, Cleveland, Ohio.
H. D. Thompson, Princeton University, Princeton, N. J.
F. C. Touton, Central High School and Junior College, St. Joseph, Mo.
A. B. Turner, College of the City of New York, New York, N. Y.
J. N. Van der Vries, University of Kansas, Lawrence, Kan.
Oswald Veblen, Princeton University, Princeton, N. J.
Evelyn Walker, Hunter College, New York, N. Y.
C. B. Walsh, Ethical Culture High School, New York, N. Y.
J. H. Weaver, High School, West Chester, Pa.
H. E. Webb, Central High School, Newark, N. J.
Louisa M. Webster, Hunter College, New York, N. Y.
A. L. Wechsler, New York City, N. Y.
Mary E. Wells, Vassar College, Poughkeepsie, N. Y.
E. E. Whitford, College of the City of New York, New York, N. Y.
F. B. Williams, Clark University, Worcester, Mass.
A. H. Wilson, Haverford College, Haverford, Pa.
E. B. Willson, Massachusetts Institute of Technology, Cambridge, Mass.
F. N. Wilson, Princeton University, Princeton, N. J..
J. W. Young, Dartmouth College, Hanover, N. H.
Mabel M. Young, Wellesley College, Wellesley, Mass.

The meeting opened with a joint session of the Association with the American Mathematical Society, Section A of the American Association for the Advancement of Science, and the American Astronomical Society, about two hundred being present. The retiring president of the American Mathematical Society, Professor Ernest W. Brown, of Yale University, gave his presidential address on "The relations of mathematics to the natural sciences." This was followed by the retiring address of Professor Armin O. Leuschner of the University of California, vice-president of Section A of the American Association. In the absence of Professor Leuschner, the address, entitled "Derivation of orbits—theory and practice," was read by Professor M. W. Haskell. This session, like those of Friday and Saturday, was held in Room 301, Hamilton Hall.

A joint dinner of these four organizations was held Thursday evening at the

Park Avenue Hotel. Following the dinner, which was attended by 142 persons, Professor L. P. Eisenhart, vice-president of Section A of the American Association, introduced the following speakers: Professor Florian Cajori, Colorado College; President R. J. Aley, University of Maine; Mr. William Bowie, U. S. Coast and Geodetic Survey; Professor J. A. Miller, Swarthmore College; Mr. G. A. Plimpton, New York City; and Professor Dunham Jackson, Harvard University.

A valuable feature in connection with the meeting was the opportunity afforded on Friday from twelve until two o'clock to visit the famous collection of portraits and medals of mathematicians gathered by Professor David Eugene Smith. The Association is under obligation to Professor Smith.

An appropriate resolution was adopted by the Council, recognizing the courtesy of the department of mathematics of Columbia University, the able work of the program committee in preparing so attractive a program, and the convenience of the facilities provided by the committee on arrangements.

The meetings of the Association continued on Friday morning. The following program was carried out, in accordance with the plan as arranged by the committee under the chairmanship of Professor David Eugene Smith, of Columbia University. At the request of President Hedrick, the chair was occupied for part of Friday morning by Vice-President Huntington and on Saturday morning by President-elect Cajori. Professor J. N. Van der Vries, chairman of the department of institutional delegates, presided at their meeting on Friday afternoon.

Friday Morning.

(1) "Discussions of Fluxions from Berkeley to Woodhouse." PROFESSOR FLORIAN CAJORI, Colorado College.

(2) "University Courses in Mathematics Intended for Teachers of Secondary Mathematics." PROFESSOR M. W. HASKELL, University of California.

(3) Discussion, led by PROFESSOR J. W. YOUNG, Dartmouth College, and PROFESSOR EDWARD KASNER, Columbia University.

Friday Afternoon.

(4) Meeting of Institutional Delegates. (See page 56.)

"A Nucleus for a Mathematical Library." DR. T. H. GRONWALL, New York, N. Y. (Introduced by Professor Oswald Veblen.)

Report of the Library Committee. PROFESSOR W. B. FORD, University of Michigan, Chairman.

Saturday Morning.

(5) "The Mathematics of Aërodynamics." PROFESSOR E. B. WILSON, Massachusetts Institute of Technology.

Discussion by PROFESSOR A. G. WEBSTER, Clark University.

The address of Professor Cajori clothed the dry bones of a somewhat remote controversy in the history of mathematics with an enlivening interest that showed his mastery as a writer of history.

Professor Haskell's paper forms an important contribution to a topic that is at present the object of much study on the part of university and college teachers. It aroused a lively discussion not merely by the two appointed to lead, but by a number of others who either criticized the views of the previous speakers or reported on plans now in operation similar to the one given by Professor Haskell.

In accordance with the policy already established of giving at our meetings one or more reports on the activities of committees of the Association, the Library Committee, appointed in June by President Hedrick, made a report on the plans which it has already formed. The report is printed in full on page 56.

Professor Wilson's paper on Saturday morning added a new phase to the programs of the Association, giving an exposition of the mathematics used in an important and rapidly growing science.

Abstracts are printed below for the papers and the principal discussions, so far as these are available. These are numbered to correspond with the numbers on the program above.

Abstracts of Papers.

(1) Professor Cajori's address, "Discussions of Fluxions from Berkeley to Woodhouse," contained brief accounts of Berkeley's *Analyst* and of three controversies carried on in Great Britain during the eighteenth century on the subject of fluxions. Particular attention was given to Berkeley's rejection of infinitely small quantities, Berkeley's criticism of Newton's derivation of the "moment" of a rectangle, and to Berkeley's "lemma": If in a demonstration an assumption is made, by virtue of which certain conclusions follow, and if afterward that assumption is destroyed or rejected, then all the conclusions that had been reached by the first assumption must also be destroyed or rejected. Berkeley's opponents would not accept this lemma. Robert Woodhouse in 1803 openly and fully accepted it. Rejoinders to Berkeley's *Analyst* were made by James Jurin of Cambridge and John Walton of Dublin. A second controversy on the nature of Newton's fluxions and limit-concept was carried on between James Jurin on one side, and Benjamin Robins and Pemberton on the other. Jurin believed that Newton's variables reached their limits; Robins insisted that a variable cannot reach its limit, and developed the calculus without using infinitely small quantities. This debate constitutes the most thorough discussion of limits of that time. The controversy lasted two years, gave rise to twenty articles which filled over 700 printed pages. Maclaurin's *Fluxions*, 1742, met deservedly with high appreciation, but except in the last part of the text, it did not use any notation for the new analysis; its rhetorical form of exposition made it unattractive reading. The advances in the logical exposition of the calculus in Great Britain were made during the eight years immediately following the publication of the *Analyst*. A third controversy took place in 1850–1852; friends of Emerson were arrayed against friends of Simpson. The discussion was carried on in the *Ladies Diary,* in some other journals less widely known, and in a pamphlet, *Truth Triumphant or Fluxions for the Ladies, showing the cause to be before the effect,* etc. This is the least important of the three controversies.

Towards the latter part of the eighteenth century the efforts at rigorous exposition slackened. Maclaurin was seldom read and Robins was forgotten. The first three editions of the *Encyclopedia Britannica* defined a "fluxion" as an "increment" acquired in "less than any assigned time." Both before and after the period of eight years, 1834–1842, there existed during the eighteenth century in Great Britain a mixture of Continental and British conceptions of the new calculus, a superposition of British symbols and phraseology upon the older Continental concepts. Newton's notation was poor and Leibniz's philosophy of the Calculus was poor. The mixture represented the temporary survival of the least fit of both systems. The subsequent course of events was in the opposite direction; the Leibnizian notation and phraseology was superposed upon the limit-concept as developed by Newton, Jurin, Robins, Maclaurin, D'Alembert and later writers. Woodhouse was the first Englishman to acknowledge openly the great services rendered by Bishop Berkeley in criticizing the philosophical conception of the calculus prevalent at that time. Woodhouse was the forerunner in Cambridge of Babbage, Peacock and the younger Herschel, in the promotion of the principles of pure D-ism in opposition to the dot-age of the university.

(2) Professor Haskell's paper is promised for printing in full in the March issue.

(3) In lieu of an abstract which was not obtainable in time for printing here, Professor Kasner set forth the appreciation of the research aspect of mathematics as the primary requirement of the teacher. This is best acquired by a course in original problems, composed mainly by the students themselves and as closely related to general science and life as possible. New concepts should be introduced by *difficult* interesting problems, easy ones to be discussed later. General scientific method and historical perspective should be emphasized. The teacher's mind must be liberal and flexible; the rough procedure of common sense, the finest logical criticism, and all intermediate stages, should be appreciated sympathetically.

Taking as his point of departure, Professor Haskell's question: "What is geometry?", Professor Young affirmed that teachers of mathematics must make clear to themselves the answer to the question: "What is mathematics?" What is needed is not a formal definition; but a working knowledge, in order that teachers may see clearly what they are trying to accomplish. Judging by prevailing practice in the teaching of algebra, where from 75 per cent. to 90 per cent. of the time is devoted to the development of proficiency in manipulating symbols, one would say that such technical skill is the teacher's conception of what constitutes algebra. Yet no wide-awake teacher would deny that such manipulatory expertness is only an incident and not the real substance of algebra. What then is the real substance of algebra? Courses in algebra must be so reorganized that there would be no need of asking the question. Pending that time, courses for prospective teachers, as outlined by Professor Haskell, must equip the teacher to formulate his answer. It is for this reason that

emphasis on the history of mathematics is justified,—but the history must be taught not primarily to gain information, but rather to develop an insight into the spirit of mathematical enquiry. One result of such insight will be to banish the obsession that mathematical thinking is *deductive*. It is not—every one who has stopped to think about the way he reaches a solution of a mathematical problem knows that in the majority of cases his thinking was not primarily deductive. This obsession is largely responsible for the unnecessary formalism with which our subject is branded. Finally, Professor Young expressed the opinion that sufficient place is not given to the applications of mathematics in Professor Haskell's scheme of courses.

(4) The meeting of institutional delegates is reported below.

(5) Professor Wilson pointed out that there are at least three stages in the growth of a branch of engineering: (1) the initiation by bold adventurers, (2) the development of a mathematical theory, (3) the codification in texts and handbooks. Aëronautical engineering is at present in stage (2), and properly qualified mathematicians like G. H. Bryan and Sir Geo. Greenhill have aided materially in advancing the art. Particular mention was made of the modifications and adaptations of the purely theoretical work which have been introduced by L. Bairstow in England and by J. C. Hunsaker in this country, by virtue of which experimental results may be combined with the methods of mathematicians to develop really accurate principles of airplane design.

Professor Wilson showed that apart from the purely elementary methods of algebra and trigonometry, the branches of mathematics used in aëronautical theory are: (1) The principles of mechanics including moving axes, (2) linear differential equations with constant coefficients, (3) theory of functions of a complex variable. Emphasis was laid on the need of so teaching the theory of functions that it becomes a practically usable branch of mathematics, and of devoting more time to the study of mechanics and mathematical physics, to the end that mathematicians might contribute more to the national defense both in periods of preparation and in crises of adversity.

This paper was discussed by Professor A. G. Webster, of Clark University, in his inimitable manner. The discussion was also participated in by Director L. A. Bauer, of the Carnegie Institution of Washington, and by Dr. W. S. Franklin, of South Bethlehem, Pa. Professor Ernest W. Brown, whose name appeared upon the stated program, was unavoidably prevented from attendance at this session.

Meeting of Institutional Delegates.

The meeting of delegates representing those colleges and universities holding institutional membership was held Friday afternoon at two o'clock. Professor J. N. Van der Vries, of the University of Kansas, had been chosen at the summer meeting as chairman of this department, and as such was in charge of the meeting. It was announced at the morning program that the meeting was to be an open

session and that all were welcome who were interested in the institutional questions which form the special sphere of the activities of this department. The general topic proposed by a special committee for consideration at this meeting was that of mathematical libraries, the best selection of books for a small library of 200, 300, or 500 volumes, and related questions. This topic grew naturally out of the discussion at the summer meeting, and the interest which it evoked is one of several indications of the growing importance attaching to the department of institutional delegates. According to the roll-call made by the secretary, the following 29 institutions were represented by official delegates:

Amherst College, Professor T. C. Esty;
Brooklyn Polytechnic Institute, Dr. J. B. Chittenden;
Brown University, Professor R. G. D. Richardson;
University of Buffalo, Professor W. H. Sherk;
Case School of Applied Science, Professor T. M. Focke;
University of Chicago, Professor H. E. Slaught;
Colorado College, Professor Florian Cajori;
Columbia University, Professor T. S. Fiske;
Cooper Union, Professor H. W. Reddick;
Creighton University, Professor W. F. Rigge;
Dartmouth College, Professor J. W. Young;
Hamilton College, Professor H. S. Brown;
University of Kansas, Professor J. N. Van der Vries;
Kenyon College, Professor R. B. Allen;
Lafayette College, Professor W. M. Smith;
University of Michigan, Professor W. B. Ford;
University of Missouri, Professor E. R. Hedrick;
Mount Holyoke College, Professor Sarah E. Smith;
College of the City of New York, Professor Joseph Allen;
N. Y. State College for Teachers, Professor Harry Birchenough;
New York University, Professor T. W. Edmondson;
Oberlin College, Professor W. D. Cairns;
Princeton University, Professor H. B. Fine;
Rockford College, Professor Bessie I. Miller;
Rutgers College, Professor Richard Morris;
Trinity College, Professor J. D. Flynn;
Wellesley College, Professor Helen A. Merrill;
Wesleyan University, Professor L. A. Howland;
Western Reserve University, Professor A. D. Pitcher.

There were also present members of the faculties of other institutions including the following: Bryn Mawr College, Clark University, Cornell University, Emory and Henry College, Goucher College, Harvard University, Hunter College, University of Indiana, Johns Hopkins University, University of Pittsburgh, Syracuse University, Tufts College, Vassar College, and Yale University.

Inasmuch as the value of Dr. Gronwall's paper consisted most of all in the list of books which he has drawn up, a list which must be seen to be fully appreciated, he did not attempt to read it but his presentation took the form of brief comments upon the principles underlying the selection of the list. Dr. Gronwall's list of books will be printed later, probably in connection with a further report of the Library Committee and at a point where its usefulness will be enhanced by its logical association with two or three other lists now in the process of formation.

Preliminary Report of the Library Committee.

The Library Committee, which consists of Professor W. B. Ford, Chairman, and Professors Florian Cajori, E. S. Crawley, Solomon Lefschetz, W. R. Longley, and R. E. Root, made the following report:

The relation which the Association has, or may be made to have in the near future, to the mathematical libraries of our schools and colleges at once presents a decidedly real and many sided problem. That the Association can render valuable service in this direction was early recognized by the President and others identified with him in the inception of the organization, and later the present committee was asked to arrive if possible at some definite plans for effectively carrying out such work. The committee can as yet make no final report, but it can set forth a number of suggestions which it has received from various sources and of which it fully approves. It is planned to develop these suggestions in the near future so as to put them upon an actual working basis. Meanwhile further suggestions will be most gratefully received, especially from those members who are engaged in schools and colleges where the need of better library advantages is distinctly felt.

The suggestions received are as follows:

1. That the committee prepare certain lists of books and publish them in the Monthly. To be more specific, it has been suggested that lists be prepared suitable for freshmen, sophomores, juniors, etc., the object being to furnish each of these classes of students with appropriate collateral reading in connection with their regular mathematical courses, and in a more general sense to furnish certain side lights upon the mathematics belonging to these various periods of study. Such lists should be relatively short and, where conditions permit, the books indicated might well be kept in the actual class room where, if properly shelved and labelled, they would at least attract attention and have a general salutary effect.

Besides the lists just mentioned, it has been suggested that a list of reference books be prepared suitable for purchase by a general college library. Such a list would naturally be relatively long, indicating what the college could well endeavor to accumulate in the course of several years. By way of general lists of a shorter nature, it might be proper also for the committee to work up a "five-foot shelf" of mathematics for colleges and possibly another such shelf for high schools.

The preparation of these various lists, if carefully done, will evidently require some time. It is the intention of the committee to proceed in this direction as

fast as possible and eventually make known its findings in the MONTHLY. How far it would be wise to confine the lists to books in the English language is a question upon which the committee would be glad to have the opinion of all interested persons.

2. It has been suggested that the Association, through its library committee, should form a sort of medium of exchange for books and periodicals. For example, one college may have material, such as duplicates of books, which it does not need but which would be of value to some other school or college. In such cases the Association should be in a position to assist the exchange, more especially in expediting and rendering it less artificial than at present. Such exchanges, though already common among colleges and universities for material in general, could doubtless be accomplished, so far as mathematics is concerned, in a much more effective way than at present, and the medium of publicity afforded by the MONTHLY could evidently be used to much advantage in this direction. If this idea proves valuable, the function of the committee would seem to be that of conducting a clearing house for such material, using the columns of the MONTHLY as may seem desirable.

In connection with this matter of exchanges, it seems clear that the Association should keep on file in its library copies of all American journals and periodicals such as are devoted to college or high-school mathematics, and that these should be ready for loaning out to members as desired. For the sake of completeness, the back volumes of such periodicals, whenever procurable, should at once be secured. Eventually, the question of obtaining foreign periodicals will also need consideration. Teachers in the small colleges do not ordinarily have access to foreign periodicals of this kind except as they borrow them from the large universities which in turn prefer not to loan material of this kind. So it seems highly desirable that, as soon as conditions abroad permit, the Association should arrange to have on file in its library a complete set of the French, Italian, German and British journals dealing with collegiate and high-school mathematics, including their history.

3. Mention has just been made of the Association library, but it must be admitted that only the beginnings of such a library are as yet in existence. It will be the evident duty of the committee to do whatever it can to build up such a library. As soon as it begins to take definite shape, rules for its administration will need to be formulated. In this connection, the precedents already established by the library of the American Mathematical Society and which have proved altogether satisfactory there will naturally suggest the course to be followed in the present instance. In particular, authors should be encouraged at all times to present copies of their publications to the library, and individuals generally should feel that whatever aid they can lend to the enterprise will be very gratefully received.

4. Various other suggestions reaching the committee but as yet remaining in a somewhat embryonic state are the following:

(*a*) That the Association through the library committee lend its influence

to the formation of mathematical reading circles throughout the country. No doubt the committee should at least be in position to suggest a suitable group of books for the study of any one topic which it may be desired to read in this way.

(*b*) That means be devised if possible whereby publishing houses will become interested in the general activities of the library and donate books to it by way of advertisement.

In conclusion, it would seem that the phenomenal interest and growth attending the first year of the Association portends a corresponding early and substantial development of its library and of the general library interests of all its members. The committee can but hope to aid in all possible ways to bring this about and at the present time any suggestions to this end beyond those mentioned above would be very gratefully received. These should be sent to the chairman, Professor W. B. Ford, 904 Forest Avenue, Ann Arbor, Mich.

Following the reading of the report by Professor Ford, a number of persons spoke. Professor Huntington emphasized the great value of the contemplated plans of the committee and the usefulness in particular of check lists of books in the libraries of our institutions of learning. Professor Richard Morris said that inspired by the action of the teachers of the public schools of New Jersey, the state librarian at Trenton had expressed his willingness to furnish books called for by the teachers. Professor F. J. Holder told of similar aid afforded by the Carnegie libraries, and Professor J. N. Van der Vries reported the services rendered to institutions of the middle west by the John Crerar Library of Chicago.

In contrast to the selection by Dr. Gronwall of a number of books in foreign languages, Professor R. B. Allen urged that most of the books for students should be in English, inasmuch as those interested in mathematics have not so large an interest or ability in languages. Professor W. R. Ransom suggested that where foreign books are put into such lists, there should be an indication of their size, difficulty, and accessibility. Professor W. A. Hurwitz remarked that while many satisfactory texts in elementary subjects exist in English, there is a dearth of texts in English for intermediate courses, and that, when there are no English books of the desired sort, the presence of foreign books in these lists will arouse publishers to a sense of the desirability of publishing such books.

Professors H. E. Slaught and H. E. Hawkes spoke of mathematical clubs and of the selection of a good list of topics for these clubs to be used in connection with appropriate references to available books. When the latter told of thirty or forty topics which have actually been used at Columbia University during the past few years, he was requested by the meeting to publish in the MONTHLY this list together with the accompanying references to accessible sources. Two or three others spoke in regard to the exchange of books and on reading circles.

It was voted that the report of the Library Committee be placed on file for publication, and that the committee be encouraged to continue in its valuable work.

It was voted that the general program committee for the next meeting of the

Association be asked to make a suitable place on its program for such topics as are of institutional interest, and that, if occasion require any decision by formal vote, an executive session of the institutional delegates be held. The meeting then adjourned to make way for the annual business meeting.

Annual Business Meeting.

The Secretary-Treasurer reported the death during the year of the following ten charter members of the Association:

L. L. Conant, John E. Sinclair Professor of Mathematics, Worcester Polytechnic Institute.
W. C. Esty, Professor of Mathematics, Emeritus, Amherst College.
F. W. Frankland, Consulting Actuary, New York, N. Y.
F. P. Hebblethwaite, former Instructor in Mathematics, Northwestern University.
A. H. Holmes, Lawyer, Brunswick, Me.
Dr. Emory McClintock, Consulting Actuary, Bay Head, N. J.
Mrs. Eva S. Maglott, Professor of Mathematics, Ohio Northern University.
J. C. Rayworth, Assistant Professor of Mathematics, Washington University.
H. A. Sayre, Professor of Mathematics, University of Alabama.
A. G. Smith, Head of the Department of Mathematics, University of Iowa.

The election of officers for the year 1917 was conducted both by mail and in person at this meeting, as provided by the constitution.

The tellers (Professor H. E. Hawkes, G. H. Ling, and W. R. Ransom) appointed by President Hedrick reported the result of the balloting as follows:

For President, Florian Cajori, Colorado College.
For Vice-Presidents, Oswald Veblen, Princeton University, and
D. N. Lehmer, University of California.
For Secretary-Treasurer: W. D. Cairns, Oberlin College.
For additional members of the Executive Council to serve until January, 1920:

E. R. Hedrick, University of Missouri,
D. E. Smith, Columbia University,
R. E. Moritz, University of Washington,
Helen A. Merrill, Wellesley College.

A full description of the interest in this election was published in the January Monthly.

The secretary-treasurer made his financial report for the year, giving an account of all business transacted for the Association up to the date of December 21, 1916. The report was approved subject to an inspection by the auditing committee (Professors R. G. D. Richardson, H. E. Slaught, and A. H. Wilson) appointed by the president. This committee made its inspection later in the day, and formally approved the report. This report is printed in full below.

TREASURER'S REPORT FOR THE YEAR 1916.

RECEIPTS.

Balance from 1915 business		$ 958.72
1916 subscriptions	$ 502.86	
1916 indiv. memberships	3,110.90	
1916 instit. memberships	271.60	
1916 initiation fees	24.00	
Sale copies of MONTHLY	49.15	
Sale reprints	18.77	
Advertising	392.00	
Exchange	2.02	
Interest State Savgs. Bk.	39.71	
Interest Peoples Bk.	13.36	
Total 1916 receipts		4,424.37
Total receipts, 1915–1916		$5,383.09
Balance on 1915–1916 business		$1,671.47
Recd. on 1917–1920 business		1,281.77
Book balance Dec. 21, 1916		$2,953.24

EXPENDITURES.

Publisher's bills		$2,688.19
Paid for reprints		13.74
President's office		126.65
Managing editor's office		108.11
Other editors' postage		16.30
Secretary-Treasurer's office:		
Postage	$229.00	
Bond	5.00	
Desk and office supplies	58.53	
Express, telegrams, freight, etc.	46.72	
Clerical work	239.35	
Printing	106.72	
Cambridge meeting	69.18	
Institutional meeting	4.13	
		758.63
Total expenditures		$3,711.62
Cash on hand		$ 3.00
Checking account		1,176.85
State Savgs. Bk. Co. account		1,260.03
Peoples Bkg. Co. account		513.36
Bank balance Dec. 21, 1916		$2,953.24

Approved by auditing committee,

R. G. D. RICHARDSON,
H. E. SLAUGHT,
A. H. WILSON.

December 29, 1916.

When the accounts were closed on December 21, 1916, for the purposes of the above record, there remained on the total business for the calendar year 1916 the following items:

BILLS RECEIVABLE.

Advertising	$ 86.86
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	$612.00

It will be seen from this report that the former management of the MONTHLY transferred to the Association $958.72. With this in mind the Council through its Committee on Finance has set aside one thousand dollars to be kept as a reserve fund. It may be noted also that the business for the calendar year 1916 alone will thus close with a probable balance a little under two hundred dollars, and that more than $1,200 has already been paid into the treasury on the business of the new year.

Meeting of the Council of the Association.

The Council met at nine o'clock Friday morning and held other short meetings between the various sessions of the Association, nine members being present. The principal business transacted is indicated herewith.

(1) The following sixteen institutions, on applications duly certified, were elected to institutional membership, making the total number now 76:

Woman's College of Alabama, Montgomery, Ala.
Trinity College, Hartford, Conn.
Boston University, Boston, Mass.
Mount Holyoke College, South Hadley, Mass.
Worcester Polytechnic Institute, Worcester, Mass.
Michigan Agricultural College, East Lansing, Mich.
Princeton University, Princeton, N. J.
The Polytechnic Institute, Brooklyn, N. Y.
Hamilton College, Clinton, N. Y.
Columbia University, New York, N. Y.
Rochester University, Rochester, N. Y.
Union University, Schenectady, N. Y.
Lafayette College, Easton, Pa.
Lehigh University, South Bethlehem, Pa.
Washington and Jefferson College, Washington, Pa.
Brown University, Providence, R. I.

(2) The following fifteen persons, on applications duly certified, were elected to individual membership, making the total number now 1,064, deducting the number of those who have died during 1916.

J. Q. McNatt, with the Colorado Fuel and Iron Co., Florence, Col.
W. C. Welling, Trinity College, Hartford, Conn.
E. B. Miller, University of Kansas, Lawrence, Kan.
G. A. Osborne, Massachusetts Institute of Technology, Boston, Mass.
C. A. Shook, Graduate School, Harvard University, Cambridge, Mass.
Eleanor C. Doak, Mount Holyoke College, South Hadley, Mass.
Vera L. Wright, University of Minnesota, Minneapolis, Minn.
C. J. Payne, State Normal School, Cape Girardeau, Mo.
P. H. Daus, University of New Mexico, Albuquerque, N. M.
J. B. Rosenbach, Graduate School, University of New Mexico, Albuquerque, N. M.
Vevia Blair, Graduate School, Columbia University, New York, N. Y.
Oscar Hoppe, with the American Circular Loom Co., New York, N. Y.
Louise D. Cummings, Vassar College, Poughkeepsie, N. Y.
E. A. Painter, The Yeates School, Lancaster, Pa.
H. M. Manning, Surgeon, U. S. Public Health Service, Charleston, S. C.

(3) A section of the Association was established for Maryland and the

District of Columbia, with the possible inclusion of Virginia. Professor Abraham Cohen, of Johns Hopkins University, is the secretary.

(4) A committee consisting of Professor Huntington, chairman, Professor Cajori and the secretary-treasurer was appointed with power to determine the time and place of the summer meeting, in conference with a similar committee of the American Mathematical Society.

(5) It was voted to appoint a committee which should in conjunction with a similar committee of the Society consider the question of possible assistance for *Revue Semestrielle* and the *Jahrbuch über die Fortschritte der Mathematik*. The committee was empowered to include also in its investigation other international projects of a kind similar to the two named. Mathematicians the country over are feeling increasingly the deplorable influence of the European war as it affects such indispensable aids as the German and French encyclopædias, the two journals above mentioned, and similar reference books. This action has been taken in order that the two great mathematical organizations of America may consider what contribution they may perhaps make in rendering assistance to these valuable journals of record.

(6) It was voted to hold the next annual meeting in Chicago in conjunction with the Chicago meeting of the American Mathematical Society.

(7) In a session following the election of officers, the Council, in pursuance of its constitutional authority to fill vacancies *ad interim*, filled the vacancy caused by the election of Professor Cajori to the presidency by the appointment of Professor E. V. Huntington, to serve until January, 1918.

(8) The members of the Committee on Publications (H. E. Slaught, managing editor, R. D. Carmichael, and W. H. Bussey) were reappointed for the year 1917.

(9) A Committee on Membership with ex-President Hedrick as chairman was authorized by the Council.

(10) The president-elect was empowered to make the necessary modifications in the existing committees of the Council and to appoint the new committees already authorized. He has accordingly appointed the following:

Committee on Sections: D. E. Smith, Chairman; E. R. Hedrick, M. B. Porter.

Committee on Membership: E. R. Hedrick, Chairman; E. V. Huntington, M. W. Haskell.

W. D. CAIRNS, *Secretary-Treasurer.*

ON THE ORIGIN OF CERTAIN TYPICAL PROBLEMS.[1]

By DAVID EUGENE SMITH.

One thing which impresses the student of mathematical problems is that several which he would naturally classify as purely fictitious and of the nature of pleasing puzzles apparently had their origin in genuine applications of mathematics to questions of real life. Of these I shall mention only four, although the list could be greatly extended.

[1] Extract from a paper on the History of Mathematical Recreations, read before the Mathematical Association of America at Cambridge, Mass., September 1, 1916.

The first of these problems, without which an algebra of to-day might by some be thought to be incomplete, so rooted is it in our traditions, is that of the pipes filling the cistern. No problem has had a longer and more continuous history, and the traveler who is familiar with the Mediterranean lands cannot fail to recognize that here is its probable origin. Not a town of any size that bears the stamp of the Roman power is without its public fountain into which or from which several conduits lead. In the domain of physics, therefore, this would naturally be the most real of all the problems that came within the purview of every man, woman, or child of that civilization. Furthermore, the elementary clepsydra[1] may also have suggested the same line of problems, the principle involved being the same.

The problem in definite form first appears in Heron's Μετρήσεις of about 100 A. D., and although there is some question as to the authorship and date of the work, there is none as to the fact that this style of problem would appeal to such a writer as he. It next appears in the writings of Diophantus, c. 275 A. D.,[2] and among the Greek epigrams attributed to Metrodorus, c. 325 A. D., and soon after this it became common property in the east as well as the west. It is found in the list attributed to Alcuin (c. 825); in the great classic of India, the *Lilāvati* of Bhāskara[3] (c. 1150); in the best-known of all the Arab works on arithmetic, the *Kholāsat-el-hisāb* of Behā-ed-dīn (1547–1622); and in numerous medieval manuscripts. When books began to be printed it was looked upon as one of the stock problems of the race, and many of the early writers gave it a prominent position, among them being men like Petzensteiner (1483), Tonstall (1522), Gemma Frisius (1540), and Robert Recorde (c. 1540).[4]

Such, then, was the origin of what was once a cleverly stated problem of daily life. There is, however, this interesting law of book writers—that most of them will steal from one another without the least scruple if they can thinly veil the theft. This problem, therefore, like dozens of others, went through many metamorphoses, of which I shall mention only a few.

In the fifteenth century, and very likely much earlier, there appeared the variant of a lion, a dog, and a wolf, or other animals, eating a sheep,[5] and this form was even more common in the sixteenth century.[6]

[1] Attributed to Plato but improved by Ctesibus of Alexandria. On the whole subject of clepsydræ see Marquardt, J., *La vie privée des Romains*, French edition, Paris, 1893, p. 458.

[2] In Bachet's edition (the Fermat edition of 1670, p. 271) appears this metrical translation:

Totum implere lacum tubulis è quatuor, uno
Est potis iste die, binis hic & tribus ille,
Quatuor at quartus.
Dic quo spatio simul omnes.

[3] See Taylor's translation, p. 50; Colebrooke translation, p. 42.

[4] In Recorde it appears for the first time in English: "Ther is a cestern with iiij. cocks, conteinyng 72 barrels of water, And if the greatest cocke be opened, the water will auoyde cleane in vj howers," etc. *Ground of Artes*, 1558 edition, folio A, 7 v.

[5] Johann Widman (1489) under the chapter title "Eyn fasz mit 3 zapffen." His form is:

"Lew Wolff Hunt Itm̄ des gleichen 1 lew vnd 1 hunt vn̄ 1 wolff diese essen mit einander 1 schaff. Vnd der lew esz das schaff allein in einer stund. Vnd d' wolf in 4 stunden. Vnd der hunt in 6 stunden. Nun ist die frag wan sy dass schaff all 3 mit einãder essen / in wie lãger zeit sy das essen." 1509 edition, folio 92; 1519 edition, folio 112.

[6] Thus Cataneo, *Le Pratiche*, 1546; Venice edition of 1567, folio 59v: "Se un Leone mangia

In the sixteenth century we also find in various books the variant of the case of men building a wall or a house, in place of pipes filling a cistern, and this form has survived to the present time. It appeared in Tonstall's exhaustive treatise, *De Arte Supputandi*, in 1522,[1] in Cataneo's well-known work of 1546,[2] and in due time became modified to the form beginning, "If A can do a piece of work in 4 days, B in 3 days," and so on.

The influence of the wine-drinking countries shows itself in the variant given by that remarkable writer Gemma Frisius (1540),[3] who states that a man can drink a cask of wine in 20 days, but if his wife drinks with him it will take only 14 days, from which it is required to find the time it would take his wife alone.

The influence of a rapidly growing commerce led one of the German writers of 1540 to consider the case of a ship with 3 sails, by the aid of the largest of which a voyage could be made in 2 weeks; with the next in size in 3 weeks, and with the smallest in 4 weeks, it being required to find the time if all three were used, several factors being evidently ignored, such as one sail blanketing the others and the speed not being proportional to the power.[4]

The agricultural interests changed it to a mill with four "Gewercken,"[5] and other interests continued to modify it further until, as is usually the case, the style of problem has tended to fall from its own absurdity. Merely mentioning one of our modern writers who modifies the problem to the case of the pipes of a gasolene tank in a motor car, I may close its varied history by referring to a writer of the early nineteenth century,[6] moved by a bigotry which we would not countenance in academic circles to-day, who proposed to substitute priests praying for souls in purgatory.

Thus we see a recreative problem, starting as an ingenuously worded practical case, becoming fictitious under changed conditions, maintaining itself for two thousand years because of its recreative feature, and almost falling by the wayside because of the absurdities which finally attached to it. It is likely to retain, however, some minor place in our schools because it is not only real within the imagination of pupils, which our technical mechanical problems usually are not, but it is interesting and illustrates a valuable mathematical principle.

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in 2. hore una pecora, & l'Orso la mangia in 3. hore, & il Leopardo la mangia in 4. hore, dimandasi cominciando a mangiare una pecora tutti e 3. a un tratto in quanto tempo la finirebbono."

This form is also found in J. Albert's work of 1540 (1561 edition, folio Nviii), in Coutereel (1631 edition, p. 352) and in the works of numerous other writers.

[1] With the statement that it is similar to the one about the cistern pipes: "Questio hæc similis est illi de cisterna tres habete fistulas: et simili modo soluenda." Folio f. 1.

[2] See folio 60v of the Venice edition of 1567.

[3] 1563 edition of his arithmetic, folio 38.

[4] "Item / 1 ein Schiff mit 3 Siegeln gehet vom Sund gen Riga / Mit dem grösten allein / in 2 wochen / Mit dem andern / in 3 wochen / Vnnd mit dem kleinsten / in 4 wochen," etc. J. Albert (1540), 1561 edition, folio Nvii.

[5] "Ein Mülmeister hat ein Müle mit vier Gewercken / Mit dem ersten mehlt er in 23 stūden 35 Scheffel / Mit dem andern 39 Scheffel / Mit dem dritten 46 Scheffel / Vnnd mit dem vierten 52 Scheffel," etc. The question then is how long it will take them together to grind 19 Wispel (1 Wispel = 24 Scheffel). *Ibid.*

[6] Hay, *The Beauties of Arithmetic*, 1816, p. 218.

The next problem to which I wish to call your attention has not maintained its place in our books although it has an honorable history of over 2,000 years; it is interesting, it is real within the realm of the pupil's imagination; but it fails for the reason that no principle is involved that is needed in secondary mathematics. The problem is the one commonly known as the Josephsspiel, or the one of the Turks and Christians. It relates that 15 Turks and 15 Christians were on a ship and that half had to be sacrificed; it being necessary to choose the victims by lot, the question is as to how they can be arranged in a circle so that, in counting round, every fifteenth should be a Turk.

It is probable that the problem goes back to the custom of *Decimatio* in the old Roman armies, the selection by lot of every tenth man when a company had been guilty of cowardice, mutiny, or loss of standards in action. Both Livy (ii, 59) and Dionysius (ix, 50) speak of it in the case of the mutinous army of the consul Appius Claudius (B. C. 471), and Dionysius further speaks of it as a general custom. Polybius (vi, 38) says that it was a usual punishment when troops had given way to panic. The custom seems to have died out for a time, for when Crassus resorted to decimation in the war of Spartacus he is described by Plutarch (Crassus, 10) as having revived an ancient punishment. It was extensively used in the civil wars and was retained under the Empire, sometimes as *vicesimatio* (every twentieth man being taken), and sometimes as *centesimatio* (every hundredth man).

Now it is very improbable that those in charge of the selection would fail to have certain favorites, and hence it is natural that there may have grown up a scheme of selection that would save the latter from death. Such customs may depart, but their influence remains in various ways. In the present great war we have frequently read of a regiment being decimated; but how few of us have thought of the origin of the expression.[1]

In its semi-mathematical form it is first referred to in the work of an unknown author, possibly Ambrose of Milan, who wrote, under the nom de plume of Hegesippus, a work *De bello iudaico*.[2] In this work he refers to the fact that Josephus, the author of the well-known history of the wars of the Jews, was saved on the occasion of a choice of this kind.[3] Indeed, Josephus himself refers to the matter of his being saved by lucky chance or by the act of God.[4]

The oldest European trace of the problem, aside from that of Hegesippus, is found in Codex Einsidelensis No. 326, of the beginning of the tenth century. It is also referred to in a manuscript of the eleventh century now in the Munich library and in Codex Bernensis No. 704, of the twelfth century. It is given in the *Ta'hbula* of Rabbi ben Esra (d. 1167) in the twelfth century, and indeed

[1] Lucas, in his *Arithmétique Amusante*, p. 17, also suggests the origin of the problem in the custom of *decimatio*.

[2] Edited by C. F. Weber and J. Caesar, Marburg, 1864. See Ahrens, Math. Unterh. u. Spiele, p. 286.

[3] "Itaque accidit ut interemtis reliquis Iosephus cum altero superesset neci." Quoted from Ahrens, l. c.

[4] *Καταλείπεται δὲ οὗτος, εἴτε ὑπὸ τύχης χρὴ λέγειν εἴτε ὑπὸ Θεοῦ προνοίας σὺν ἑτερῳ.*

it is tó this writer that Elias Levita, who seems first to have given it in printed form (1518), attributes its authorship.

The problem, as it came to be stated, related that Josephus at the time of the sack of the city of Jotapata by Vespasian, hid himself with forty other Jews in a cellar. It becoming necessary to sacrifice some of the number, a method analogous to the old Roman method of *decimatio* was adopted, but in such way as to preserve himself and a special friend. It is on this account that the Germans still call the problem by the name of Josephsspiel.

Chuquet (1484) mentions the problem, as does at least one other writer of the fifteenth century.[1] When, however, printed works on algebra and higher arithmetic began to appear, it became well known. The fact that such writers as Cardan[2] and Ramus[3] gave it prominence was enough to assure its coming to the attention of scholars.[4]

Like so many curious problems, this one found its way to the Far East, appearing in the Japanese books as relating to a mother-in-law's selection of the children to be disinherited. With characteristic Japanese humor, however, the woman was described as making an error in her calculations so that her own children were disinherited and her step-children received the estate.[5]

The third problem of which I think the origin is worth our attention is the common one of the testament. It relates that a man about to die made a will bequeathing $\frac{1}{3}$ of his estate to his widow in case an expected child was a son, the son to have $\frac{2}{3}$; and $\frac{2}{3}$ to the widow if the child was a daughter, the daughter to have $\frac{1}{3}$. The issue was twins, one a boy and the other a girl, and the question was as to the division of the estate.

The problem in itself is of no particular interest, being legal rather than mathematical; but I mention it because it is a type and is by no means isolated. Under both the Roman and the Oriental influence these inheritance problems played a very important rôle in such parts of analysis as the ancients had developed. In the year 40 B. C. the *lex Falcidia* required at least $\frac{1}{4}$ of an estate to go to the legal heir. If more than $\frac{3}{4}$ was otherwise disposed of, this had to be reduced by the rules of partnership. Problems involving this "Falcidian fourth" were therefore common under the Roman law, just as problems involving the widow's dower right were and are common in the English law and in this country.

The problem as I have stated it appears in the writings of Juventius Celsus, a celebrated jurist of about 75 A. D., who wrote on testamentary law; in those of Salvianus Julianus, a jurist in the reigns of Hadrian (117–138), and Antoninus Pius (138–161), and in those of Cæcilius Africanus (c. 100), celebrated for his knotty legal puzzles.[6]

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[1] Anonymous MS. in Munich. See *Bibl. Math.*, 1893, p. 32; M. Curtze, *ibid.*, IX (2), 33; VIII (2), 116; X (2), 29; *Abhandlungen zur Geschichte der Math.*, III, 123.

[2] In his *Arithmetica* of 1539.

[3] In his edition of 1569, p. 125.

[4] It is also in Thierfelder's arithmetic (1587, p. 354), in Wynant van Westen's *Mathemat. Vermaecklyckh* (1644 edn. I, p. 16), in Wilken's arithmetic of 1669 (p. 395), and in many other early works.

[5] See Smith and Mikami, *History of Japanese Mathematics*.

[6] Coutereel (Eversdyck edition of 1658, p. 382) traces the problem back to lib. 28, title 2,

In the Middle Ages it was a favorite conundrum, and in the early printed arithmetics it is often found in a chapter on inheritances which reminds us of the Hindu mathematical collections.[1] It went through the same later development that characterizes most problems and finally fell on account of its very absurdity. That is, Widman (1489) takes the case of triplets, one boy and two girls,[2] and in this he is followed by Albert (1540) and Rudolff (1526).[3] Cardan (1539) complicates it by supposing 4 parts to go to the son and 1 part to the mother, or 1 part to the daughter and 2 parts to the mother, and in some way decides on an 8, 7, 1 division.[4] Texeda (1545) supposes 7 parts to go to the son and 5 to the mother, or 5 to the daughter and 6 to the mother,[5] while other writers of the sixteenth century complicate the problem even more.[6] The final complications of the "swanghere Huysvrouwe" or "donna grauida" are found in some of the Dutch books, and these and the change in ideas of propriety account for the banishment of the problem from books of our day.[7] The most sensible remark about the problem to be found in any of the early books is given in the words of the "Scholer" in Robert Recorde's *Ground of Artes* (c. 1540): "If some cunning lawyers had this matter in scanning, they would determine this testament to be quite voyde, and so the man to die vntestate, because the testament was made vnsufficient."[8]

The fourth problem to whose origin and development I wish to direct special attention is the one of pursuit. It would be difficult to conceive of a problem that would seem more real, since we commonly overtake a friend in walking, or are in turn overtaken. It would therefore seem very certain that this problem is among the ancient ones in what was once looked upon as higher analysis. We have a striking proof that this must be the case in the famous paradox of Achilles and the Tortoise, the history of which has been so carefully and entertainingly worked out by our colleague, Professor Cajori. It is a curious fact, however, that it is not to be found in the Greek collections, although it must also be said that we have not a single work on the Greek *logistice* (λογιστική) extant, so that

law 13 of the *Digest* of Julianus. He gives the usual 4, 2, 1 division as followed by Tartaglia, Rudolff, Forcadel, Ramus, Trenchant, vander Schuere, Mellema, and vander Gucht. Coutereel, however, argues for the 4, 3, 2 division, and in this he has the support of Anth. Smijters. Peletier gives 2, 2, 1, and Chauvet gives 9, 6, 4. Brief historical notes appear in other books, as in the Schonerus edition of Ramus (1586 edition, p. 186).

[1] Thus we have "Ein Testament" (Widman), "Erbteilung vnd vormundschaft" (Riese), "Erf-Deelinghe" (Vander Schuere), and "Erbtheilugs-Rechnung" (Starcken).

[2] Edition of 1558, folio 97. He then divides the property in the proportion 4, 2, 1, 1.

[3] Unger, p. 109.

[4] *Arithmetica*, cap. 66, ex. 87.

[5] Folio Xliij.

[6] Ghaligai (1552, folio 65), Köbel (1518, folio Fij), Riese (*Rechnung nach lenge*, 1550, folios 43, 100), Trenchant (1571, 1578 edition, p. 328), Vander Schuere (1600, folio 96), Peletier (1607 edition, p. 244), Coutereel (1631 edition, p. 358), Starcken (1714 edition, p. 444), Tartaglia (*Tvtte l'opere d'aritmetica*, 1592 edition, II, p. 136).

[7] "Soo ontfangt sy ter tijdt haerder baringhe eenen Sone met een Dochter / eñ een Hermaphroditus, dat is / half Man / half Vrouwe." Vander Schuere, 1600, folio 98. In this case he divides 3175 guldens thus: d. 254, m. 508, s. 1524, h. 889. The same problem appears in Clausberg, *Demonstrative Rechen-Kunst*, 1772.

[8] 1558 edition, folio X 8.

it may have been common without our knowing of the fact. It appears, however, among the *Propositiones ad acuendos juvenes* attributed to Alcuin, in the form of the hound pursuing the hare,[1] and thereafter it was looked upon as one of the stock questions of European mathematics. I have run across it in an Italian manuscript of c. 1440, it is in Petzensteiner's work of 1483,[2] Calandri used it in 1491,[3] Pacioli gives it in his Suma of 1494,[4] and most of the writers of any prominence in the sixteenth century embodied it in their lists.[5]

In those centuries when commercial communication was wholly by means of couriers who traveled regularly from city to city, a custom still determining the name of *correo* for a postman in certain parts of the world, the problem of the hare and hound naturally took on the form of, or perhaps paralleled, the one of the couriers. This problem was not, however, always one of pursuit, since the couriers might be traveling either in the same direction or in opposite directions.[6] This variant of the stock problem is purely Italian, for even the early German writers give it with reference to Italian towns.[7] As a matter of course also, it was varied by substituting ships for couriers,[8] while our modern text-book writers show their lack of originality by merely substituting automobiles for ships.

It was natural to expect that the problem should have a further variant, namely, the one in which the couriers should not start simultaneously. In this form it first appeared in print in Germany in 1483,[9] in Italy in 1484,[10] and in England in 1522.[11]

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[1] "De cursu canis ac fuga leporis."

[2] Folio 54; Unger, p. 106.

[3] "Una lepre e inanzi aun chane 3000 passi et ogni 5 passi delcane sono *p* 8 diquegli della lepre uosapere inquanti passi elcane ara giũto lalepre."

[4] "Vna lepre e dinançe a vn cane passa .60. e per ogni passa .5. che fa el cane la lèpre ne fa .7. e finalmente el cane lagiongni [la giongi in the edition of 1523, from la giúgnere, to overtake her] dimando in quanti passa el cane giõgera la lepre." Folio 42v. He says that the problem is not clear because we do not know whether the "passa .60." are leaps of the dog or of the hare, showing that he felt bound to take the stock problem as it stood without improving upon the phraseology. Indeed, we have few such marked examples of plagiarism, in that era of universal literary theft, as Pacioli's Sũma.

[5] Thus Rudolff (*Kunstlich rechnung*, 1526, 1534 edition, folio Nvj); Köbel (*Rechenbuch*, 1531, 1549 edition, folio 88, under the title "Von Wandern über Landt," with a picture in which the hare is quite as large as the hound); Cardan (*Arithmetica*, 1539, cap. 66); Wentzel (1599, p. 51); Ciacchi (*Regole generali d'Abbaco*, Firenze, 1675, p. 130); Coutereel (*Cyffer-Boek*, 1690 edition, p. 584), and many others.

[6] Various types are given in Pacioli's *Sũma* of 1494, folio 39.

[7] Thus Petzensteiner (1483, Folio 53), in his chapter "Von wandern," makes the couriers go to "rum" (Rome), thus: "Es sein zween gesellen die gand gen rum. Eyner get alle tag 6 meyl der ander geth an dem ersten tage 1 meyl an dem andern zwue etc. unde alle tag eyner meyl mer dan vor. Nu wildu wissen in wievil tagen eyner als vil hat gangen als der ander." Günther, *Geschichte*, p. 304; Müller, *Deutsche Blätter*, VI, p. 88.

[8] Thus Calandri (1491) says: "Una naue ua da Pisa a Genoua in 5 di: unaltra naue uiene dageno ua a pisa in 3 di. uo sapere partendosi in nun medesimo tempo quella da Pisa per andare a Genoua et quella da Genoua *p* andare a pisa in quanti di siniscon terrano insieme."

[9] Petzensteiner's arithmetic, printed at Bamberg.

[10] Borghi's arithmetic.

[11] Tonstall's *De arte supputandi*, folio 4, "Cvrsor ab Eboraco Londinvm proficiscens," etc. See also Cardan (*Arithmetica*, 1539, cap. 66, with various types); Ghaligai (1521, 1552 edi-

The invention of clocks with minute hands as well as hour hands gave the next variant, as to when both hands would be together—a relatively modern form of the question, as is also the astronomical problem of the occurrence of the new moon. The latest form, however, has to do with the practical question of a railway timetable, but here graphic methods naturally take the place of analysis so that of all the variants those of the couriers and the clock hands seem to be the only ones that will survive. Neither is valuable *per se,* but each is interesting, each is real within the range of easy imagination, and each involves a valuable mathematical principle—a fairly refined idea of function, and so it is probable that each will persist in spite of the present transitory period of the attempted debasement of elementary mathematics.

AN INVERSION OF THE COMPLETE QUADRILATERAL.

By J. W. CLAWSON, Ursinus College.

It is the purpose of this paper to point out an interesting example of the method of inversion. If a complete quadrilateral with some of its related lines and circles be inverted with respect to the quadrilateral's Wallace point, a new complete quadrilateral with some of its related circles and lines results. It is remarkable that this new quadrilateral is inversely similar to the original one, as will appear from II below.

Four straight lines (Fig. 1) ARB, BCP, CQA, PQR form a complete quadrilateral with A, P; B, Q; C, R for opposite vertices. The lines taken three by three also determine four triangles ABC, AQR, BRP, CPQ. It is well known that the circles circumscribing* these triangles are concurrent at a point O, the *Wallace point* of the quadrilateral,[1] that the circumcenters of the triangles are concyclic on l, the *circumcentric circle,*[2] that the orthocenters of the triangles are

tion, folio 64); Albert (1540, 1561 edition, folio Pi); Baker (1568, 1580 edition, folio 36); Coutereel (1631 edition, p. 371, and Eversdyck edition of 1658, p. 403); Trenchant (1566, 1578 edition, p. 280); Wentsell (1599, p. 51); Peletier (1549, 1607 edition, p. 290), Vander Schuere (1600, folio 179); Schonerus (notes on Ramus, 1586 edition, p. 174), and many others. Köbel's rather quaint German is interesting: "Zwen Burger vsz Oppenheym / einer Sō Heynrich / der ander Contz vō Treber gnant / wolten mit einander gen Rom gehn / vn̄ Heinrich was alt / vn̄ mocht einn tag nit mehr dan̄ zehen meiln gehn / Aber Contz vō Treber was jung vnnd starck / der mocht einen tag 13. meilen gehn / Deszhalben gieng Son Heynrich neun tag eh ausz Oppenheym dann Contz von Treber / Also war Son Heynrich Contzen 90. meilen furgangen / eh Contz angehaben hat auszugehn.

"Nun ist die frag / inn wie vil tagen Contz von Treber / Son Heynrichen übergangen / vnnd die zwen zusamen kommen seind." See also his *Zwey rechenbüchlin,* Frankfort, 1537 edition, folio 84.

* This figure is slightly distorted. The circles should join exactly through the points R and Q.

[1] "SCOTICUS," Leybourn's *Mathematical Repository,* 1804, Vol. I, p. 170. MACKAY, *Proc. Edin. Math. Soc.,* Vol. IX.

[2] DAVIES, *Math. Repos.,* 1835, Vol. VI, Question 555 answered.

[1, 2] STEINER, Gergonne's *Annales,* 1828, Vol. XVIII, pp. 302, 303, 1°, 2°, 3°, 4°.

collinear on o, the *orthocentric line*,[3] and that the feet of the perpendiculars from O on the four lines are collinear on p, the *pedal* line[4] of the quadrilateral.

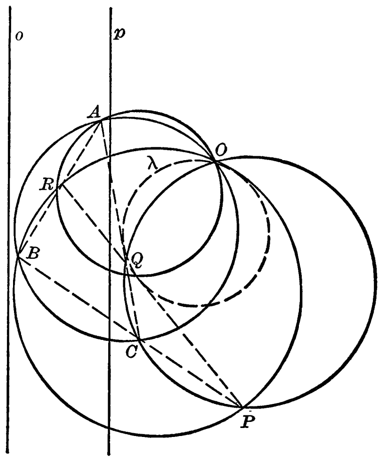


FIG. 1.

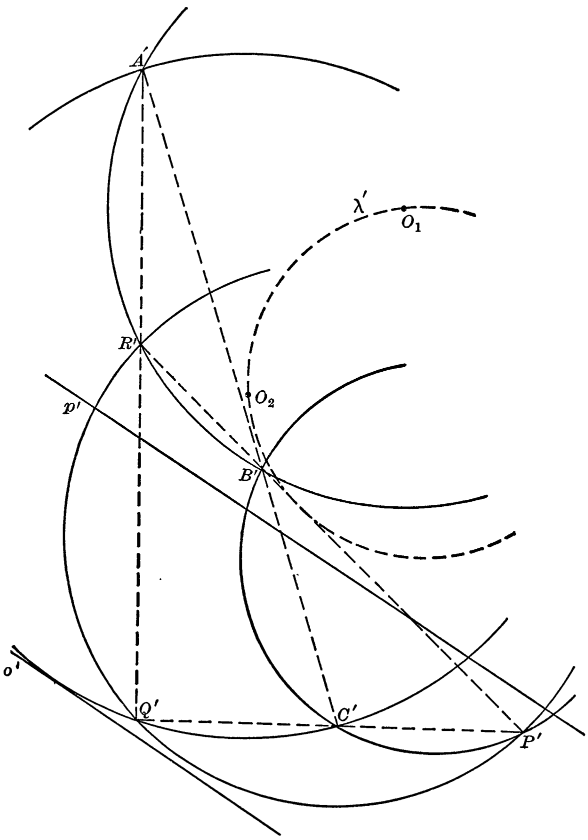


FIG. 2.

Invert the figure with O as center of inversion and with any radius. The full lines of Fig. 1 invert into broken lines of Fig. 2 and vice versa.

I. The circles ABC, AQR, BRP, CPQ of Fig. 1 invert into four straight lines of Fig. 2, $A'B'C'$, $A'R'Q'$, $P'B'R'$, $P'Q'C'$, which form a new quadrilateral with P', A'; Q', B'; R', C' for opposite vertices. The straight lines ARB, BCP, AQC, PQR invert into the four circles, concurrent at O' (not shown on this figure), which circumscribe the triangles $A'R'B'$, $B'C'P'$, $C'A'Q'$, $P'Q'R'$.[5]

II. The new quadrilateral is inversely similar to the old, the center and axis of similitude being respectively O and the bisector of the angle AOP. (This line also bisects angles BOQ and COR.)

For angle A is equal to (or supplementary to) the angle between the lines joining A' to the centers of the circles $A'R'B'$ and $A'C'Q'$. Denote these centers

[1,3,4] CASEY, *Sequel to Euclid*, ed. of 1886, pp. 35, 36.

[5] See MCCLELLAND, *Geometry of the Circle*, p. 256, Ex. 12.

by O_1 and O_2. Then $\angle O_1A'O_2 = \angle O_1O'O_2 = \angle O_1O'A' + \angle O_2O'A' = \angle O'B'A' - \angle O'C'A' = \angle C'O'B' = \angle P'$. Thus $\angle A = \angle P'$. Similarly any angle in the figure $APBQCRO$ can be proved equal to the corresponding angle in the figure $P'A'Q'B'R'C'O'$.

III. The orthocentric line of either quadrilateral inverts into the circumcentric circle of the other.

For the pedal line, p, passes through the intersection of BCP and a line through O perpendicular to that line. Hence its inverse is a circle passing through the intersection of the circle $O'B'C'P'$ and a straight line through O' orthogonal to that circle. Hence the inverse of p is a circle passing through the opposite extremities of the diameters through O' of the circles $A'R'B'$, $B'C'P'$, $C'A'Q'$, $P'Q'R'$. Hence the inverse of the orthocentric line, o, which is parallel to p and twice as far from O,[6] is a circle touching the inverse of p at O' and with a radius half as large; that is, it is the circumcentric circle.[7]

IV. Many properties of a complete quadrilateral lead, on inversion, to new properties of this figure. However the results are often complicated and of little interest. One example will be given.

The theorem: "The circles on the three diagonals of the quadrilateral as diameters are coaxial, the radical axis being the orthocentric line,"[8] inverts into:

"The three circles, (1) through A, P orthogonal to the circle OAP, (2) through B, Q orthogonal to OBQ, (3) through C, R orthogonal to OCR, are coaxial; and the circumcentric circle belongs to the same coaxial system."

A PROBLEM IN PROBABILITY.[1]

By C. S. JACKSON, R. M. Academy, Woolwich, England.

1. A problem first proposed by De Moivre and extended by Simpson was thrown into the following form by Laplace: If the numerical result of a single trial is equally likely to have any value between 0 and b, the chance that after n trials the sum of the results obtained shall be less than a is

$$\text{(i)} \qquad \frac{1}{b^n n!}\{a^n - n_1(a-b)^n + n_2(a-2b)^n \cdots\},$$

n_r denoting $n!/[r!(n-r)!]$ and the series being continued as long as $a - rb$ is positive.[2] In the following note an alternative mode of investigating (i) is used, which is intended to illustrate how each term of the formula arises.

2. Let $x_1 \cdots x_n$ be n positive items, each equally likely to have any value

[6] STEINER, *loc. cit.* DAVIES, *loc. cit.* CASEY, *loc. cit.*

[7] Cf. McCLELLAND, *loc. cit.*

[8] DURELL, *Plane Geometry for Advanced Students*, Part I, Theorem 86, p. 188.

[1] The proof sheets of this article never reached us from the author, having probably been lost in ocean transit. EDITORS.

[2] See TODHUNTER, *History, etc., of Probability*, pp. 84, 208, 542.

between 0 and a, where a lies between the values rb and $(r+1)b$. The chance k_0 that their sum s is less than a is

$$a^{-n}\int_0^a dx_1 \int_0^{a-\xi_1} dx_2 \int_0^{a-\xi_2} dx_3 \cdots \int_0^{a-\xi_{n-1}} dx_n,$$

where $\xi_r = x_1 + x_2 \cdots + x_r$.

This is a well-known integration, or may be worked out by putting

$$z_1 = a - \xi_1, \qquad z_2 = a - \xi_2, \qquad \cdots, \qquad z_n = a - \xi_n,$$

when it becomes

$$a^{-n}\int_0^a dx_1 \int_0^{z_1} dz_2 \cdots \int_0^{z_{n-1}} dz_n = a^{-n}\frac{a^n}{n!} = \frac{1}{n!}.$$

Again, the chance that $s < a$ and, at any rate, each of m specified items, say $x_1 \cdots x_m$, is greater than b, whatever the others may be, is

$$a^{-n}\int_b^{a-(m-1)b} dx_1 \cdots \int_b^{a-(m-r-1)b-\xi_r} dx_{r+1} \cdots \int_b^{a-\xi_{m-1}} dx_m \cdots \int_0^{a-\xi_s} dx_{s+1} \cdots \int_0^{a-\xi_{n-1}} dx_n,$$

the limits being obtained by noticing that $x_1 > b$ and $x_1 < a - (m-1)b$, because at least $(m-1)b$ must be left to provide for $x_2 \cdots x_m$ each exceeding b. We may put for $\lambda > m$

$$z_\lambda = a - \xi_\lambda, \qquad \text{so that} \qquad \int_0^{a-\xi_{\lambda-1}} dx_\lambda = \int_0^{z_{\lambda-1}} dz_\lambda;$$

while for $\lambda \leqq m$ we put

$$z_\lambda = a - (m-\lambda)b - \xi_\lambda, \qquad \text{so that} \qquad \int_b^{a-(m-\lambda)b-\xi_{\lambda-1}} dx_\lambda = \int_0^{z_{\lambda-1}} dz_\lambda;$$

and then the integral becomes

$$a^{-n}\int_0^{a-mb} dz_1 \int_0^{z_1} dz_2 \cdots \int_0^{z_{n-1}} dz_n = \frac{(a-mb)^n}{a^n n!}.$$

3. The m specified items might be chosen in n_m ways, whence we would write

$$k_m = \frac{(a-mb)^n}{a^n m!(n-m)!},$$

and proceed to analyze k_m into components according to the exact number of items which exceed b.

If u_0 = the chance that $s < a$, when *none* of the items $x_1 \cdots x_n$ exceeds b,

u_1 = the chance that $s < a$, when *one* of the items $x_1 \cdots x_n$ exceeds b,

. . . .

u_s = the chance that $s < a$, when *exactly* s of the items $x_1 \cdots x_n$ exceeds b,

and so on (the set ending with u_r, for it is impossible for more than r items to exceed b), then

(ii) $$k_m = u_m + (m+1)_m u_{m+1} \cdots + r_m u_r.$$

The cases which give rise to u_{m+l} are each counted $(m + l)_m$ times in k_m. Putting $m = 0, 1, \cdots r$ in turn in (ii) we obtain

$$k_0 = u_0 + u_1 + \cdots + u_r. \tag{0}$$

$$k_1 = u_1 + 2_1 u_2 + \cdots + r_1 u_r. \tag{1}$$

. . . .

$$k_m = u_m + (m+1)_m u_{(m+1)} + \cdots + r_m u_r. \tag{m}$$

. . . .

$$k_{(m+l)} = u_{(m+l)} + \cdots + r_{(m+l)} u_r. \tag{m + l}$$

. . . .

$$k_r = r_r u_r. \tag{r}$$

To solve these equations, multiply equation m by 1, $\cdots$, equation $(m + l)$ by $(-1)^l (m + l)_m$, $\cdots$, and add.

The coefficient of u_{m+l} in the result is

$$(m+l)_m - (m+l)_{m+1}(m+1)_m + \cdots + (-1)^s (m+l)_{(m+s)}(m+s)_m \cdots + (-1)^l (m+l)_m,$$

which is

$$(m+l)_m\{1 - l_1 + \cdots + (-1)^s l_s \cdots + (-1)^l\} = (m+l)_m (1-1)^l = 0.$$

Thus,

(iii) $u_m = k_m - (m+1)_m k_{m+1} + (m+2)_m k_{m+2} \cdots + (-1)^{r-m} r_m k_m, \cdots$

and, in particular,

$$u_0 = k_0 - k_1 + k_2 \cdots + (-1)^r k_r,$$

where, as already shown,

$$k_m = \frac{(a - mb)^n}{a^n m!(n-m)!}.$$

4. Now the probability being u_0 that $s < a$ and that each of the items $x_1 \cdots x_n$ less than b, and the *a priori* probability being $(b/a)^n$ that $x_1 \cdots x_n$ shall each be less than b, then the probability that, when $x_1 \cdots x_n$ are each given less than b, their sum s shall be less than a is

$$\left(\frac{a}{b}\right)^n \times u_0$$

or

(iv) $$\frac{1}{b^n n!}\{a^n - n_1(a-b)^n + n_2(a-2b)^n \cdots\}$$

5. Again, from (iii),

$$u_m = \frac{(a - mb)^n}{a^n \cdot m!(n - m)!} - (m + 1)_m \frac{(a - mb - b)^n}{a^n(m + 1)!(n - m - 1)!} + \cdots$$
$$+ (-1)^s(m + s)_m \frac{(a - mb - sb)^n}{a^n(m + s)!(n - m - s)!} \cdots,$$

and the *a priori* probability being $n_m[(a - b)^m b^{n-m}/a^n]$ that exactly m items exceed b, the chance that, when m items exceed b, their sum s shall be less than a is

$$\text{(v)} \quad \frac{1}{n!(a - b)^m b^{n-m}} \{(a - mb)^n - (n - m)(a - mb - b)^n \cdots$$
$$+ (-1)^s(n - m)_s(a - mb - sb)^n \cdots\}.$$

This last result (v) is the chance that if, out of n positive items, m are equally likely to have any value between b and a, and the remainder to have any value less than b, then their sum shall be less than a.

6. The Hon. R. J. Strutt gave an interesting application of formula iv in the *Philosophical Magazine*, 6 series, Vol. 1, p. 311. The sum of the numerical departures from integral values of nine well-determined atomic weights is .809. If we suppose that an individual departure is equally likely to have any value between 0 and .5, the chance of the sum of nine departures being less than .809 proves to be .001159. The smallness of this value, he infers, gives some support to the well-known hypothesis that the atomic weights should be integers.

A SIMPLE GEOMETRICAL PARADOX.

PROPOSED BY J. L. COOLIDGE, Harvard University.

Suppose that we have an algebraic surface

$$x = \frac{f_1(u, v, w)}{f(u, v, w)}, \qquad y = \frac{f_2(u, v, w)}{f(u, v, w)}, \qquad z = \frac{f_3(u, v, w)}{f(u, v, w)},$$
$$F(u, v, w) = 0.$$

We shall assume that this surface has no singular curve, an assumption which still leaves us in what we may call the *general* case, since the discriminant of a polynomial in three variables does not vanish identically. Let us cut this surface by an arbitrary plane which does not pass through any isolated singularity which the surface may possess, a *general* plane we might say. The coördinates of the points of the curve of intersection are algebraic functions of a single parameter, and the same is true of the sine of the angle which the given plane makes with the tangent plane to the surface at the points of the curve.

Suppose first, that this algebraic function is not a constant. It must, then,

have one or more zeros. At such a point, the sine of the angle, and hence the angle itself, must be zero and the surface touches the plane. This, however, is absurd; no surface can touch every plane not through an isolated singular point. Let us, then, suppose that the algebraic function is a constant. The surface will then meet every plane at a constant angle, every plane through a normal will cut it everywhere orthogonally. Hence each two normals are coplanar, the normals are all concurrent or parallel, and the surface is either a sphere or a plane.

CONCLUSION. *Every algebraic surface which has no singular curve is a sphere or plane.*

ORGANIZATION OF A MINNESOTA SECTION OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

On November 8, 1916, a committee consisting of Messrs. G. N. Bauer, R. M. Barton and G. W. Hartwell, representing the Minnesota members of the Mathematical Association of America, sent out a circular letter over the state inviting all teachers of college mathematics to meet at the University of Minnesota on Friday, December 1, 1916, for the purpose of considering the possibility and wisdom of organizing a Minnesota section of the Mathematical Association of America.

Twenty-three teachers attended this meeting sixteen of whom were already, or have since become, members of the Association. These members are as follows: Sister M. Magna, of St. Benedict's College, St. Joseph; Miss E. G. Berger, of St. Catherine's College, St. Paul; Wm. C. Etzel, College of St. Thomas, St. Paul; Paul E. Kretzmann, Concordia College, St. Paul; L. E. Lunn, Supt. of Schools, Heron Lake; Thos. C. Wollan, Fergus Falls; G. N. Bauer, C. McCormick, Wm. O. Beal, H. H. Dalaker, A. L. Underhill, Vera L. Wright, H. L. Slobin, W. H. Bussey, R. M. Barton, and W. D. Reeve, of the University of Minnesota.

In the morning session Dr. Bauer presented the report of the committee in which he discussed some of the problems confronting college teachers in Minnesota. It was then decided by unanimous vote to form a Minnesota Section of the Mathematical Association of America and a committee was chosen to nominate officers for the ensuing year. The meeting then adjourned to a luncheon arranged in Alice Shevlin Hall at the University.

At the beginning of the afternoon session the nominating committee recommended Dr. G. N. Bauer for President, W. D. Reeve for Secretary-Treasurer and Dr. C. N. Gingrich as the third member of an executive committee. In addition the committee selected J. S. Mikesh and Miss E. G. Berger to act with the executive committee as a committee on policy for the section. The recommendations of the nominating committee were accepted.

It was further agreed to hold two meetings of the section each year, one in the spring and one in the autumn, the time to be set definitely by the executive committee.

The following program was then given: "Cultural value of college mathematics," J. S. Mikesh; "Report of current research in transcendental curves and numbers," Dr. H. L. Slobin; "Thoughts on a natural number system," L. E. Lunn, Heron Lake; "A solution of the differential equation $dy/dt + (\alpha + \beta \cos t)y = \rho \cos t$," W. O. Beal; "Unification of mathematics in the high school and college," W. D. Reeve.

W. D. REEVE, *Secretary.*

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

NEW BOOKS RECEIVED.

ELEMENTS OF ANALYTIC GEOMETRY. By Alexander Ziwet and Louis Allen Hopkins. The Macmillan Company, New York, 1916. viii + 272 pages. $1.60.

FIRST YEAR MATHEMATICS. By George W. Evans and John A. Marsh. Charles E. Merrill Co., New York, 1916. 253 pages. $0.90.

PROJECTIVE ORNAMENT. By Claude Bragdon. The Manas Press, Rochester, N. Y., 1915. 79 pages. $1.50.

A PRIMER OF HIGHER SPACE. By Claude Bragdon. The Manas Press, Rochester, N. Y., 1913. 79 pages. $1.25.

FOUR DIMENSIONAL VISTAS. By Claude Bragdon. Alfred A. Knopf, New York, 1916. 134 pages. $1.25.

A SHORT COURSE IN ELEMENTARY MECHANICS FOR ENGINEERS. By Clifford Newton Mills. D. Van Nostrand Co., New York, 1916. xi + 127 pages. $1.00.

INTRODUCTION TO MATHEMATICS. Junior High School Series. By Robert L. Short and William H. Elson. D. C. Heath and Co., Boston, 1916. vii + 200 pages. $1.00.

NEW PLANE AND SOLID GEOMETRY. By Edward Rutledge Robbins. American Book Company, New York, 1916. viii + 460 pages.

Differential Calculus. By H. B. PHILLIPS. John Wiley and Sons, New York, 1916. v + 162 pages.

This small compact volume contains a brief course on differential calculus in 139 pages followed by 14 pages of supplementary exercises and 9 pages devoted to answers to problems and an index. It contains all that can be covered in a semester course of 3 hours a week. It may even be made to serve as the basis of a four- or five-hour course. The following quotation from the preface tells what was the author's idea in writing the book: "In this text on differential calculus I have continued the plan adopted for my Analytic Geometry, wherein a few central methods are expounded and applied to a large variety of examples to the end that the student may learn principles and gain power. In this way the differential calculus makes only a brief text suitable for a term's work and leaves for the integral calculus, which in many respects is far more important, a greater proportion of time than is ordinarily devoted to it."

The reviewer finds the following things about the book worthy of comment: (1) In the "Introduction" (Chapter I), along with the usual definition of infinitesimals, is given the idea of the *order of infinitesimals.* (2) "Derivatives and Differentials" are taken up side by side in Chapter II. (3) Applications of

differential calculus to "Rates" (Chapter IV) and "Maxima and Minima" (Chapter V) are given right after "Differentiation of Algebraic Functions" (Chapter III). "Transcendental Functions" are not introduced until Chapter VI. (4) Maxima and Minima are treated without the use of second derivatives. (5) The number e is introduced in Chapter VI as the number satisfying the relation

$$\lim_{x \doteq 0} \frac{e^x - 1}{x} = 1.$$

and the approximate value of e is not computed until Chapter X on "Series and Approximations" is reached. (6) Chapter VII contains "Geometrical Applications" to both plane and solid geometry and they are so arranged that the latter may be omitted if it is desired. (7) There are numerous applications to the simpler problems of mechanics in the plane and in space.

The chapter headings not already mentioned are Chapter VIII, "Velocity and Acceleration in a Curved Path," Chapter IX, "Rolle's Theorem and Indeterminate Forms," and Chapter XI, "Partial Differentiation."

The supplementary exercises at the end of the book (pages 140–153) are to be used as "material for review and to provide problems for which answers are not given.

The book seems to be one which will teach the student not only the mechanical part of differential calculus but also the true value of the processes involved.

A. L. UNDERHILL.

UNIVERSITY OF MINNESOTA.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

ALGEBRA.

475. Proposed by E. B. ESCOTT, Kansas City, Mo.

A man makes a contract to purchase a house, making a cash payment down and agreeing to make monthly payments of a dollars, interest being charged at six per cent., the balance of the monthly payments being credited on the principal. Find a formula for M_n, the balance due after n payments.

476. Proposed by W. HAROLD WILSON, University of Illinois.

Prove that, if $x_h \neq x_j$, $h, j = 1, 2, 3 \cdots n$, $h \neq j$, then

$$\sum_{i=1}^{n} \frac{x_i^{n-1}}{\prod_{h=1}^{n} (x_i - x_h)'} = 1,$$

where the prime indicates the omission of zero factors in the denominators.

GEOMETRY.

508. Proposed by J. E. ROWE, State College, Penn.

The trilinear coördinates of the vertices of the Brocard triangle are $(s_3^3, s_1s_2s_3, s_1^3)$, $(s_2^3, s_1^3, s_1s_2s_3)$, and $(s_1s_2s_3, s_3^3, s_2^3)$, where s_i $(i = 1, 2, 3)$ are the sines of the angles of the funda-

mental triangle. Show that the Brocard triangle and the fundamental triangle are in perspective, and that the trilinear coördinates of the center of perspectivity are s_i^{-3} ($i = 1, 2, 3$) instead of s_i^3 which are incorrectly given in Clebsch's *Vorlesungen über Geometrie*, p. 323.

509. Proposed by NORMAN ANNING, Chilliwack, B. C.

A picture whose coördinates are (0, 0), (50, 0), (50, 50), and (0, 50) is repeated on a smaller scale as part of itself with the coördinates (7, 0), (31, 7), (24, 31), (0, 24). Locate the vanishing point.

CALCULUS.

423. Proposed by J. B. REYNOLDS, Lehigh University.

Show that the envelope of all circles with their centers on the circle $x^2 + y^2 = a^2$ and tangent to the x-axis is the two-arched epicycloid.

424. Proposed by OSCAR S. ADAMS, Washington, D. C.

What is the value of

$$\frac{\Gamma'(1)}{\Gamma(1)} - \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})}?$$

MECHANICS.

340. Proposed by PAUL CAPRON, U. S. Naval Academy.

A rigid straight line l passes through a fixed point O, but is otherwise free to move in a plane. If C is the instantaneous center of rotation for l, prove that CO is always perpendicular to l and that, if (O being used as pole) $\rho = f(\theta)$ represents the locus of any point P on l, OC is always equal to $(d/d\theta)f(\theta)$.

341. Proposed by PAUL CAPRON, U. S. Naval Academy.

A pole l feet long, with one end on the ground, touches the top of a wall a feet high and slides in a vertical plane perpendicular to the wall. Show that its instantaneous center of rotation is at the intersection of the vertical where it touches the ground with the perpendicular to its axis where it touches the wall, and that the locus of this center is a parabola having the latus rectum a.

NUMBER THEORY.

259. Proposed by E. E. WHITFORD, College of the City of New York.

If p is relatively prime to 10, and if any multiple of p consisting of n digits has its digits permuted cyclically, the number thus formed is also a multiple of p; the number n to be determined by the congruence $10^n \equiv 1 \pmod{p}$. For example, 481, 814, and 148 are each multiples of 37.

260. Proposed by ALBERT A. BENNETT, University of Texas.

Let $\binom{n}{r}$ denote as usual the binomial coefficient $n!/[r!(n-r)!]$, where $\binom{n}{0} = 1$, but where n, r, $(n - r)$ are always to be supposed to be positive integers or zero. Let us define $k_i(m, n)$ as $\sum_j \binom{m-i+j}{i-j}\binom{n-j}{j}$. Prove that the following recursion formulas are consistent:

$$\sum_i (-1)^i k_i(m, n) C_{m+n-i} = \binom{m+n}{m}$$

and determine $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, $C_5 = 42$, $C_6 = 132$, $C_7 = 429$, $C_8 = 1{,}430$, etc. Prove also that these quantities satisfy the following relations, as well:

$$\sum_i (-1)^i C_{m-n-i} \binom{m-i}{i} = 0 \text{ for each } n \text{ where } 2n \leq m.$$

SOLUTIONS OF PROBLEMS.

ALGEBRA.

462. Proposed by H. S. UHLER, Yale University.

Show how to transform A into S, where these symbols denote the equivalent formulæ for the general case of Calculus Problem No. 363, pages 52 and 54 in the February, 1916, MONTHLY:

$$A \equiv 4\pi R^2 - 4nR^2 \sin^{-1}\left(\frac{R \sin \pi/n}{\sqrt{R^2 - a^2}}\right) + 2anR \sin^{-1}\left[\frac{2a(\tan \pi/n)(\sqrt{R^2 - a^2 \sec^2 \pi/n})}{R^2 - a^2}\right],$$

$$S \equiv 4nR\left\{a \sin^{-1}\left(\tan\frac{\pi}{n} \cdot \frac{a}{\sqrt{R^2 - a^2}}\right) - R \sin^{-1}\left[\tfrac{1}{2}\sin\frac{2\pi}{n} \cdot \frac{R - \sqrt{R^2 - a^2 \sec^2 \pi/n}}{\sqrt{R^2 - a^2}}\right]\right\}.$$

SOLUTION BY GEO. W. HARTWELL, Hamline University.

$$A \equiv 4nR\left\{R\left[\frac{\pi}{n} - \sin^{-1}\left(\frac{R \sin \pi/n}{\sqrt{R^2 - a^2}}\right)\right] + \tfrac{1}{2}a \sin^{-1}\left[\frac{2a(\tan \pi/n)(\sqrt{R^2 - a^2 \sec^2 \pi/n})}{R^2 - a^2}\right]\right\}$$

$$\equiv 4nR(a\gamma + R\beta).$$

Let

$$\sin^{-1}\left(\frac{R \sin \pi/n}{\sqrt{R^2 - a^2}}\right) = \alpha \quad \text{and} \quad \frac{\pi}{n} - \alpha = \beta.$$

Then,

$$\sin\beta = \sin\frac{\pi}{n}\cos\alpha - \cos\frac{\pi}{n}\sin\alpha.$$

$$\cos\alpha = \sqrt{1 - \frac{R^2 \sin^2 \pi/n}{R^2 - a^2}} = \sqrt{\frac{R^2 \cos^2 \pi/n - a^2}{R^2 - a^2}} = \frac{\cos \pi/n \sqrt{R^2 - a^2 \sec^2 \pi/n}}{\sqrt{R^2 - a^2}}.$$

Hence,

$$\sin\beta = \sin\frac{\pi}{n}\cos\frac{\pi}{n}\left[\frac{\sqrt{R^2 - a^2 \sec^2 \pi/n} - R}{\sqrt{R^2 - a^2}}\right],$$

or

$$\beta = -\sin^{-1}\left[\tfrac{1}{2}\sin\frac{2\pi}{n} \cdot \frac{R - \sqrt{R^2 - a^2 \sec^2 \pi/n}}{\sqrt{R^2 - a^2}}\right].$$

Let

$$\sin^{-1}\frac{2a(\tan \pi/n)(\sqrt{R^2 - a^2 \sec^2 \pi/n})}{R^2 - a^2} = \gamma.$$

Then,

$$\sin\tfrac{1}{2}\gamma = \sqrt{\tfrac{1}{2}(1 - \cos\gamma)} = \sqrt{\frac{1}{2}\left(1 - \sqrt{1 - \frac{4a^2 \tan^2 \pi/n\,(R^2 - a^2 \sec^2 \pi/n)}{(R^2 - a^2)^2}}\right)}$$

$$= \sqrt{\frac{1}{2}\left(1 - \frac{\sqrt{R^4 - 2a^2R^2(1 + 2\tan^2 \pi/n) + a^4(1 + 4\sec^2 \pi/n \tan^2 \pi/n)}}{R^2 - a^2}\right)}$$

$$= \sqrt{\frac{1}{2}\left(1 - \frac{R^2 - a^2(1 + 2\tan^2 \pi/n)}{R^2 - a^2}\right)} = \frac{a \tan \pi/n}{\sqrt{R^2 - a^2}};$$

hence,

$$\tfrac{1}{2}\gamma = \sin^{-1}\frac{a \tan \pi/n}{\sqrt{R^2 - a^2}}.$$

Making these substitutions, we have $A \equiv S$.

Also solved by the PROPOSER.

463. Proposed by H. O. HANSON, East Elmhurst, N. Y.

Find the sum of the series

$$\binom{2n}{0} + 2\binom{2n-1}{1} + 2^2\binom{2n-2}{2} + \cdots + 2^n\binom{n}{n},$$

where $\binom{n}{r}$ denotes the coefficient of x^r in the expansion of $(1+x)^n$.

SOLUTION BY EDWIN R. SMITH, State College, Pa.

Consider the expansion of $1/(1-x-2x^2)$ into a series of ascending powers of x. First,

$$\frac{1}{1-x-2x^2} = \frac{1}{1-(x+2x^2)} = 1 + (x+2x^2) + (x+2x^2)^2 + \cdots$$

$$= 1 + \binom{1}{0}x + \left[\binom{2}{0} + \binom{1}{1}2\right]x^2 + \left[\binom{3}{0} + \binom{2}{1}2\right]x^3 + \cdots$$

$$+ \left[\binom{2n}{0} + 2\binom{2n-1}{1} + 2^2\binom{2n-2}{2} + \cdots + 2^n\binom{n}{n}\right]x^{2n} + \cdots.$$

A second expansion can be obtained as follows:

$$\frac{1}{1-x-2x^2} = \frac{1}{1+x}\cdot\frac{1}{1-2x}$$

$$= (1 - x + x^2 - x^3 + \cdots)(1 + 2x + 2^2x^2 + 2^3x^3 + \cdots)$$

$$= 1 - (1-2)x + (1-2+2^2)x^2 - (1-2+2^2-2^3)x^3 + \cdots$$

$$+ (1 - 2 + 2^2 - 2^3 + \cdots + 2^{2n})x^{2n} - \cdots$$

$$= 1 - \frac{1-2^2}{1+2}x + \frac{1+2^3}{1+2}x^2 - \cdots + \frac{1+2^{2n+1}}{1+2}x^{2n} - \cdots.$$

Comparing in the two series the expressions for the coefficient of x^{2n} there is obtained

$$\binom{2n}{0} + 2\binom{2n-1}{1} + 2^2\binom{2n-2}{2} + \cdots + 2^n\binom{n}{n} = \frac{1+2^{2n+1}}{1+2} = \tfrac{1}{3}(2^{2n+1}+1),$$

which is the required sum.

Also solved by A. M. KENYON, N. P. PANDYA, and the PROPOSER.

464. Proposed by GEORGE Y. SOSNOW, Newark, N. J.

Find the general term and the sum of n terms of the series 1, 4, 15, 56, $\cdots$, where

$$U_n = 4U_{n-1} - U_{n-2}.$$

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

Changing the notation so that for U_n we may write U_{n+2}, we have

$$U_{n+2} - 4U_{n+1} + U_n = 0,$$

an equation in finite differences. Integrating, noticing that the roots of

$$m^2 - 4m + 1 = 0,$$

are

$$m_1 = 2 + \sqrt{3},$$

$$m_2 = 2 - \sqrt{3},$$

(1) $$U_n = C_1(2+\sqrt{3})^n + C_2(2-\sqrt{3})^n.$$

When $n = 1$, $U_1 = 1$, and when $n = 2$, $U_2 = 4$; then (5) gives for determining C_1 and C_2,

$$C_1(2+\sqrt{3}) + C_2(2-\sqrt{3}) = 1,$$
$$C_1(2+\sqrt{3})^2 + C_2(2-\sqrt{3})^2 = 4,$$

giving $C_1 = \frac{1}{6}\sqrt{3}$ and $C_2 = -\frac{1}{6}\sqrt{3}$.

(1) then becomes $$U_n = \tfrac{1}{6}\sqrt{3}\{(2+\sqrt{3})^n - (2-\sqrt{3})^n\},$$

the general term. The usual theory for the sum of n terms gives

$$S_n = C + C_1 m_1 \frac{m_1^n}{m_1 - 1} + C_2 m_2 \frac{m_2^n}{m_2 - 1}.$$

Substituting the values of m_1, m_2, C_1, C_2, we have

$$(2) \qquad S_n = C + \tfrac{1}{6}\sqrt{3}\left\{\frac{(2+\sqrt{3})^{n+1}}{\sqrt{3}+1} + \frac{(2-\sqrt{3})^{n+1}}{\sqrt{3}-1}\right\}.$$

When $n = 1$, $S_n = 1$, and (2) gives $C = -\frac{1}{2}$, and this in (2) gives the required sum.

Also solved by AMELIA BENSON, G. W. HARTWELL, E. B. ESCOTT, A. M. HARDING, N. P. PANDYA, and O. S. ADAMS.

GEOMETRY.

489. Proposed by NATHAN ALTSHILLER, The University of Oklahoma.

The parallels to the asymptotes a, b of a given hyperbola, drawn from a variable point of the curve, meet a and b in P, Q respectively. The line PQ envelops an hyperbola whose asymptotes are a and b.

I. SOLUTION BY E. J. OGLESBY, Williamsburg, Virginia.

Take the asymptotes a, b as the axes of coördinates. Then the equation of the hyperbola may be taken as $xy = c^2$ and the coördinates of the variable point on the hyperbola as $(ct, c/t)$ in terms of the parameter t. P is the point $(ct, 0)$, and Q is $(0, c/t)$.

The equation of PQ may be written

$$(1) \qquad t^2 y - ct + x = 0.$$

We find the envelope of (1) by applying the condition that this equation shall have equal roots in the parameter t.

Hence, we have

$$(-c)^2 - 4yx = 0, \quad \text{or} \quad xy = c^2/4,$$

which is an hyperbola having a and b as asymptotes.

II. SOLUTION BY THE PROPOSER.

The tangents a, b to a given conic at the points A, B, are met by the lines BM, AM joining A and B to a variable point M of the curve, in the points P, Q respectively. The line PQ envelops a conic having a double contact with the given curve at the points A, B.

Indeed, the lines AM, BM describe two projective pencils, hence their sections by the lines a, b are two projective ranges.

$$(P\cdots) \barwedge B(M\cdots) \barwedge A(M\cdots) \barwedge (Q\cdots).$$

Consequently the line PQ envelops a conic tangent to a and b. To the point (ab) considered as an element of a and b in turn correspond, in the ranges $(Q\cdots)$ and $(P\cdots)$, the points B and A, these points are therefore the points of contact of a and b with the envelope, which proves the proposition.

If for A, B are taken some remarkable points of the conic, special cases of this general proposition are obtained. For example, if A be the point at infinity of a parabola, and B its vertex, the

proposition takes the following form: *The diameter passing through a variable point of a parabola, meets the tangent at the vertex in the point P. The parallel through P to the line joining M to the vertex of the parabola, envelops another parabola having the same vertex and the same axis as the given curve.*

The proposed problem is another special case of this general proposition, namely when both A and B are at infinity.

The duals of the three propositions are, in order:

The points of intersection of two fixed tangents to a given conic, with a variable tangent to the same curve, are projected from the points where the fixed tangents touch the conic. The point of intersection of the two projecting lines describes a conic having a double contact with the given curve.

From a variable point of the tangent at the vertex of a given parabola, are drawn the diameter and the tangent to the curve. The point of intersection of the diameter with the parallel to the tangent through the vertex of the curve, describes a parabola having the same axis and the same vertex as the given curve.

The parallels to the asymptotes of a given hyperbola drawn through the points of intersection of the latter lines with a variable tangent to the curve, intersect in a point whose locus is an hyperbola having the same asymptotes as the given curve.

Also solved by O. S. ADAMS, CLARA L. BACON, J. W. CLAWSON, A. M. HARDING, HORACE OLSON, PAUL CAPRON, G. W. HARTWELL, R. M. MATHEWS, and N. P. PANDYA.

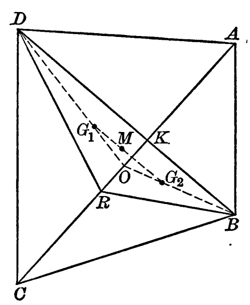
490. Proposed by ELMER E. MOOTS, University of Arizona.

In any quadrilateral $ABCD$, let AC and BD be the diagonals intersecting at K. On AC, lay off CR equal to AK. Join B and R. Connect the middle point G of BR with D. On GD lay off GM equal to $\frac{1}{3}GD$. Show that M is the center of gravity of the quadrilateral.

SOLUTION BY A. M. HARDING, University of Arkansas.

It is evident that M is the center of gravity of the triangle BDR. Hence it will be sufficient to prove that the triangle BDR and the quadrilateral $ABCD$ have the same center of gravity.

Let O be the mid-point of RK, then it will also be the mid-point of CA. Then G_1 is the center of gravity of the triangles RDK and CDA, and G_2 is the center of gravity of the triangles RBK and CBA where $OG_1 = \frac{1}{3}OD$ and $OG_2 = \frac{1}{3}OB$.



Since

$$\frac{\triangle\, RDK}{\triangle\, RBK} = \frac{\triangle\, CDA}{\triangle\, CBA},$$

is follows that the center of gravity of the triangle BDR will also be the center of gravity of the quadrilateral $ABCD$.

Also solved by J. W. CLAWSON, O. S. ADAMS, and N. P. PANDYA.

491. Proposed by N. P. PANDYA, Sojitra, India.

In a triangle $mx = b$ and $nx = c$, determine a relation between m, n, x, A and s, and solve it for x.

SOLUTION BY J. A. COLSON, Searsport, Maine.

Since $b = mx$ and $c = nx$, we have $\sin^2 A/2 = (s - b)(s - c)/bc = (s - mx)(s - nx)/mnx^2$. Hence, $mnx^2 \sin^2 A/2 = s^2 - (m + n)sx + mnx^2$, or $mnx^2 \cos^2 A/2 - (m + n)sx + s^2 = 0$.

Solving this quadratic for x, we have

$$x = \frac{(m + n)s \pm s\sqrt{\{(m + n)^2 - 4mn \cos^2 A/2\}}}{2mn \cos^2 A/2}.$$

492. Proposed by FRANK V. MORLEY, Student, Haverford College.

Let a_i ($i = 1, 2, 3, 4$) be four points on a circle, and let the symmedian point of the triangle formed by omitting a_i be s_i. Prove that the four points s_i have the same diagonal triangle as the four points a_i.

SOLUTION BY J. E. ROWE, Pennsylvania State College.

We choose that system of homogeneous coördinates in which the coördinates of a point are proportional to $\alpha/a : \beta/b : \gamma/c$, where a, b, c are the lengths of the sides of the reference triangle and α, β, γ the lengths of the $\perp$'s from the sides to the point. It may easily be shown that in this system of coördinates the equation of the circle circumscribing the reference triangle is

$$x_2x_3 + x_1x_3 + x_1x_2 = 0. \tag{1}$$

Let the coördinates of the four a's be

$$a_1 \equiv b_1, b_2, b_3; \quad a_2 \equiv 1, 0, 0; \quad a_3 \equiv 0, 1, 0; \quad a_4 \equiv 0, 0, 1.$$

There is evidently no loss of generality in this selection of the a's, and they will all lie on (1) if only

$$b_2b_3 + b_1b_3 + b_1b_2 = 0. \tag{2}$$

The coördinates of P_1 the intersection of the lines a_1a_3 and a_2a_4 are $b_1, 0, b_3$; similarly the coördinates of P_2 the intersection of the lines a_1a_4 and a_2a_3 are $b_1, b_2, 0$; and the coördinates of P_3 the intersection of the lines a_1a_2 and a_3a_4 are $0, b_2, b_3$. That is, $P_1P_2P_3$ is the diagonal triangle of the a's.

The equations of the tangents to (1) at the points a_i are

$$\begin{aligned} T_1 &\equiv (b_2 + b_3)x_1 + (b_1 + b_3)x_2 + (b_1 + b_2)x_3 = 0, \\ T_2 &\equiv x_2 + x_3 = 0, \\ T_3 &\equiv x_1 + x_3 = 0, \\ T_4 &\equiv x_1 + x_2 = 0. \end{aligned}$$

The tangents to (1) at two of the points a intersect in a point, and this point and a third a determine a line. Any set of three a's yields three such lines which are concurrent through the symmedian point of the three a's. In this way we find that the coördinates of the symmedian points are

$$s_1 \equiv 1, 1, 1; \; s_2 \equiv b_1, b_1 + 2b_2, b_1 + 2b_3; \; s_3 \equiv b_2 + 2b_1, b_2, b_2 + 2b_3; \; s_4 \equiv b_3 + 2b_1, b_3 + 2b_2, b_3.$$

By reason of (2) the determinant

$$\begin{vmatrix} b_1 & b_1 + 2b_2 & b_1 + 2b_3 \\ b_3 + 2b_1 & b_3 + 2b_2 & b_3 \\ b_1 & 0 & b_3 \end{vmatrix} = 0.$$

Hence, the points s_2, s_4, and P_1 are collinear. In the same way it may be shown that s_1, s_3, and P_1 are collinear. From the symmetry of the coördinates of the P's and the s_i it follows that $s_2s_3P_2$, $s_1s_4P_2$, $s_1s_2P_3$, and $s_3s_4P_3$ are collinear sets of three points, and this shows that $P_1P_2P_3$ is the diagonal triangle of the s_i.

Also solved by J. W. CLAWSON and J. W. HASLEY.

CALCULUS.

410. Proposed by J. A. BULLARD, U. S. Naval Academy.

(*a*) Find the area of the loop of the curve $x^{2q+1} + y^{2q+1} = (2q + 1)ax^q y^q$. (For $q = 1$ we have the folium of Prob. 379.)

(*b*) Find the area between the curve and its asymptote. (From Johnson's *Integral Calculus*.)

SOLUTION BY THE PROPOSER.

The required integration is simplified by the use of polar coördinates. If we let $y = mx$ and express the equation in parametric form we have merely to note that the parameter m is $\tan\theta$ and to substitute in the integral $\frac{1}{2}\int r^2\, d\theta$. Thus in the case of the folium (see p. 343, Vol. XXII) we have

$$A = \frac{1}{2}\int_0^{\pi/2} r^2 d\theta = \frac{1}{2}\int_0^{\infty} x^2 dm = \frac{9a^2}{2}\int_0^{\infty} \frac{m^2 dm}{(1 + m^3)^2} = -\frac{3a^2}{2}\left[\frac{1}{(1 + m^3)}\right]_0^{\infty} = 3a^2/2.$$

(*a*) The above equation becomes in polar coördinates

$$r = \frac{(2q + 1)a \tan^q \theta \sec\theta}{1 + \tan^{2q+1}\theta};$$

or in parametric form

$$x = \frac{(2q + 1)am^q}{1 + m^{2q+1}}, \qquad y = \frac{(2q + 1)am^{q+1}}{1 + m^{2q+1}},$$

where $m = \tan\theta$. The loop is generated when θ varies from 0 to $\pi/2$. Thus

$$A = \frac{1}{2}\int_0^{\pi/2} r^2 d\theta = \frac{(2q + 1)^2 a^2}{2}\int_0^{\pi/2} \frac{\tan^{2q}\theta \sec^2\theta d\theta}{(1 + \tan^{2q+1}\theta)^2}$$

$$= \frac{(2q + 1)^2 a^2}{2}\int_0^{\infty} \frac{m^{2q} dm}{(1 + m^{2q+1})^2} = -\frac{(2q + 1)a^2}{2}\left[\frac{1}{1 + m^{2q+1}}\right]_0^{\infty} = \frac{(2q + 1)a^2}{2}.$$

(*b*) The equation of the asymptote is $x + y = (-1)^q a$ or

$$r = \frac{(-1)^q a \sec\theta}{1 + \tan\theta}.$$

The asymptote forms a triangle of area $a^2/2$ with the coördinate axes, and this triangle separates the remainder of the required area into two equal parts. That part in the second quadrant when q is odd but in the fourth quadrant when q is even, is given by

$$A_1 = \frac{1}{2}\int_{3\pi/4}^{\pi} (r_1^2 - r_2^2) d\theta = \frac{a^2}{2}\int_{-1}^{0}\left[\frac{1}{(1 + m)^2} - \frac{(2q + 1)^2 m^{2q}}{(1 + m^{2q+1})^2}\right] dm = \frac{a^2}{2}\left[-\frac{1}{1 + m} + \frac{2q + 1}{1 + m^{2q+1}}\right]_{-1}^{0}$$

$$= \frac{a^2}{2}\left[\frac{2q + m - m^2 + m^3 - \cdots + m^{2q-1} - m^{2q}}{1 + m^{2q+1}}\right]_{-1}^{0}$$

$$= \frac{a^2}{2}\left[\frac{2q - (2q - 1)m + (2q - 2)m^2 - \cdots + 2m^{2q-2} - m^{2q-1}}{1 - m + m^2 - \cdots - m^{2q-1} + m^{2q}}\right]_{-1}^{0}$$

$$= \frac{a^2}{2}\left[2q - \frac{(2q + 1)(2q)/2}{2q + 1}\right] = \frac{qa^2}{2}.$$

Then

$$A = 2A_1 + a^2/2 = \frac{(2q + 1)a^2}{2}.$$

Thus the area of the loop is equal to the area between the curve and its asymptote.

In the case of the folium ($q = 1$) the coördinate axes trisect the area between the curve and its asymptote.

Also solved by O. S. ADAMS, HORACE OLSON, PAUL CAPRON, and A. M. HARDING.

MECHANICS.

304. Proposed by B. F. FINKEL, Drury College.

A spherical shell, inner radius r and outer radius R, has within it a perfectly smooth solid sphere of the same material and with radius $r_1 < r$. If the inner surface of the spherical shell is also perfectly smooth, determine the motion, after the time t, of the shell and sphere down a rough inclined plane, inclination α.

II. Solution by William Hoover, Columbus, Ohio.

Let the radii of gyration of the shell and sphere, k', k'', be given by

$$k'^2 = \frac{2}{5}\frac{R^5 - r^5}{R^3 - r^3},$$

$$k''^2 = \frac{2}{5} r_1^2;$$

C_0T_0 the radius of the shell to the tangent point on the inclined plane initially; E_0 the common point initially of the inner surface of the shell and sphere; CT_0 the position of C_0T_0 after any time from the beginning of motion of the system; D the center of the sphere at the same time t; CT the radius of the shell to the tangent point T of shell and inclined plane at the same instant; φ = angle TCT_0; DE = the radius of the sphere to the tangent point of the inner surface of the shell and the sphere; θ = the angle DE makes with the vertical through D;

$$s = TT_0 = CC_0 = R\phi;$$

CH = a perpendicular from C upon C_0E_0 cutting the latter at H;

$$r - r_1 = r';$$

C_0 the coördinate origin; C_0C the x-axis; C_0T_0 the y-axis; x, y the coördinates of D; then $\angle C_0CH = \alpha$; $\angle C_0CD = \pi/2 - \theta + \alpha$; and then, F being the foot of the perpendicular from D upon CC_0,

$$x = s - CF = R\varphi - r'\cos DCF = R\varphi - r'\sin(\theta - \alpha);$$

$$y = DF = r'\sin DCF = r'\cos(\theta - \alpha).$$

The dynamic conditions for the motion of the sphere can be most clearly indicated by noticing that the initial point E_0 remains in contact with the inner surface of the shell, while the sphere has an angular velocity $\dot\theta$, $\dot\varphi$ being that of the shell.

Let M, m, be the masses of the shell and of the sphere; T, V, the kinetic energy, and potential energy; then the kinetic potential equation for the system is

$$T = \tfrac{1}{2}M(k'^2\dot\varphi^2 + \dot s^2) + \tfrac{1}{2}m(\dot x^2 + \dot y^2 + k''^2\dot\theta^2)$$

$$= Mgs\sin\alpha + mg(r'\cos\theta + s\sin\alpha) + C = V.$$

But

$$\dot x = R\dot\varphi - r'\cos(\theta - \alpha)\dot\theta,$$

$$\dot y = -r'\sin(\theta - \alpha)\dot\theta,$$

and

$$\dot s = R\dot\varphi,$$

whence, on substitution and reduction,

$$T = \tfrac{1}{2}\{M(k'^2 + R^2) + mR^2\}\dot\varphi^2 + \tfrac{1}{2}m\{-2Rr'\cos(\theta - \alpha)\dot\theta\dot\varphi + (r'^2 + k''^2)\dot\theta^2\}$$

$$= g\{(M + m)R\varphi\sin\alpha + mr'\cos\theta\} + C = V.$$

The Lagrangian equations

$$\frac{d}{dt}\frac{dT}{d\dot\varphi} - \frac{dT}{d\varphi} = \frac{dV}{d\varphi},$$

$$\frac{d}{dt}\frac{dT}{d\dot\theta} - \frac{dT}{d\dot\theta} = \frac{dV}{d\theta},$$

applied to the last result, give after simplifications,

$$\{(M+m)R^2+Mk'^2\}\ddot{\varphi}-mRr'\cos(\theta-\alpha)\ddot{\theta}+mRr'\sin(\theta-\alpha)\dot{\theta}^2=g(M+m)R\sin\alpha,$$

$$Rr'\cos(\theta-\alpha)\ddot{\varphi}-(r'^2+k''^2)\ddot{\theta}=gr'\sin\theta.$$

Eliminating $\ddot{\varphi}$,

$$[(r'^2+k''^2)\{(M+m)R^2+Mk'^2\}+mR^2r'^2\cos^2(\theta-\alpha)]\ddot{\theta}-mR^2r'^2\sin(\theta-\alpha)\cos(\theta-\alpha)\dot{\theta}^2$$
$$=-gr'[\{(M+m)R^2+Mk'^2\}\sin\theta+(M+m)R^2\sin\alpha\cos(\theta-\alpha)].$$

Multiplying by $2\dot{\theta}$ and integrating

$$[(r'^2+k''^2)\{(M+m)R^2+Mk'^2\}+mr'^2R^2\cos^2(\theta-\alpha)]\dot{\theta}^2$$
$$=gr'[2\{(M+m)R^2+Mk'^2\}\cos\theta-2(M+m)R^2\sin\alpha\sin(\theta-\alpha)]+C',$$

which is of the same general form as (7), p. 351, this MONTHLY for November, 1916.

NUMBER THEORY.

235. Proposed by W. D. CAIRNS, Oberlin College.

Prove that $n=1$ is the only positive integer for which n^4+4 is a prime.

SOLUTION BY WM. E. PATTEN, Government Institute of Technology, Shanghai, China.

$$n^4+4=(n^4+4n^2+4)-4n^2=(n^2+2)^2-(2n)^2=(n^2+2n+2)(n^2-2n+2).$$

Therefore, n^4+4 is a prime, if at all, only for those values of n which make either $n^2+2n+2=1$, or $n^2-2n+2=1$, since each of the factors of n^4+4 given above is integral in value when n is integral, and both are positive when n is positive.

(1) When $n^2+2n+2=1$, then $n=-1$.

(2) When $n^2-2n+2=1$, then $n=+1$. When $n=+1$, then $n^4+4=5$, a prime.

Therefore, n^4+4 is a prime for $n=1$, and for no other positive integral values of n.

Also solved by ELMER SCHUYLER, FRANK IRWIN, HORACE OLSON, ELIJAH SWIFT, H. H. CLARK, ELIZABETH B. DAVIS, NORMAN ANNING, L. G. WELD, and the PROPOSER.

236. Proposed by V. M. SPUNAR, Chicago, Illinois.

Find integral values of x, y, z, such that

$$xy+z=\square,\quad yz+x=\square,\quad\text{and}\quad xz+y=\square.$$

SOLUTION BY ARTEMAS MARTIN, LL.D., Washington, D. C.

Assume $x=n^2$, $y=(n+1)^2$; then the given equation becomes

$$n^2(n+1)^2+z=\square,\quad (n+1)^2z+n^2=\square,\quad n^2z+(n+1)^2=\square.$$

Put

$$n^2(n+1)^2+z=a^2.$$

Assume $a=n^2+n+b$, and the last equation becomes

$$n^2(n+1)^2+z=a^2=(n^2+n+b)^2,$$

from which we immediately find

$$z=b(2n^2+2n+b).$$

Substituting in

$$(n+1)^2z+n^2=\square,$$

we have

$$b(n+1)^2(2n^2+2n+b)+n^2=\square=c^2,$$

which is satisfied by $b = 2$; for then

$$c^2 = 4n^4 + 12n^3 + 17n^2 + 12n + 4 = (2n^2 + 3n + 2)^2,$$

and $z = 4(n^2 + n + 1)$. This value of z, with the assumed values, $x = n^2$, $y = (n + 1)^2$, satisfies all the proposed conditions.

$$xy + z = n^2(n + 1)^2 + 4(n^2 + n + 1) = (n^2 + n + 2)^2,$$
$$yz + x = 4(n + 1)^2(n^2 + n + 1) + n^2 = (2n^2 + 3n + 2)^2,$$
$$xz + y = 4n^2(n^2 + n + 1) + (n + 1)^2 = (2n^2 + n + 1)^2.$$

If $n = 1$, then $x = 1$, $y = 4$, $z = 12$.
If $n = 2$, then $x = 4$, $y = 9$, $z = 28$.
If $n = 3$, then $x = 9$, $y = 16$, $z = 52$.
And so on, indefinitely.

The values of x, y, z just found will also satisfy the conditions

$$xy + x + y = \square, \quad xz + x + z = \square, \quad \text{and} \quad yz + y + z = \square.$$

Also solved by ELIZABETH B. DAVIS and H. N. CARLETON.

237. Proposed by NORMAN ANNING, Chilliwack, B. C.

Prove that for three numbers x, y, z,

$$9\Sigma(x - y)^4 = \Sigma(2x - y - z) = 2\square.$$

SOLUTION BY E. F. CANADAY, University of South Dakota.

This problem is evidently misprinted. If we write it

$$9\Sigma(x - y)^4 = \Sigma(2x - y - z)^4 = 2\square,$$

a solution is possible. To prove

$$9[(x - y)^4 + (y - z)^4 + (z - x)^4] = (2x - y - z)^4 + (2y - z - x)^4 + (2z - x - y)^4 = 2\square,$$

we put

$$(x - y) = a, \quad (y - z) = b, \quad \text{and} \quad (z - x) = -(a + b).$$

Then

$$9[a^4 + b^4 + (-a - b)^4] = (2a + b)^4 + (b - a)^4 + (-a - 2b)^4 = 9(2a^4 + 4a^3b + 6a^2b^2 - 4ab^3 + 2b^4) = 16a^4 + 32a^3b + 24a^2b^2 + 8ab^3 + b^4 + b^4 - 4ab^3 + 6a^2b^2 - 4a^3b + a^4 + a^4 + 8a^3b + 24a^2b^2 + 32ab^3 + 16b^4 = 2[9(a^4 + 2a^3b + 3a^2b^2 + 2ab^3 + b^4)] = 18a^4 + 36a^3b + 54a^2b^2 + 36ab^3 + 18b^4 = 2\square.$$

$$2[3(a^2 + ab + b^2)]^2 = 2\square.$$

Also solved by the PROPOSER.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence, Kansas.

REPLIES.

20. Some of our readers would like to have a simple account, without proofs, of just what has been accomplished toward the proof of the theorem that the equation $x^n + y^n = z^n$ is impossible in integers when $n > 2$.

Readers interested in the above question will be glad to learn that a better and more complete article than that contemplated as an answer to the question

is soon to appear in the *Annals of Mathematics*. The paper is being prepared by Professor L. E. Dickson, of the University of Chicago, and will be a somewhat extended account of the more important results in proper historical setting.

Reference may also be made to A. Fleck's six-page article on Fermat's last theorem in Auerbach and Rothe's *Taschenbuch für Mathematiker und Physiker*, 3. Jahrgang, 1913 (Teubner), pp. 103–109; Benno Lind's forty-three-page article, "Über das letzte Fermatsche Theorem" in *Abhandlungen zur Geschichte der mathematischen Wissenschaften*, Heft XXVI_2, pp. 21–65, 1910; and Lipke's four-page review of Lind's article (*Bull. Amer. Math. Soc.*, Vol. XVIII, pp. 194–198) in which references are given to the criticisms of Lind's work published in *Archiv der Mathematik und Physik*.

33. Under what conditions or to what extent is Mr. Iwerson's construction a useful or practical approximation to a true ellipse? What criterion can be given to measure definitely the degree of approximation?

Mr. Iwerson's approximate construction for an ellipse by ruler and compasses alone, having given the axes, was given in the November, 1916, issue of the MONTHLY, pp. 354, 355. The following corrections should be made: In the figure, B' should be B. In the last two lines, Ox should be OY, and Oy should be OX.

REPLY BY PAUL CAPRON, U. S. Naval Academy, Annapolis, Md.

In the figure above referred to, let $XO = OX' = a$, $YO = OY' = b$. Then $X'A = a$, $X'B = (a^2 - b^2)/\sqrt{a^2 + b^2}$. Let $k = AB$ and $l = NN' = NR$. Then $k = a - (a^2 - b^2)/\sqrt{a^2 + b^2}$, and $l = 2a - k = a + (a^2 - b^2)/\sqrt{a^2 + b^2}$. Since $X'N'P'$ and $N'NR$ are equilateral triangles,

$$RO = (l - k)\frac{\sqrt{3}}{2} = \frac{a^2 - b^2}{\sqrt{a^2 + b^2}} \cdot \sqrt{3}, \qquad \text{and} \qquad l - RO = a - \frac{a^2 - b^2}{\sqrt{a^2 + b^2}}(\sqrt{3} - 1).$$

In order that the arc drawn with R as center and l as radius (through N and N') may pass through Y, $l - RO$ must be equal to b. This is the case (if $b < a$) when and only when $a = b\sqrt{3}$.

Let $b/a = x$ (if the eccentricity is e, $e^2 + x^2 = 1$). The proportional error in the length of the minor axis is

$$E_1 = \frac{1}{b}[l - RO - b] = \frac{1 - x}{x}\left[1 - \frac{1 + x}{\sqrt{1 + x^2}}(\sqrt{3} - 1)\right].$$

The proportional errors in the radii of curvature at the ends of the axes are: at the end of the major axis,

$$E_2 = \frac{a}{b^2}\left(k - \frac{b^2}{a}\right) = \frac{a^2 - b^2}{b^2}\left(1 - \frac{a}{\sqrt{a^2 + b^2}}\right) = \frac{1 - x^2}{x^2}\left(1 - \frac{1}{\sqrt{1 + x^2}}\right);$$

at the end of the minor axis,

$$E_3 = \frac{b}{a^2}\left(l - \frac{a^2}{b}\right) = \frac{a-b}{a^2}\left[\frac{b(a+b)}{\sqrt{a^2+b^2}} - a\right] = (1-x)\left[\frac{x(1+x)}{\sqrt{1+x^2}} - 1\right].$$

$$\frac{dE_1}{dx} = \left[(\sqrt{3}-1)\frac{1+3x^2}{(1+x^2)^{3/2}} - 1\right], \qquad \frac{dE_2}{dx} = \frac{1}{x^3}\left[\frac{2+3x^2-x^4}{(1+x^2)^{3/2}} - 2\right],$$

$$\frac{dE_3}{dx} = \frac{1-3x^2-2x^4}{(1+x^2)^{3/2}} + 1.$$

Of the six variables, E_2 and dE_2/dx vanish for no real value of x between 0 and 1; the values, aside from 0 and 1, which cause the variables to vanish, are given by the following equations:

$$E_1 = 0; \qquad (x - \tfrac{1}{3}\sqrt{3})(x - \sqrt{3}) = 0,$$

$$\frac{dE_1}{dx} = 0; \qquad x^6 + 3(6\sqrt{3} - 11)x^4 + 3(4\sqrt{3} - 7)x^2 + (2\sqrt{3} - 3) = 0,$$

$$\frac{dE_2}{dx} = 0; \qquad x^4 - 10x^2 - 7 = 0,$$

$$E_3 = 0; \qquad x^4 + 2x^3 - 1 = 0,$$

$$\frac{dE_3}{dx} = 0; \qquad 4x^6 + 11x^4 + 2x^2 - 9 = 0.$$

It is said that architects find it troublesome to draw a shapely oval opening; the ellipse is the most satisfactory curve in itself, but there is much labor involved in making offsets for several parallel curves in the design of the mouldings. With Mr. Iwerson's approximation, it would merely be necessary, with four fixed centers, to use appropriately lengthened radii. The most serious defect in the shapeliness of the broken line is the abrupt discontinuity in the curvature at the four points where the radius of curvature is altered in the ratio k/l.

$$\frac{k}{l} = \frac{[\sqrt{1+x^2} - (1-x^2)]^2}{x^2(3-x^2)}.$$

(If $x < \frac{1}{2}$, $\dfrac{k}{l} = x^2\left(\dfrac{3}{4} + \dfrac{x^2}{12} + \dfrac{107}{400}x^4 - \dfrac{101}{200}x^6\right)$, nearly enough to give three decimal places.)

When $x = \dfrac{1}{\sqrt{3}}$, $(E_1 = 0)$, $\dfrac{k}{l} = 0.268$; when $x = \frac{3}{4}$, $\dfrac{k}{l} = \dfrac{13}{27}$; when $x = \frac{1}{2}$, $\dfrac{k}{l} = 0.197$.

For consecutive values of x, at intervals of 0.1:

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
k/l	0.000	0.008	0.030	0.068	0.123	0.197	0.291	0.412	0.559	0.752	1.000

In order to estimate this error on somewhat the same basis as the errors E_1, E_2, E_3, the second of the following tables includes, under the caption $\pm E_4$, the value of

$$\frac{l - \frac{1}{2}(l + k)}{\frac{1}{2}(l + k)} = -\frac{k - \frac{1}{2}(l + k)}{\frac{1}{2}(l + k)} = \frac{1 - k/l}{1 + k/l}.$$

The first of the following tables shows certain critical values of E_1, E_2, E_3 and their derivatives; the second shows consecutive values of E_1, E_2, E_3, $\pm E_4$ and the eccentricity.

e	x	E_1	$\frac{dE_1}{dx}$	E_2	$\frac{dE_2}{dx}$	E_3	$\frac{dE_3}{dx}$
1.000	0.000	$+\infty$	$-\infty$	0.500	0.000	−1.000	2.000
0.817	0.577	0.000	−0.147	0.268	−0.642	−0.089	0.380
0.696	0.717					0.000	0.099
0.693	0.721	−0.008	0.000				
0.515	0.857					0.030	0.000
0.000	1.000	0.000	+0.035	0.000	−0.586	0.000	−0.414

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
E_1	$+\infty$	1.789	0.555	0.207	0.073	0.018	−0.004	−0.008	−0.007	−0.004	0.000
E_2	0.500	0.491	0.466	0.426	0.375	0.317	0.254	0.188	0.123	0.060	0.000
E_3	−1.000	−0.801	−0.518	−0.439	−0.243	−0.165	−0.071	−0.008	+0.025	0.027	0.000
$\pm E_4$	1.000	0.985	0.942	0.872	0.781	0.671	0.549	0.418	0.283	0.141	0.000
e	1.000	0.995	0.980	0.954	0.917	0.866	0.800	0.714	0.600	0.484	0.000

The constructions made from these data show that Mr. Iwerson's approximation is very close for a medium eccentricity, e. g. for $e = 0.92$, $x = 0.4$.

It is possible to calculate in advance a length to be used instead of b for OY, so that the circle with center R shall pass through the end of the minor axis. If $b/a = x$, $\frac{1}{2}(1 - x)(\sqrt{3} - 1) = p$, $OY = ya$, where

$$y^4 - (2 + p^2)y^2 + (1 - p^2) = 0;$$

or $Oy = a \tan \phi$, where $\cos 2\phi \cdot \sec \phi = p$.

This might be worth while in case e is large, E_1 objectionable and E_2 and E_3 of no consequence.

As to what errors are tolerable and what objectionable, that is of course entirely a matter of circumstances.

Discussions.

I. Relating to the Order of Operations in Algebra.

By N. J. Lennes, University of Montana.

§ 1. *The Rules as Given in the Books.*

Subtraction and division are defined as the inverse operations of addition and multiplication. The commutative and associative laws of addition and multiplication are, therefore, extended in the same manner to both subtraction and division. In the case of addition alone or of multiplication alone it is agreed that, when no symbols of aggregation occur, the operations are to be performed from left to right. Thus, $a + b + c$ means $(a + b) + c$, and $a \times b \times c = (a \times b) \times c$. Without such an understanding the associative laws would have no meaning.

The Commutative and Associative Laws. In case symbols of addition and subtraction both occur (and no other symbols), it is agreed that each symbol applies only to the term immediately following it, and that the operations are to be performed from left to right.

Thus, $8 - 2 + 4 = (8 - 2) + 4 = 10$, and *not* $8 - (2 + 4) = 2$. From this usage it follows that terms connected by $+$ and $-$ signs may be *commuted,* but they may not be *associated,* except when a $+$ sign precedes the group in question. Thus, $8 + 4 - 2 = 8 + (4 - 2)$, but $8 - 4 - 2$ is not equal to $8 - (4 - 2)$.

In case the signs of multiplication and division occur with no signs of addition and subtraction intervening, and in case no symbols of aggregation are used, then it is likewise agreed (in the theoretical development in the books) that each symbol applies only to the factor (or divisor) immediately following it, and that the operations are to be performed in order from left to right.

Thus, $8 \div 2 \times 4 = (8 \div 2) \times 4 = 16$, and *not* $8 \div (2 \times 4) = 1$. As in the case of addition and subtraction, it results from this agreement that the commutative law applies to the operations of multiplication and division, while the associative law does not apply, except when the sign $\times$ precedes the group in question.

That is, $8 \div 4 \times 2 = 8 \times 2 \div 4$, but $8 \div 4 \times 2$ is not equal to $8 \div (4 \times 2)$. As remarked by Chrystal, under these conventions, the associative and commutative laws for addition and subtraction are *formally identical* with these laws for multiplication and division. (*Text Book of Algebra,* Part I, page 17.)

Following this theoretical development most of the current text-books give a rule like the following:

A series of operations involving multiplication and division alone shall be performed in the order in which they occur from left to right.

§ 2. *Actual Usage.*

The Above Rule Contrary to Actual Usage. The rule stated above is agreed to by practically all those writers on algebra who make any mention of the matter at all. Chrystal gives a detailed development and writers on elementary algebra have in general followed him. It would, however, follow from this rule for carrying out multiplications and divisions in order from left to right, that

$$9a^2 \div 3a = (9a^2 \div 3) \times a = 3a^3.$$

But I have not been able to find a single instance where this is so interpreted. The fact is that the rule requiring the operations of multiplication and division to be carried out from left to right *in all cases,* is not followed by *anyone.* For example, in case an indicated product follows the sign $\div$ the whole product is *always* used as divisor, except in the theoretical statement of the case.

Writers meet the situation in different ways:

(*a*) Some always use the fractional form to indicate division, this being equivalent to a symbol of aggregation. Thus, $ab/cd = (ab) \div (cd)$.

(*b*) Some write out the words in full, thus: "divide this expression by that expression."

(*c*) Some use the sign $\div$ to mean that the whole product, following the sign $\div$, shall be the divisor.

The most important exception to (*c*) occurs in the development of the theory in such a text as Chrystal. In Chrystal, $\div\, u \times v$ is sometimes used to mean $(\div\, u)v$. In such cases, however, the notation $\div\, u \times v$ and not $\div\, uv$ is used.

Chrystal in one case writes $2^2 \div 3^2 \times 5^2 = (2^2 \div 3^2) \times 5^2$. (Note the sign $\times$ to indicate multiplication.) He also writes $pa/pb = pa \div (pb)$. (Note the parenthesis.) This comes nearer consistency than is usually the case. However, in no case does Chrystal write $9a^2 \div 3a$ as the equivalent of $(9a^2 \div 3) \times a$. He overcomes the difficulty by *never using the sign* $\div$ *with a product after it.*

The followers of Chrystal have too often blindly copied his theory, but have not taken pains, as he did, to avoid inconsistency in usage. Examples of such inconsistency in theory and usage could be multiplied *ad infinitum.* One text, which has been in very wide use, states (in developing the theory)

$$60 - 40 \div 5 \times 3 - 20 = 60 - \frac{40}{5} \times 3 - 20,$$

but on the next page we read:

$$10bc \div 12a = \frac{10bc}{12a}.$$

The Established Usage. When an indicated product follows the sign $\div$ the whole product is, by overwhelming preponderance of actual usage, to be regarded as the divisor. Hence, the true rule as to the order of operations when both

multiplications and divisions are involved is not the one stated above, but the following:

All multiplications are to be performed first and the divisions next.

That is, $9a^2 \div 3a = 3a$ and not $3a^3$.

The multiplications may be taken in any order, but the divisions are to be taken in the order in which they occur from left to right.

That is, the associative law holds for the former but not for the latter.

Thus, $3 \times 5 \times 2 = (3 \times 5) \times 2$ or $= 3 \times (5 \times 2)$; but, $16 \div 4 \div 2 = (16 \div 4) \div 2$ and does not $= 16 \div (4 \div 2)$.

Compare the corresponding rules for addition and subtraction in § 1.

Mathematical Idioms. It might be agreed that, for the sake of simplicity and logical coherence, the past tense of the verb *to drink* should be *drinked,* but even so, English speaking people would continue to say *drank,* and not drinked. Precisely, for the same reason, all who know anything about the language of algebra regard $9a^2 \div 3a$ as equal to $3a$ and not $3a^3$, and, therefore, the rule just given is the correct one as determined by actual usage. When a mode of expression has become wide-spread, one may not change it at will. It is the business of the lexicographer and grammarian to record, not what he may think an expression should mean (no matter how far-fetched the usual or idiomatic usage may seem), but what it is *actually understood to mean by those who use it.* The language of algebra contains certain idioms and in formulating the grammar of this language we must note them. For example, that $9a^2 \div 3a$ is understood to mean $3a$ and not $3a^3$ is such an idiom. The matter is not logical but historical.

II. RELATING TO AN EXTENSION OF WILSON'S THEOREM.

By Elizabeth Brown Davis, U. S. Naval Observatory.

From Wilson's theorem we have the congruence,

$$(p-1)!+1 \equiv 0, \quad (\text{mod. } p),$$

which may be written,

$$(p-1)(p-2)!+1 \equiv 0, \quad (\text{mod. } p). \tag{1}$$

Subtracting (1) from $p(p-2)! \equiv 0$, (mod. p), we have

$$(p-2)!-1 \equiv 0, \quad (\text{mod. } p).$$

This may be written

$$(p-2)(p-3)!-1 \equiv 0, \quad (\text{mod. } p). \tag{2}$$

Subtracting (2) from $p(p-3)! \equiv 0$, (mod. p), we have

$$2(p-3)!+1 \equiv 0, \quad (\text{mod. } p),$$

or

(3) $$2(p-3)(p-4)!+1\equiv 0, \quad (\text{mod. } p).$$

Subtracting (3) from $2p\,(p-4)!\equiv 0, \quad (\text{mod. } p)$, we have

$$3!(p-4)!-1\equiv 0, \quad (\text{mod. } p).$$

Proceeding in like manner, we obtain successively,

$$4!(p-5)!+1\equiv 0, \quad (\text{mod. } p),$$

$$5!(p-6)!-1\equiv 0, \quad (\text{mod. } p),$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

$$\left(\frac{p-1}{2}\right)!\left(\frac{p-1}{2}\right)!\pm 1\equiv 0 \quad (\text{mod. } p),$$

the constant term being $+1$ when $(p-1)/2$ is even and -1 when $(p-1)/2$ is odd. Hence the theorem:

If p is prime, and a is any integer less than $p-1$, then

$$a!(p-1-a)!+(-1)^a\equiv 0, \quad (\text{mod. } p).$$

Wilson's theorem is the special case $a=0$, of which the above is the more general form, it being understood that $0!=1!/1=1$, and that

$$(-1)^0=(-1)^1/(-1)=+1.$$

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Indiana.

At the College of the City of New York, Dr. P. H. LINEHAN has been promoted to an assistant professorship of mathematics.

Dr. CORA B. HENNEL has been promoted from an instructorship to an assistant professorship of mathematics at Indiana University.

Dr. DANIEL BUCHANAN has been made professor of mathematics and astronomy at Queen's University, Kingston, Ontario.

Professor H. L. RIETZ, of the University of Illinois, has been appointed a member of the joint committee of the American Association of University Professors with the trustees of the Carnegie Foundation for the Advancement of Teaching to report upon the proposed changes in the scope of the foundation.

"The relation of mathematics to the natural sciences" is the subject of a six-page discussion in *Science*, December 15, 1916, by Professor T. E. MASON, of Purdue University.

Professor C. SMITH, head master of Sidney Sussex College, Cambridge, died on November 13, 1916, at the age of seventy-two years. Professor SMITH was known to a large number of teachers of mathematics in America through his very excellent series of college texts on algebra, conic sections, and solid analytic geometry.

In *School and Society*, December 16, 1916, Professor G. A. MILLER, of the University of Illinois, has an interesting paper on "History and use of mathematical text-books." After an introductory discussion of the general features of good and bad text-books, Dr. MILLER devotes some space to the use made of text-books, incidentally touching different methods of teaching. He also emphasizes the importance of accuracy and clearness, and the value of historical notes in mathematical text-books.

The thirty-eighth meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on December 22 and 23, 1916, at which eleven research papers were presented by representatives of the following universities: Chicago, Illinois, Minnesota, Nebraska, Purdue and Wisconsin, and Rose Polytechnic Institute. The members present enjoyed an informal dinner together at the Quadrangle Club on Friday evening. Among topics informally discussed were (1) The contract between the Mathematical Association of America and the *Annals of Mathematics* for the enlargement of that journal and the publication in it of expository and historical articles; (2) the need in this country of careful consideration of the whole question of the history of mathematics and of combined and systematic effort in developing investigation in this line; and (3) the desirability of holding ourselves in readiness to assist the publishers of the *Revue Semestrielle* and the *Fortschritte* in case it becomes necessary, on account of the war conditions, in order to continue the publications.

The twenty-third annual meeting of the American Mathematical Society was held at Columbia University, New York, on December 27–28, 1916. Thirty-three papers were presented by representatives from twelve educational institutions including two from Canada. The session of Thursday afternoon was a joint meeting of the American Mathematical Society, the Mathematical Association of America, Section A (Mathematics and Astronomy) of the American Association, and the American Astronomical Society. At this meeting Professor E. W. BROWN, of Yale University, President of the American Mathematical Society, delivered his retiring address on "The relation of mathematics to the natural sciences," and the vice-presidential address, Section A of the American Association, on "Derivation of orbits—theory and practice," which was to have been delivered by Professor A. L. LEUSCHNER, of the University of California, was read by Professor M. W. HASKELL, of the University of California.

At the holiday meeting of the American Association for the Advancement of Science held in New York City, great interest was manifested in the programs of Section D, Engineering. On Wednesday forenoon a joint session of Section D and Section I was devoted to discussion of the social and economic question of "The advisability of adopting the metric standard of weights and measures in the United States." The program of Section D for Thursday morning was given to the presentation of papers on "Sanitary engineering"; in the afternoon of the same day a program of nine papers was provided on the general subject of "Highway engineering education"; and on the evening of the same day a joint program was devoted to the subject of "Highway engineering." On Friday the general program of Section D was continued, and in the evening the program of the section was concluded by a joint session of Section D with the American Society of Civil Engineers, the American Society of Mechanical Engineers, the American Institute of Mining Engineers, and the American Institute of Electrical Engineers, at which the topic for discussion was "The inter-relationship of engineering and pure science." In the program of Friday afternoon, Professor D. J. McAdam, of Washington and Jefferson College, presented a paper on "Mathematical education for civil engineers."

In the report of the meeting of the Kansas Section in the January issue, the Secretary of the Section inadvertently gave the wrong name as new Chairman. It should have been Professor B. L. Remick of the Kansas Agricultural College. (State.)

The editor of "Notes and News" for the Monthly would greatly appreciate the coöperation of members of the Mathematical Association in supplying him with items suitable for insertion in these columns. Kindly send us notices of promotions, resignations, deaths, departmental activities, mathematical clubs, local programs, summer session schedules, and notes on important mathematical contributions of general interest. Please send all such notices to D. A. Rothrock, Bloomington, Indiana.

Notes on the Annual Meeting of the Association.

The truly national character of the Mathematical Association of America is strongly shown in the List of Officers and Charter Members just published and distributed to all subscribers to the Monthly. The geographical distribution of members shows clearly the wisdom of holding the first two national meetings of the Association in the East, and the attendance upon these meetings, in Cambridge and in New York, has fully justified the selection of these meeting places. A second look at the directory will show equally clearly the desirability of holding the next national meetings in the Middle West. The Council in New York made provision toward this end by appointing a committee to act in connection with the American Mathematical Society with respect to holding a joint summer meeting in 1917 somewhere west of New York, and by voting to hold the next

annual meeting in December, 1917 in Chicago. These meetings should draw large numbers from all parts of the country and especially from the Middle West and South. The decision has since been made to hold the summer meeting in Cleveland.

The Association was exceedingly fortunate in the accommodations provided for its meetings at the Massachusetts Institute of Technology and at Columbia University. Hamilton Hall, which is the headquarters of the department of mathematics at Columbia University, proved admirably adapted to Association purposes, both for the larger gatherings and for committees and council meetings. As a side attraction, nothing could have been more interesting than the collection of portraits and medals of mathematicians belonging to Professor David Eugene Smith and put on exhibition by him between the sessions on Friday. Special credit is due to the committee on program and arrangements for the orderly and smooth progress of all matters pertaining to the meetings.

One matter in connection with meetings of the Association, whether national or sectional, deserves careful consideration, namely, the opening of all sessions sharply at the time announced. At Cambridge this was done rather successfully; in New York the result was not so successful, owing partly to uncontrollable circumstances and partly to the distractions arising from the many interesting meetings in progress on the campus. It might be a worthy New Year's resolution for the Association to establish a definite policy of starting all its meetings, including the meetings of sections, exactly on time, and of holding definitely to the program schedule in all respects.

The interest in establishing sections still grows apace. The Minnesota Section, reported in this issue, was duly organized and held its first meeting early in December. The Maryland Section has just been organized and was admitted by the council at the New York meeting. Hearty coöperation and much enthusiasm were reported in connection with this section. The Committee on Sections has also authorized the establishment of a section in Kentucky. Preliminary steps were taken at a small gathering in Chicago on December 23 toward the organization of a section in Illinois and more definite plans are soon to be put into action. Other sections are under consideration in various parts of the country. The Council Committee on Sections is likely to become one of the most important in the near future, not only in respect to the organization of new sections but also in coördinating the work of the sections, in stimulating their activities, and in preparing and distributing types of programs and discussions, and possibly in providing representative speakers to visit the section meetings, to give assistance in local plans, and to stir up activity and enthusiasm.

The plan of coöperation with the Annals of Mathematics, which had already been unanimously approved by a mail vote of the Council, was still further perfected in joint conferences between the committee of the Association and the editors of the Annals. It was agreed that the plan should go into operation with the final number of the present volume of the Annals, which is to be issued early in May and is to contain an expository paper by Professor L. E. Dickson,

of the University of Chicago, on Fermat's last theorem and other matters of historical interest in connection with algebraic number theory. A communication setting forth the details of this plan will be issued early in March by the editors of the Annals and the committee of the Association. Only warm commendation of this plan was heard on every hand at the New York meeting. An informal referendum showed that between sixty and seventy persons were ready to subscribe to the Annals under the half-rate offer to be extended to members of the Association. This communication will be sent to all members of the Society as well as to the members of the Association.

About fifty scientific societies are affiliated with the American Association for the Advancement of Science. Once in four years at the annual meeting of the American Association there is held a convocation of these affiliated societies, such as took place in New York City during the last week of December, 1916. As an inducement to members of the affiliated societies to join the American Association, the entrance fee of five dollars was remitted by a special order during the year 1916. This privilege has now been further extended by a special resolution at the New York meeting, to include all new members of any affiliated society who may be elected to membership in the American Association within one year of their admission to the affiliated society. This privilege would then be applicable to the charter members of the Mathematical Association of America up to April first, 1917, since the charter membership list closed on April first, 1916. The dues in the American Association are three dollars per year and all members receive the weekly journal *Science*.

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VOLUME XXIV MARCH, 1917 NUMBER 3

DIRECTED ANGLES IN ELEMENTARY GEOMETRY.

BY R. A. JOHNSON, Western Reserve University.

In many of the theorems of the modern elementary geometry, the treatment of angles seems somewhat unsatisfactory. In the statement of theorems, one is often confronted with the dilemma of a choice between an inaccurate statement, and one so verbose and involved as to be unwieldy. Again, many proofs, as given in the texts,[1] are insufficient because they apply only to particular positions of the figure. A very common example is the following. If A, B, C, D are four points on a circle, the angles ABC, ADC are equal or supplementary, according as B and D are on the same side of AC, or on opposite sides. This theorem is repeatedly used in proofs; but in a given case, when we know only that the points are concyclic, and have no data as to their order on the circle, how are we to decide which of the two possibilities is the correct one? Apparently the usual custom is to draw a single figure, and decide by inspection of the figure, trusting that the proof so obtained can be modified to fit all possible figures. Not only is such a method entirely unscientific, but in cases where the figure is at all complicated, the determination of the number of possibilities and the corresponding modifications of the proof are practically impossible.

As a simple illustration, let us consider Simson's theorem, so-called. "If from any point P on the circumcircle of the triangle ABC, PX, PY, PZ be drawn perpendicular to the sides, the points X, Y, Z will be collinear." (This statement, and the following proof, are taken from Lachlan, *l. c.*, § 120.)

"Join ZX, YX. Then since the points P, X, Z, B are concyclic, the angle PXZ is the supplement of the angle ABP. And since P, Y, C, X are concyclic, the angle YXP is the supplement of the angle YCP, and is equal to the angle ABP, because P, C, A, B are concyclic."

[1] Such books as Lachlan, *Modern Pure Geometry;* Casey, *A Sequel to Euclid;* McClelland, *Geometry of the Circle;* etc., are here referred to.

"Hence the angles PXZ, YXP are supplementary, and therefore ZX, XY are in the same straight line."

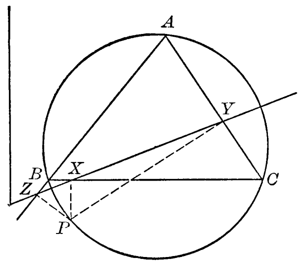


FIG. 1.

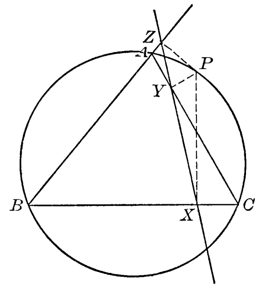


FIG. 2.

Now in Fig. 1, which resembles the one in the text, each of these assertions is true; but in Fig. 2, which illustrates equally well the theorem as stated, most of them are untrue; angles stated to be supplementary are equal in three different cases. Hence the proof fails to be adequate for all cases of the theorem. In fact, it will doubtless require some thought on the part of the reader to decide how many different cases Lachlan ought to consider; and this is an unusually simple example.

The scheme developed below has been devised to meet these difficulties; to make exact statements and general and rigorous proofs possible. So far as the writer is aware, such a method has not been used in elementary work, though the underlying ideas are obviously familiar ones in analytic geometry.

Definitions. Let us first agree to distinguish between a positive and a negative direction of rotation. Following the usual custom, we shall regard an angle as positive when it is generated by rotation in anti-clockwise direction, and negative in the other direction.

The symbol $\angle\, BAC$ shall mean the angle between the half-lines AB, AC, a signless quantity, just as in elementary geometry, so that

$$\angle\, BAC = \angle\, CAB.$$

If l, l' are any two lines, the symbol $\measuredangle\, l, l'$ shall mean *that angle through which l must be rotated in a positive direction in order that it may come to coincide with l'.* Similarly the symbol $\measuredangle\, BAC$ means *the positive angle through which the line AB, taken as a whole, must be rotated, to coincide with the line AC taken as a whole;* and this without regard to the position of B and C on these lines.

For example, if A, B, C are any points on a line, and D a point not on that line, we have, whatever the order of A, B, C,

$$\measuredangle\, ABD = \measuredangle\, CBD.$$

The quantity thus defined and denoted by $\measuredangle\, ABC$ is called *the directed angle from AB to BC.*

Further be it noted as an immediate consequence of the definition that two

directed angles are regarded as equivalent if they differ only by multiples of a straight angle.

The *addition* of directed angles is defined by the following laws, seen to be consistent with the definition. $\measuredangle\, l_1, l_2 + \measuredangle\, l_2, l_3 = \measuredangle\, l_1, l_3$; $\measuredangle\, l_1, l_2 + \measuredangle\, l_3, l_4 = \measuredangle\, l_1, l_5$, where l_5 is a line so located that $\measuredangle\, l_2, l_5 = \measuredangle\, l_3, l_4$.

From these definitions we have the following relations as bases of operations with directed angles.

THEOREM I. $\measuredangle\, l, l' + \measuredangle\, l', l = 180°$.

THEOREM II. *If l_1 is parallel to l_1', and l_2 to l_2', then $\measuredangle\, l_1, l_2 = \measuredangle\, l_1', l_2'$.* Again, *if l_1 is* perpendicular *to l_1', and l_2 to l_2', then $\measuredangle\, l_1, l_2 = \measuredangle\, l_1', l_2'$.*

THEOREM III. *For any four lines $\measuredangle\, l_1, l_2 + \measuredangle\, l_3, l_4 = \measuredangle\, l_1, l_4 + \measuredangle\, l_3, l_2$.* For, $\measuredangle\, l_1, l_2 = \measuredangle\, l_1, l_4 + \measuredangle\, l_4, l_2$, and $\measuredangle\, l_3, l_4 = \measuredangle\, l_3, l_2 + \measuredangle\, l_2, l_4$.

THEOREM IV. *A necessary and sufficient condition that three points A, B, C lie on a line is that for any other point D we have*

$$\measuredangle\, ABD = \measuredangle\, CBD.$$

For, if AB and CB are equally inclined to BD, they coincide, and conversely.

THEOREM V. *The necessary and sufficient condition that four points A, B, C, D lie on a circle is that $\measuredangle\, ABD = \measuredangle\, ACD$.*

For this equation means that (*a*) if B and C are on the same side of AD, then $\angle\, ABD$ and $\angle\, ACD$ are equal; and (*b*) if B and C are on opposite sides of AD, then $\angle\, ABD$ is equal to the supplement of $\angle\, ACD$. Hence the present theorem follows from the theorem quoted in the first paragraph above.

It would be hard to overestimate the importance of this last theorem. Let us illustrate by proving Simson's theorem, using the same notation as previously, but any figure which may be drawn.

Proof. Since PXB, PZB are right angles, P, B, X, Z lie on a circle (in what order we do not know nor care).

Hence, $\measuredangle\, PXZ = \measuredangle\, PBZ$.

Similarly, P, X, Y, C are concyclic, and $\measuredangle\, PXY = \measuredangle\, PCY$. But $\measuredangle\, PBX$ is identically the same as $\measuredangle\, PBA$, and $\measuredangle\, PCY$ the same as $\measuredangle\, PCA$. Hence $\measuredangle\, PXZ = \measuredangle\, PBA$, and $\measuredangle\, PZY = \measuredangle\, PCA$. But since A, B, C are concyclic, $\measuredangle\, PBA = \measuredangle\, PCA$, and $\measuredangle\, PXZ = \measuredangle\, PXY$, which, by theorem IV, shows that X, Y, Z are collinear.

It is obvious that this is an entirely general proof. It is a little more verbose than need be, in order to bring out the method clearly. We now apply similar methods to a few more well-known theorems.

THEOREM. *If a point is marked on each side of a triangle (or its extension), and the circles drawn, each of which passes through a vertex of the triangle and the points marked on the adjacent sides, these circles pass through a point, and the lines from this point to the marked points make equal angles with the sides.*

Let the triangle be $A_1A_2A_3$ (Fig. 3), let P_1, P_2, P_3 be marked on A_2A_3, A_3A_1, A_1A_2 respectively. Let circles $A_1P_2P_3$, $A_2P_3P_1$ be drawn, and meet at P.

Then

$$\measuredangle\, PP_2, PP_3 = \measuredangle\, P_2A_1P_3 = \measuredangle\, A_3A_1A_2,$$

$$\measuredangle\, PP_3, PP_1 = \measuredangle\, P_3A_2P_1 = \measuredangle\, A_1A_2A_3.$$

Adding,

$$\begin{aligned}\measuredangle\, PP_2, PP_1 &= \measuredangle\, A_3A_1, A_1A_2 + \measuredangle\, A_1A_2, A_2A_3 \\ &= \measuredangle\, A_3A_1, A_2A_3 = \measuredangle A_1A_3A_2 = \measuredangle\, P_2A_3P_1.\end{aligned}$$

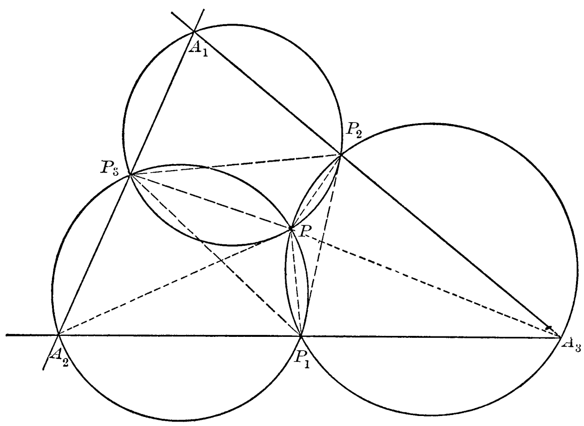


FIG. 3.

Whence, by theorem V, P, P_1, P_2, A_3 are concyclic, and the theorem is proved. Incidentally we see that $\measuredangle\, PP_1, A_2A_3 = \measuredangle\, PP_2, A_3A_1 = \measuredangle\, PP_3, A_1A_2$.

THEOREM. *In the same figure,* $\measuredangle\, A_2PA_3 = \measuredangle\, P_2P_1P_3 + \measuredangle\, A_2A_1A_3$.

The proof consists of splitting $\measuredangle\, A_2PA_3$ into two parts,

$$\measuredangle\, A_2PA_3 = \measuredangle\, A_2PP_1 + \measuredangle\, P_1PA_3.$$

Now

$$\measuredangle\, A_2PP_1 = \measuredangle\, A_2P_3P_1 = \measuredangle\, A_2A_1P_1 + \measuredangle\, A_1P_1P_3,$$

and

$$\measuredangle\, P_1PA_3 = \measuredangle\, P_1P_2A_3 = \measuredangle\, P_2P_1A_1 + \measuredangle\, P_1A_1A_3,$$

and we get the desired result by adding. Of course there are similar expressions for $\measuredangle\, A_3PA_1$ and $\measuredangle\, A_1PA_2$. To see the difficulties encountered by attacking this figure without due care, the reader should note McClelland, pages 40–41.

The above is called the theorem of Miquel.[1] The point is called the Miquel point for the set of points P_1, P_2, P_3.

COROLLARIES. (1) If P is a fixed point, it is the Miquel point of infinitely many triangles inscribed in $A_1A_2A_3$. These triangles are all directly similar, with P as center of similitude.

(2) If P lies on the circumcircle of $A_1A_2A_3$, then P_1, P_2, P_3 are collinear, and conversely (Simson's theorem).

[1] Cf. J. L. Coolidge, *Geometry of the Circle*, 1916, p. 85.

For $\measuredangle P_2P_1P_3 = 0$ if and only if $\measuredangle A_2PA_3 = \measuredangle A_2A_1A_3$.

(3) Among the triangles having a given point P for Miquel point is the pedal triangle of P, *i. e.*, the triangle whose vertices are the feet of the perpendiculars from P to the sides of $A_1A_2A_3$. The angles of the pedal triangle of any point are therefore given by the formulas $\measuredangle P_3P_1P_2 = \measuredangle A_2A_1A_3 + \measuredangle A_3PA_2$, etc.

(4) Conversely, if three circles are concurrent at a point, it is possible in an infinite number of ways to draw a triangle having one vertex on each circle and one side passing through each of the intersections of the circles two by two. All such triangles are similar.

Other corollaries suggest themselves readily.

We close with another fundamental theorem, of much less importance, and an application of it to a rather difficult theorem of Steiner.

THEOREM VI. *If O is the center of the circle through A, B, C, then*

$$\measuredangle OAB = \measuredangle ACB + 90°.$$

For, let AO meet the circle again at D. By the rule for adding angles,

$$\measuredangle OAB = \measuredangle ADB + \measuredangle DBA = \measuredangle ACB + 90°.$$

Now corollary 2 above may be re-stated in the following familiar form:

THEOREM. *The circumcircles of the four triangles of a complete quadrilateral meet in a point.*

For if $P_1P_2P_3$ is a transversal of triangle $A_1A_2A_3$, we have seen that the four circles $A_1A_2A_3$, $A_1P_2P_3$, $A_2P_3P_1$, $A_3P_1P_2$ are concurrent. And obviously any complete quadrilateral may be regarded as a triangle and a transversal.

THEOREM (Steiner). *The centers of the four circumcircles lie on a circle which also passes through this point.*

Let P be the intersection of the four circles named above, and let their centers, in the order named, be O, C_1, C_2, C_3. Then $C_1O \perp A_1P$, $C_3O \perp A_3P$, and hence

$$\measuredangle C_1OC_3 = \measuredangle A_1PA_3 = \measuredangle A_1A_2A_3.$$

Similarly for C_2, and we see that the four centers are concyclic. To show that this circle passes through P is not so simple. The triangles C_1PC_3 and $AC_1P_2C_3$ lie symmetrically with regard to C_1C_3; hence

$$\measuredangle C_1PC_3 = \measuredangle C_3P_2C_1 = \measuredangle C_3P_2P_1 + \measuredangle P_3P_2C_1. \quad \text{(addition formula)}$$

But

$$\measuredangle C_3P_2P_1 = \measuredangle P_2A_3P_1 + 90°$$

and

$$\measuredangle P_3P_2C_1 = \measuredangle PA_1P_2 + 90°. \qquad \text{(theorem VI)}$$

Hence

$$\measuredangle C_1PC_3 = \measuredangle P_2A_3P_1 + \measuredangle PA_1P_2 = \measuredangle P_3A_1, A_3P_1 = \measuredangle A_1A_2A_3 = \measuredangle C_1OC_3$$

and the proof is completed.

MATHEMATICS IN THE NEW INTERNATIONAL ENCYCLOPEDIA.[1]

By G. A. MILLER, University of Illinois.

The *New International Encyclopedia* has won an eminent position among the American works of reference and is found in a large number of public and private libraries. A second edition has recently appeared in twenty-four large volumes, and the eminence of the men on its editorial staff tends to inspire confidence with respect to accuracy and wise selection of material. The present note does not aim to shake this confidence as regards the useful articles relating to mathematics. On the contrary, it aims to increase the interest in these articles by directing attention to a few desirable corrections and especially to the need of a critical study of some of the statements before accepting them as final authority.

The critical mathematical reader can easily convince himself that certain corrections would greatly improve this standard work. By consulting the article under the word *equation* he will find near the end thereof several surprising statements to the effect that the expressions

$$\sqrt{p^2 - 4q}, \qquad \sqrt{g^2 + 4h^3}$$

are called the discriminants of the two equations

$$x^2 + px + q = 0, \qquad x^3 + 3hx + g = 0$$

respectively. While it is well known that the expression "discriminant of an equation in one unknown" has not always been defined in the same manner by recent noteworthy authors, yet all of these authors seem to agree on the point that such a discriminant is a rational function of the coefficients.[2] The fact that the radical signs noted above are found also in volumes dated 1904 and 1912 respectively of the encyclopedia under consideration makes it more difficult to attribute their appearance in the latest edition to an oversight.

With this clear instance where corrections are needed fresh in mind one may be inclined to approach the rest of this article on the equation in a critical spirit and to observe a considerable number of statements which one would like to change. In fact, the sentence beginning in the eleventh line of this article seems to exhibit too narrow a spirit for a big work. It reads as follows: "The expression $2 + 5 = 7$ expresses an equality, but it is not an equation as the word is technically used in mathematics." If one turns to such a standard work as E. Borel's

[1] This note is a part of a paper read before the Iowa Association of Mathematics Teachers, November 3, 1916.

[2] In the *New Standard Dictionary*, 1915, the following two definitions of the term discriminant are given. "The integral function of the coefficients of an algebraic equation that becomes zero only when the equation has equal roots. The discriminant is equal to the continued product of the squares of all the differences of the roots." Even by means of the quadratic equation $ax^2 + bx + c = 0$ it can easily be verified that these two definitions of the term discriminant are contradictory whenever $a \neq 1$. Two definitions which appear in Webster's *New International Dictionary*, 1916, under the word "discriminant" are contradictory for the same reason.

Die Elemente der Mathematik, translated by P. Stäckel, 1908, one finds on page 7 the following identity

$$145 = 145$$

as an illustrative example of an equation (Gleichung). Many good writers speak of "identical equations" and of "conditional equations," and it would seem that an article in a standard work of reference should recognize this fact.

One does not need to read many lines in the article under consideration before reaching the statement "and in the theory of equations, so called, they [the coefficients] stand for real quantities." It is true that equations with real coefficients usually receive most attention in our elementary works on the theory of equations, but the developments frequently include explicitly the case when the coefficients are complex. In particular, the fundamental theorem of algebra is stated in Dickson's *Elementary Theory of Equations,* 1914, page 47, in the following form: "Every equation with complex coefficients $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n = 0$ has a complex (real or imaginary) root." Hence it would appear that the article in question could be improved by cancelling the statement quoted near the beginning of the present paragraph.

Without trying to direct attention to all the statements in the article under consideration which the critical reader might be inclined to modify, we shall refer to the following, appearing near the end of the first column: "In case there is not a sufficient number of relations given to enable the roots of an equation to be determined, exactly or approximately, the equation is said to be *indeterminate; e. g.,* in the equation $x + 2y = 10$ any of the following pairs of values satisfies the equation: (0, 5), (1, 4.5), (2, 4), (3, 3.5), $\cdots$, (10, 0), (11, $-$ 0.5), $\cdots$." Since such number pairs are called roots of the indeterminate equation in question[1] it is somewhat difficult to see why an indeterminate equation should be characterized as one in which not a sufficient number of relations is given to enable the roots to be determined.

These quotations from a single article may suffice to direct attention to the fact that the reader of the mathematical articles in the *New International Encyclopedia* cannot afford to accept as final all the statements contained therein. It should, however, not be inferred that this particular article, which is also marred by a number of typographical errors, is a fair sample of the mathematical articles in the work under consideration. In view of the importance of the subject treated in this article it seems desirable to endeavor to aid the young reader by directing attention to its shortcomings, especially since such a reader is usually inclined to exercise too little caution in accepting results found in what are commonly regarded as standard works of reference.

In the interesting article on *geometry,* contained in volume 9 of the encyclopedia under consideration, there appears on page 610, column 2, the following perplexing sentence: "Riemann and Helmholtz formulated assumptions for a geometry in space of n-ply manifoldness and with constant curvature and observed that on the sphere, whose curvature is constant and positive, the sum of the

[1] Cf. *Encyclopédie des Sciences Mathématiques,* tome I, Vol. 2, p. 55.

angles of a triangle is less than a straight angle, this characterizing the space of the geometry of Bolyai and Lobachevsky." One might at first be inclined to regard the word "less" in this quotation as due to an oversight and to replace it by the word "greater." If this were done there would evidently be trouble with the rest of the sentence since in the Lobachevsky geometry the sum of the angles of a triangle is actually less than two right angles.

That the article on *determinants,* contained in volume 6 of the work under review, is apt to give an incorrect impression of the fundamental difference between the concepts implied by the terms matrix and determinant, as used by the most careful modern writers, results directly from the following sentences which appear near the beginning of this article: "The first or square form of notation is called the array notation. If there are more columns than rows, the form is called a *matrix*." Probably the critical reader will also be surprised when he meets the following sentence near the close of the article in question: "The theory as a whole has been most systematically treated by F. Brioschi (1824–97), well known as the editor of the *Annali di Matematica,* whose masterly treatise on determinants is a standard (French and German translations, 1856)." F. Brioschi's little treatise was the first book on determinants if we except the monograph by W. Spottiswoode which appeared a little earlier, and it seems very strange that the theory as a whole should be said to have been most systematically treated in the former of these two works when one bears in mind the more modern and more comprehensive treatises by E. Pascal, G. Kowalewski, and others.

The critical mathematical considerations which have been suggested thus far do not relate to what is commonly understood by mathematical history. In view of the emphasis on historical questions in many of the articles in the encyclopedia under review it may be of interest to suggest also a few considerations relating to modifications along this line. To begin with an unusually strong case we may refer to the fact that in the article on *complex number* it is stated that "the first appearance of the imaginary is found in the *Stereometria* of Heron of Alexandria (third century B. C.)." If we turn to the name "Hero or Heron of Alexandria" in the same encyclopedia we meet the following sentence: "The most recent investigation by Schmidt leads to the conclusion that he may have lived in the first century A. D., but other writers, who, it must be said, have not considered the question so fully, have usually placed him in the first or even in the second century B. C." These two statements clearly imply different dates to be associated with the work of an important Greek mathematician. It may be noted in this connection that the same too early date (third century B. C.) for Heron of Alexandria is also found in the *New Standard Dictionary* (1915) and in the *Century Dictionary* (1914).

Another historical statement which seems to need modification appears in the article on *algebra,* in volume 1, page 402, of our encyclopedia, and is as follows: "It was only after the opening of the nineteenth century that Abel, by the use of the theory of groups discovered by Galois, gave the first satisfactory

proof of the fact, anticipated by Gauss and announced by Ruffini, that it is impossible to express the solution of a general equation as algebraic functions of the coefficients when the degree exceeds the fourth." As the first publication by Galois on the theory of groups appeared after the death of Abel it is difficult to see how his proof could have been based upon the discoveries of Galois. At any rate the sentence as quoted above would naturally lead the reader to think that Abel based his work upon work done earlier by Galois and hence it is unsatisfactory in its present form.

We shall refer here to only one more sentence which seems to convey an incorrect impression in regard to a historical fact of considerable mathematical interest. This sentence appears in the article on the Italian mathematician *Cardan*, volume 4, page 536, and is as follows: "The publication of the *Ars Magna* stimulated mathematical research and hastened the general solution of the biquadratic equation, of which Cardan himself had solved special cases." If we read this statement in the light of the fact that Ferrari's general solution of the biquadratic equation actually appeared in the *Ars Magna* it may possibly have some meaning, but it is evident that the beginner would be apt to draw entirely incorrect conclusions therefrom.

The few modifications which have been suggested could scarcely be supposed to be of general interest to mathematics teachers if they did not relate to an excellent work of reference which is very extensively used by college and university students. It is evidently highly desirable that such works be as clear and accurate as possible. We can scarcely expect that the publishers will make special efforts to attain these ends unless the public actually demands them. Hence publicity given to shortcomings, especially where such publicity tends to the discovery of many other important improvements, seems desirable. Such publicity may also tend to inspire caution in the use of even the most reliable works of reference, a caution which needs to be cultivated on the part of most young mathematicians.

NEW RULES OF QUADRATURE.

By P. J. DANIELL, Rice Institute.

The rules given in this paper are developed from Euler's summation formula.[1] This formula has been used in the past chiefly as a means of converting a series into an integral. Nevertheless it has several advantages as a source from which to obtain rules of quadrature. Three such rules are stated here, and the author believes that the second and third are new, while even the first has not received the attention which it deserves.

Rule 1.

$$\int_a^b y\,dx = h[\tfrac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \tfrac{1}{2}y_n] + \frac{h^2}{12}[y_0' - y_n'] + R_1.$$

[1] BROMWICH, *Theory of Infinite Series*, Chap. X, p. 238.

Here the interval (a, b) is divided into n equal parts of length h, $y_0, y_1, y_2 \cdots$ are the ordinates at $a, a + h, \cdots$ and y_0', y_n' are the derivatives of y at a and b. If R_1 denotes the remainder, then

$$R_1 = -\frac{1}{4!}\int_a^b y^{\mathrm{IV}}(x)\varphi_1(x)dx,$$

where $y^{\mathrm{IV}}(x)$ denotes the fourth derivative of y with respect to x, and $\varphi_1(x)$ denotes the function

$$\varphi_1(x) = (x - a - rh)^2(a + rh + h - x)^2, \qquad a + rh \leqq x \leqq a + rh + h.$$

To prove this it is only nccessary to integrate R_1 by parts four times.

$$\int_{a+rh}^{a+rh+h} \varphi_1(x)dx = h^5 \int_0^1 t^2(1 - t)^2dt = \frac{h^5}{30}.$$

Then

$$| R_1 | \leqq \frac{h^4}{6!}(b - a) \max | y^{\mathrm{IV}}(x) | .$$

Comparison with Simpson's Rule. To find the error in Simpson's rule, let n be even; then by a similar process, taking $n/2$ intervals of length $2h$,

$$\int_a^b ydx = 2h[\tfrac{1}{2}y_0 + y_2 + y_4 + \cdots + \tfrac{1}{2}y_n] + \frac{4h^2}{12}[y_0' - y_n'] + R_2;$$

where

$$R_2 = -\frac{1}{4!}\int_a^b y^{\mathrm{IV}}(x)\varphi_2(x)dx,$$

and

$$\varphi_2(x) = (x - a - 2rh)^2(a + 2rh + 2h - x)^2, \qquad a + 2rh \leqq x \leqq a + 2rh + 2h.$$

Eliminating $[y_0' - y_n']$, we obtain Simpson's rule with remainder R', where $R' = \frac{1}{3}(4R_1 - R_2)$, and

$$\frac{1}{3}\int_{a+2rh}^{a+2r+2h} [\varphi_2(x) - 4\varphi_1(x)]dx = \tfrac{2}{3}h^5 \int_0^1 [t^2(2 - t)^2 - 4t^2(1 - t)^2]dt = \tfrac{4}{15}h^5.$$

Then

$$| R' | \leqq 4\frac{h^4}{6!}(b - a) \max | y^{\mathrm{IV}}(x) | .$$

Thus the "maximum error," according to rule 1, is only one quarter of that in Simpson's rule.

Closed Curves. For closed curves Simpson's rule is not applicable except by a separate treatment of different portions. But by an application of rule 1 we can obtain an exceedingly simple rule.

The area of a segment between a chord and the curve, considering the chord as a single interval, will be one twelfth of the square of the chord multiplied by

the change in the tangent of the angle between the curve and the chord. If the chord is small compared to the radius of curvature this change will be approximately the angular change in the direction of the curve. If then we have a number of equal successive chords, the total area between them and the curve will be one twelfth of the square of chord times the total angular change in direction of the curve. For a closed curve this will be 2π, hence we obtain the following:

Rule 2. *Step off equal chords along the curve coming back, if possible, to the starting point. Then if O is any point inside the polygon thus formed,* we have

$$Area = \tfrac{1}{2}\, chord \times \left[sum\ of\ perpendiculars\ from\ O\ on\ chords + \frac{\pi}{3} \times chord \right].$$

The first part of this expression is the area of the chord polygon while the latter part is

$$\tfrac{1}{12} \times \text{chord}^2 \times 2\pi = \tfrac{1}{2} \times \text{chord} \times \left[\frac{\pi}{3} \times \text{chord} \right].$$

If the chord polygon is not quite closed we can add to the above expression the area of the remaining sector regarded as a triangle. If the closing chord equals a small fraction λ of the equal chords, then λ times the perpendicular from O on the closing chord is to be added to the sum of the perpendiculars in the rule. Otherwise we may step round again, obtaining the value of twice the area and replacing 2π by 4π. Since the rule is only approximate we may replace $\pi/3$ in the bracket by $1\frac{1}{20}$, thus

$$Area = \tfrac{1}{2}\, chord \times [sum\ of\ perpendiculars\ from\ 0 + 1\tfrac{1}{20}\ of\ chord].$$

Rule 3.

$$\int_a^b y dx = \frac{h}{15}(7y_0 + 16y_1 + 14y_2 + 16y_3 + \cdots + 16y_{2n-1} + 7y_{2n}) + \frac{h^2}{15}(y_0' - y_{2n}').$$

Here the interval is divided into an even number $2n$ of parts.

$$\int_a^b y dx = h(\tfrac{1}{2}y_0 + y_1 + y_2 + \cdots + \tfrac{1}{2}y_{2n}) + \frac{h^2}{12}(y_0' - y_{2n}') - \frac{h^4}{6!}(y_0''' - y_{2n}''') + S_1,$$

where

$$S_1 = -\frac{1}{6!}\int_a^b y^{\text{VI}}(x)\psi_1(x)dx, \qquad \psi_1(x) = z^2(z + \tfrac{1}{2}),$$

and

$$z = (a + rh + h - x)(x - a - rh), \qquad a + rh \leqq x \leqq a + rh + h.$$

To prove this integrate S_1 by parts six times.

Taking n intervals of length $2h$ each,

$$\int_a^b y dx = 2h(\tfrac{1}{2}y_0 + y_2 + y_4 + \cdots + \tfrac{1}{2}y_{2n}) + \frac{4h^2}{12}(y_0' - y_{2n}') - \frac{16h^4}{6!}(y_0''' - y_{2n}''') + S_2,$$

where

$$S_2 = -\frac{1}{6!}\int_a^b y^{\mathrm{VI}}(x)\psi_2(x)dx, \qquad \psi_2(x) = z_2{}^2(z_2 + \tfrac{1}{2})$$

and

$$z_2 = (a + 2rh + 2h - x)(x - a - 2rh), \qquad a + 2rh \leqq x \leqq a + 2rh + 2h.$$

Eliminating $[y_0''' - y_{2n}''']$ between these two, we obtain rule 3 with remainder $S' = \frac{1}{15}(16S_1 - S_2)$, and

$$\frac{1}{15}\int_{a+2rh}^{a+2rh+2h}[\psi_2(x) - 16\psi_1(x)]dx$$

$$= \frac{2}{15}h^7\int_0^1[t^2(2 - t)^2(\tfrac{1}{2} + 2t - t^2) - 16t^2(1 - t)^2(\tfrac{1}{2} + t - t^2)]dt$$

$$= \frac{4}{7 \times 25} \times 2h^7.$$

Hence

$$| S' | \leqq \frac{32}{25}\frac{h^6}{8!}(b - a) \max | y^{\mathrm{VI}}(x)| .$$

Examples. (1) Using Rule 2, to find π by means of the regular polygon of 24 sides inscribed in a circle of unit radius, we have

$$\text{chord} = 2 \sin 7°.5, \qquad \text{and} \qquad \text{perpendicular} = \cos 7°.5.$$

Hence,

$$\pi = \sin 7°.5\,[24 \cos 7°.5 + 21/20 \times 2 \sin 7°.5] = 12 \sin 15° + 1.05(1 - \cos 15°).$$

Using

$$\sin 15° = \tfrac{1}{4}(\sqrt{6} - \sqrt{2}), \qquad \text{and} \qquad \cos 15° = \tfrac{1}{4}(\sqrt{6} + \sqrt{2})$$

we find $\pi = 3.14162$ instead of 3.14159.

(2) To approximate the value of $\log_e 2$. We have

$$\int_1^2 \frac{dx}{x} = \log_e 2 = .69315.$$

Using two intervals we get from $\int_1^2 \frac{dx}{x}$ by Simpson's rule, .69444, with the error .00129; by Rule 1, .69271, with the error .00044; and by Rule 3, .69306, with the error .00009.

(3) To approximate the value of π from $\int_0^{\frac{1}{2}} \sqrt{1 - x^2}\, dx$. We have

$$\int_0^{\frac{1}{2}} \sqrt{1 - x^2}\, dx = \frac{\pi}{12} + \frac{1}{8}\sqrt{3}.$$

Using two intervals we get the value of π from $\int_0^{\frac{1}{2}} \sqrt{1-x^2}\,dx$ by Simpson's rule, 3.14093, with the error .00066; by Rule 1, 3.14178, with the error .00019; and by Rule 3, 3.14161, with the error .00002.

SOME METRICAL PROPERTIES OF THE PENTAHEDROID IN A SPACE OF FOUR DIMENSIONS.

By M. H. SZNYTER, University of California.

The purpose of this paper is to establish for the pentahedroid in a four-dimensional space, theorems similar to those found in ordinary solid geometry, dealing with the tetrahedron. The terms line, point and plane are used with the same significance as in three-dimensional geometry. By hyperplane we shall mean that three-dimensional element which consists of any four points not points of one plane, all points collinear with any two of them or with any two obtained by this process.[1] We shall first develop the simpler theorems, then we shall consider the pentahedroid with its tangent hyperspheres.

THEOREM 1.[2] *Let $A_1A_2A_3A_4A_5$ be a pentahedroid cut by a hyperplane α in such a way that the edge A_1A_2 lies on one side of α and the face $A_3A_4A_5$ on the other side. Then the following cases will appear.*

1. *If α is parallel to the line A_1A_2 and to the plane $A_3A_4A_5$, the section will be a prism.*
2. *If α is parallel to the line A_1A_2 but not to the plane $A_3A_4A_5$, the section will be a truncated prism.*
3. *If α is parallel to the plane $A_3A_4A_5$ but not to the line A_1A_2, the section will be a frustrum of a pyramid.*
4. *If α is parallel neither to the plane $A_3A_4A_5$ nor to the line A_1A_2, the section will be a truncated pyramid.*

For the hyperplane α cuts the tetrahedrons $A_2A_3A_4A_5$ and $A_1A_3A_4A_5$ in planes which cut a triangle from each $A_3'A_4'A_5'$ and $A_3''A_4''A_5''$. Since α passes between the edges A_1A_2 and the edges A_3A_4, A_4A_5 and A_3A_5, it must cut the other three tetrahedrons in planes which cut quadrilaterals from them. The section cut out from the pentahedroid will be a three-dimensional figure having for its boundaries two triangular faces and three quadrilateral faces. In cases 1 and 2 the lateral edges of the section are parallel and the section is prismatic. In case 1 the triangular faces are parallel while in case 2 they are not. Hence the former gives a prism while the latter gives a truncated prism. In cases 3 and 4, the lateral edges of the section will meet, if produced, at the point P which is the point of intersection of A_1A_2 with α, and the section is pyramidal. When α is parallel to plane $A_3A_4A_5$, the triangles $A_3'A_4'A_5'$ and $A_3''A_4''A_5''$ are parallel;

[1] MANNING: *Geometry of Four Dimensions*, p. 24.
[2] *Ibid.*, p. 228.

the section is thus a frustrum of a pyramid, as in case 3. When α is not parallel to the plane $A_3A_4A_5$, the planes of the triangles $A_3'A_4'A_5'$, $A_3''A_4''A_5''$ meet if produced in the line g, the intersection of α and $A_3A_4A_5$. This section is therefore a truncated pyramid.

THEOREM 2. *The lateral volumes of two similar pentahedroids are to each other as the cubes on their homologous sides.*

THEOREM 3. *Pentahedroids are to each other as the products of their hyperplane bases by their altitudes; hence pentahedroids which have equivalent bases are to each other as their altitudes and pentahedroids which have equivalent altitudes are to each other as their bases.*

THEOREM 4. *Pentahedroids which have a hyperspace angle of one equal to a hyperspace angle of the other are to each other as the products of the edges of the equal hyperspace angles.*

From theorems 3 and 4 we get at once,

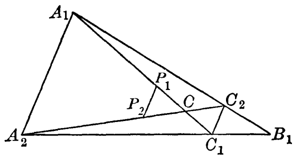
THEOREM 5. *Similar pentahedroids are to each other as the fourth powers of their homologous edges.*

THEOREM 6. *The lines drawn from the five vertices of a pentahedroid to the centers of gravity of the opposite cells cut each other in two segments in the ratio of 4 : 1.*

Consider the pentahedroid $A_1A_2A_3A_4A_5$ with C_k the center of gravity of the cell opposed to the vertex A_k. Since we know that the segments A_1C_1, A_2C_2, $\cdots$ are concurrent at the center of gravity of the pentahedroid we may denote th s point by C. The following relations shall hold:

$$A_1C : CC_1 = 4 : 1, \quad A_2C : CC_2 = 4 : 1, \quad \cdots.$$

For upon the segment A_1C_1 consider a point P_1 on the same side of C as A_1 and such that CP_1 equals $\frac{1}{4}CA_1$. Consider four other points P_k similarly situated upon the other four segments A_kC_k.



Then the segments A_1C_2 and A_2C_1 meet at a point B_1 which is the center of gravity of the face $A_3A_4A_5$. For C_2 is the center of gravity of the cell $A_1A_3A_4A_5$ and hence A_1C_2 passes through the center of gravity of the face $A_3A_4A_5$. Likewise C_1 is the center of gravity of the cell $A_2A_3A_4A_5$ and hence A_2C_1 also passes through the center of gravity of the face $A_3A_4A_5$. We now have two triangles $A_1A_2B_1$ and A_1A_2C in the plane determined by A_1B_1 and A_2B_1.

The segment C_1C_2 is parallel to A_1A_2 and is equal to $\frac{1}{4}A_1A_2$, for

$$C_1B_1 = \tfrac{1}{4}A_2B_1 \quad \text{and} \quad C_2B_1 = \tfrac{1}{4}A_1B_1.$$

Similarly the segment P_1P_2 is parallel to A_1A_2 and is equal to $\frac{1}{4}A_1A_2$. Thus

P_1P_2 and C_1C_2 determine a parallelogram, from which relation we obtain at once,

$$CC_1 = P_1C = \tfrac{1}{4}A_1C \quad \text{and} \quad CC_2 = P_2C = \tfrac{1}{4}A_2C$$

or $A_1C : CC_1 = 4 : 1$, etc., which proves our theorem.

PROBLEM 1. *To find the radius of the inscribed and of the circumscribed hyperspheres of any regular pentahedroid in terms of its edge.*

Consider the regular pentahedroid $A_1A_2A_3A_4A_5$ with an edge equal to a certain finite magnitude K. Its ten edges, ten faces, dihedral angles and cells are all equal among each other for we know that the regular pentahedroid is congruent to itself in sixty different ways. It is first necessary to determine the radii of circles inscribed in and circumscribed about the plane faces and also the radii of the spheres inscribed and circumscribed about the five cells of the pentahedroid.

Consider one of the regular faces, $A_1A_2A_3$. By plane geometry the altitude of this triangle is

$$a = \sqrt{K^2 - \tfrac{1}{4}K^2} = \frac{\sqrt{3}}{2}K.$$

Since the three altitudes intersect at the center of gravity of this triangle, the radius of the inscribed circle is one third of the altitude and the radius of the circumscribed circle is two thirds of the altitude. Thus

$$r_i = \frac{1}{3}\frac{\sqrt{3}}{2}K = \frac{\sqrt{3}}{6}K, \quad r_c = \frac{2}{3}\frac{\sqrt{3}}{2}K = \frac{\sqrt{3}}{3}K.$$

Consider one of the regular cells as $A_1A_2A_3A_4$. Its altitude A is measured by the length of one leg of a right-angled triangle whose hypotenuse is the edge K and whose other leg is the radius of the circle circumscribed about the regular face $A_1A_2A_3$. Hence

$$A = \sqrt{K^2 - \frac{3K^2}{9}} = \frac{\sqrt{6}}{3}K.$$

The altitudes of the tetrahedron meet at its center of gravity and hence the radius of the inscribed sphere is one fourth the length of the altitude of the cell and the radius of the circumscribed sphere is three fourths the altitude. Therefore,

$$R_i = \frac{1}{4}\frac{\sqrt{6}K}{3} = \frac{\sqrt{6}}{12}K, \qquad R_c = \frac{3}{4}\frac{\sqrt{6}}{3}K = \frac{\sqrt{6}}{4}K.$$

In the pentahedroid $A_1A_2A_3A_4A_5$, the altitudes are concurrent at the center of gravity of the pentahedroid, therefore the radius of the inscribed hypersphere is equal to one fifth of an altitude and the radius of the circumscribed hypersphere is equal to four fifths of an altitude. Let α represent any altitude of the pentahedroid and ρ_i, ρ_c the radii of the inscribed and circumscribed hyperspheres respectively. As α is measured by the length of one leg of a right-angled

triangle whose hypotenuse is the edge K and whose other leg is the radius R_c of the sphere circumscribed about one of its cells, therefore,

$$\alpha = \sqrt{K^2 - \frac{6}{16}K^2} = \frac{\sqrt{10}}{4}K$$

and

$$\rho_1 = \frac{1}{5}\alpha = \frac{\sqrt{10}}{20}K,$$

radius of the inscribed hypersphere

$$\rho_c = \frac{4}{5}\alpha = \frac{\sqrt{10}}{5}K,$$

radius of the circumscribed hypersphere.

Turning now to the second part of our problem, namely the tangent hyperspheres of the pentahedroid, it is first necessary to establish a system of coördinates for any given point of hyperspace with reference to a fixed pentahedroid. This system may be called pentahedroidal and will be based upon the distances of any point in hyperspace from the five cells of the pentahedroid. It is necessary to find a relation between these distances. They will then be sufficient to determine a single point in hyperspace.

Consider the pentahedroid $A_1A_2A_3A_4A_5$. Let V_k be the volume of the cell opposite the summit A_k and x_k the distance of any point P from the hyperplane V_k. Then the necessary and sufficient condition that any five quantities x_1, x_2, x_3, x_4, x_5 represent the coördinates of the point P with respect to the pentahedroid is that they satisfy the following equation,

$$V_1x_1 + V_2x_2 + V_3x_3 + V_4x_4 + V_5x_5 = 4H, \tag{1}$$

where H is the hypervolume of the pentahedroid. The five hyperplanes of the pentahedroid divide hyperspace into thirty-one regions, as follows:

1. The interior $A_1A_2A_3A_4A_5$ of the pentahedroid.
2. The region α having only the vertex A_1 in common with the pentahedroid and lying on the same side of $A_2A_3A_4A_5$ that A_1 lies. There are five regions of this sort, one opposed to each vertex of the pentahedroid.
3. The region β bounded by three hyperplanes having the edge A_2A_3 in common and two hyperplanes having each only one point in common with the edge A_2A_3. There are ten regions of this sort, one opposite each edge of the pentahedroid.
4. The region γ bounded by two hyperplanes having the face $A_2A_3A_4$ in common and by three hyperplanes having each only a line in common with the face $A_2A_3A_4$. There are ten regions of this sort, one relative to each face.
5. The region δ formed by removing the pentahedroid from the interior of one of its tetrahedroidal angles. There are five such regions formed by taking each of the points A_1, A_2, $\cdots$ in turn.

Then for any point lying in any of the regions so determined, the perpendicular distances from it to the five hyperplanes of the pentahedroid shall satisfy equation (1).

Consider P any point within the pentahedroid and x_1, x_2, x_3, x_4, x_5 the distances from this point to the five cells of the pentahedroid which is now divided into five smaller pentahedroids formed by the five cells taken in turn with the point P as a common vertex. The hypervolume of the original pentahedroid is equal to the sum of the hypervolumes of the five smaller pentahedroids. Since the hypervolume of any pentahedroid is equal to one fourth the product of its hyperplane base by its altitude[1] we have at once,

$$4H = V_1x_1 + V_2x_2 + V_3x_3 + V_4x_4 + V_5x_5.$$

If the point P is within the region α, the pentahedroid $A_1A_2A_3A_4A_5$ is equal to the pentahedroid $PA_2A_3A_4A_5$ minus the sum of the other four pentahedroids determined by P with the other four cells of the original pentahedroid. The altitudes of the five pentahedroids having P for a vertex are $x_1, -x_2, -x_3, -x_4, -x_5$.[2] Hence

$$4H = V_1x_1 - (-V_2x_2) - (-V_3x_3) - (-V_4x_4) - (-V_5x_5)$$

or

$$V_1x_1 + V_2x_2 + V_3x_3 + V_4x_4 + V_5x_5 = 4H.$$

When P is within the region β, the five altitudes of the five pentahedroids determined by P are $-x_1, x_2, x_3, -x_4$ and $-x_5$. Thus

$$4H = V_2x_2 + V_3x_3 - (-V_4x_4) - (-V_5x_5) - (-V_1x_1),$$

or

$$V_1x_1 + V_2x_2 + V_3x_3 + V_4x_4 + V_5x_5 = 4H.$$

For P in the region γ the altitudes of the five pentahedroids determined by it are $-x_1, x_2, x_3, x_4, -x_5$ and for ρ in the region δ the altitudes are $-x_1, x_2, x_3, x_4$, and x_5. The relations between the hypervolumes become

$$4H = V_2x_2 + V_3x_3 + V_4x_4 - (-V_1x_1) - (-V_5x_5) \text{ for } \gamma,$$

$$4H = V_2x_2 + V_3x_3 + V_4x_4 + V_5x_5 - (-V_1x_1) \qquad \text{for } \delta,$$

whence

$$V_1x_1 + V_2x_2 + V_3x_3 + V_4x_4 + V_5x_5 = 4H.$$

From the above consideration it is evident that if a point P lies within any one of the regions determined by the five hyperplanes of a pentahedroid, its distances to the five hyperplanes satisfy a certain relation which is expressed by equation (1). To complete the establishment of the system it is necessary to show that if any five quantities x_1, x_2, x_3, x_4 and x_5 satisfy equation (1) they determine one and only one point in hyperspace.

[1] MANNING: *Geometry of Four Dimensions*, pp. 277–278.

[2] Any distance x_k shall be positive or negative depending on whether it is on the same side or opposite side of the hyperplane V_k that the vertex A_k is.

Consider the hyperplanes P_1, P_2, P_3, P_4 and P_5 at the distances x_1, x_2, x_3, x_4 and x_5 from the hyperplanes V_1, V_2, V_3, V_4 and V_5 and such that P_k is parallel to V_k. Any four hyperplanes will meet at a point in hyperspace.

Suppose P_1, P_2, P_3 and P_4 meet at Q and let g_1, g_2, g_3, g_4 and g_5 be the coördinates of the point φ with respect to the pentahedroid. From (1) we have

$$V_1g_1 + V_2g_2 + V_3g_3 + V_4g_4 + V_5g_5 = 4H.$$

From the construction of the hyperplanes P_1, P_2, P_3, P_4, we have

$$g_1 = x_1, \qquad g_2 = x_2, \qquad g_3 = x_3, \qquad g_4 = x_4.$$

Therefore

$$V_1x_1 + V_2x_2 + V_3x_3 + V_4x_4 + V_5g_5 = 4H.$$

Also

$$V_1x_1 + V_2x_2 + V_3x_3 + V_4x_4 + V_5x_5 = 4H.$$

Therefore $g_5 = x_5$ and Q is a single point of hyperspace.

THEOREM 7. *There are at most sixteen hyperspheres tangent to the five cells of a pentahedroid.*

1. If there be a hypersphere inscribed within the pentahedroid, the coördinates of the center C, besides being all equal, must satisfy equation (1). If R be the radius of the hypersphere, then

$$x_1 = x_2 = x_3 = x_4 = x_5 = R,$$

and hence

$$RV_1 + RV_2 + RV_3 + RV_4 + RV_5 = 4H,$$

$$R(V_1 + V_2 + V_3 + V_4 + V_5) = 4H,$$

$$R = \frac{4H}{V_1 + V_2 + V_3 + V_4 + V_5}. \tag{2}$$

Since H, V_1, V_2, V_3, V_4, V_5 are all positive, (2) gives a single positive value of R, hence there is a single hypersphere inscribed in the pentahedroid, tangent to its five cells.

2. If there exists a hypersphere tangent to the cells with its center within the region α, the coördinates of the center will satisfy the following,

$$x_1 = -x_2 = -x_3 = -x_4 = -x_5 = R$$

and

$$RV_1 - RV_2 - RV_3 - RV_4 - RV_5 = 4H,$$

whence

$$R = \frac{4H}{V_1 - (V_2 + V_3 + V_4 + V_5)}. \tag{3}$$

As any four cells of the pentahedroid are together greater than the fifth, the right-hand member of (3) is negative, and since a negative value of R is inadmissible, there is no hypersphere tangent to the cells with its center in the region α. Therefore there are no tangent hyperspheres in the other four regions of the type α.

3. If there be tangent hyperspheres in the regions β and γ, the following must hold:

$$-x_1 = x_2 = x_3 = -x_4 = -x_5 = R,$$

$$-x_1 = x_2 = x_3 = x_4 = -x_5 = R,$$

and

$$R = \frac{4H}{V_2 + V_3 - (V_1 + V_4 + V_5)} \text{ in } \beta, \tag{4}$$

$$R = \frac{4H}{V_2 + V_3 + V_4 - (V_1 + V_5)} \text{ in } \gamma. \tag{5}$$

The value of R is now dependent on whether the sum of two cells of a pentahedroid is less than, equal to or greater than the sum of the other three cells. Thus if $V_2 + V_3 - (V_1 + V_4 + V_5)$ is negative in (4) then $V_1 + V_4 + V_5 - (V_2 + V_3)$ in (5) will be positive giving a positive R in γ for a negative value of R in β. Thus the number of tangent hyperspheres in the regions β and γ together will always be ten unless some of these expressions happen to be zero.

4. Finally when the center of the tangent hypersphere lies within the region δ,

$$-x_1 = x_2 = x_3 = x_4 = x_5 = R,$$

$$R = \frac{4H}{V_2 + V_3 + V_4 + V_5 - V_1}. \tag{6}$$

As the sum of any four cells of a pentahedroid is greater than the fifth, (6) gives a single positive value for R and hence there is one hypersphere tangent to the cells with its center in the region δ. There are five such hyperspheres, one in each of the five regions of the type δ.

Thus we have one tangent hypersphere within the pentahedroid, ten within the regions β and γ taken together and five in the regions δ, mak'ng the number of tangent hyperspheres equal in general to sixteen.

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

Plane and Solid Geometry. By WEBSTER WELLS and WALTER W. HART. D. C. Heath and Company, Boston, 1916. viii + 467 pages.

It is very important that the introduction to a course in plane geometry should be well done. With inexperienced teachers, at least, the introduction to the course will usually be the one that is given in the text. The introductory chapter of a geometry may be expected to make a clear statement of certain definitions and assumptions and to illustrate them; to offer some suggestions that will tend to make the students feel a need for geometry; and to show the need of formal proofs and to give an introduction to them. In the present text

the definitions are well illustrated with simple exercises. Not much is done in the introduction to show the uses of geometry. More exercises in construction would make the introduction more interesting and more effective in fixing new ideas.

The following sentences from the preface suggest the attitude of the authors in the selection and the arrangement of material: "In each Book, the fundamentally important theorems are given first. These theorems present a *safe and sane minimum course.* These are followed in each Book by one or more groups of theorems or applications which are strictly supplementary—material which has either long appeared in geometries in some form or has been introduced in recent years to add to the pupils' interest."

Care has been taken to make the book teachable. Teachers examining this text will be interested in the summaries such as appear on pages 27, 62, and 113; in the references to supplementary exercises in the footnotes; in the treatment of loci and inequalities; and in the suggested proofs which are left for the students to finish. The exercises are numerous and appear to be well graded. There is a table of trigonometric ratios and some problems applying them.

Certain definitions need revision. The definition of a circle given in § 16 is: "A circle is a closed curve all points of which are equidistant from a point within called the center." The circle need not be a plane figure by this definition. The definition of ratio, § 212, reads as follows: "The ratio of two magnitudes of the same kind is the quotient of their numerical measures in terms of a common measure." It seems likely that students will be confused by this definition. The following definition of the limit of a variable is given in § 401: "A limit of a variable is a constant such that the numerical value of the difference between the constant and the variable becomes and remains less than any small positive number." The "small positive number" must of course be pre-assigned.

In three important respects the book has departed little from the traditional. The exercises are mainly formal, with a comparatively small number of applications; not much has been done in applying algebra to the solution of geometric problems; and the notions of motion and symmetry, which are useful and which have become prominent in the teaching of elementary geometry in recent years, have received slight attention.

E. H. Taylor.

Eastern State Normal School,
Charleston, Illinois.

Fundamental Conceptions of Modern Mathematics: Variables and Quantities, with a Discussion of the General Conception of the Functional Relation. By Robert T. Richardson and Edward H. Landis. Chicago and London, The Open Court Publishing Co., 1916.

In their preface the authors describe their work as "essentially one of constructive criticism . . . the first attempt made on any extensive scale to examine critically the fundamental conceptions of mathematics as embodied in the current definitions." The present volume, which is to be followed by a second, is concerned with variables and quantities.

No one interested in recent efforts to bring precision and order into the conceptions of mathematics can fail to commend the purpose of the joint authors or to agree with them in their view of the unsatisfactory character of the usual definitions. On the other hand, the book fails, I think, to present any convincing and novel constructive results. The ideas of the authors are not presented either with the clearness or the acumen necessary to make them effective and are not as new as the authors suppose. For example, the main thesis of the authors regarding functional relation, to wit, that it often involves likeness of order of the corresponding quantities of the variables has been expressed with far greater exactness and logical completeness by Russell in his "Principles of Mathematics," Chapter XXXII (On the Correlation of Series); while, on the other hand, the argument (pages 151, 152) by means of which they seek to restrict the notions of variable and function to relations between *ordered* aggregates is not likely to convince "the Peano school" against whom it is directed. Philosophically speaking, a variable denotes *any* term standing in a given relation and a function denotes the relation in question. The definitions of Richardson and Landis simply restrict the broader definitions to more special cases. But the broader definitions should be insisted upon for the sake of the logical unification which they confer upon the various branches of mathematics. And when it comes to the point of giving a constructive definition of variable, the authors resign the effort on the plea that it would require a too lengthy discussion of the philosophical doctrine of the categories (page 154 et seq.).

The consideration of quantity yields an equally disappointing result. Of all the concepts of mathematics requiring clarification, this is the most flagrant; yet, so far as I can see, the present authors only add to the confusion.

More in accord with what the reviewer would regard as sound doctrine is the stand taken against the symbol theory of pure mathematics. For, even before any real objects satisfying the variables in a mathematical expression have been found, the expression is not a mere symbol, but indicates certain operations or relations between *possible* objects. Good remarks are also to be found concerning the distinction between independent and dependent variables (page 175).

Taken as a whole, what the book lacks is exactly what it aims to embody—a philosophical background for the criticism of mathematical ideas. One representative illustration of this is the naïve discussion of similarity and identity on pages 5 and 6. But after all, this is only the first and preliminary volume.

DeWitt H. Parker.

Department of Philosophy,
University of Michigan.

The Arithmetical Philosophy of Nicomachus of Gerasa. By George Johnson, Ph.D. (University of Pennsylvania dissertation.) New Era Printing Company, Lancaster, Pa. 1916. 49 pages.

This pamphlet presents a partial translation (the complete translation is deposited in the library of the University of Pennsylvania) of the famous *Intro-*

duction to Arithmetic of Nicomachus, the earliest extant separate essay at a systematization of the Greek science of Arithmetikê, treating of numbers as such, as distinguished from Logistikê, which treated of numerical reckoning from a practical standpoint. Although the *Introduction* was no such landmark in the history of mathematical advance as the works of Euclid, or Archimedes, or Diophantus, or of some few other Greeks, it is still an extremely important document in educational mathematics and continued to exercise a great influence in the Latin versions of Apuleius and Boethius.

Dr. Johnson's dissertation is not a pretentious performance, and its aim seems to be description only, for he has not added to the translation, which covers a selection of passages designed to show the general argument of the *Introduction*, any independent treatment of Nicomachus himself, his philosophy, or his number theory. The commentary which he presents is confined to summaries and explanatory matter connecting the translated passages (for the footnote to the title is the only one in the book), and is largely based on Nesselmann and Cantor. Recognizing the generally descriptive nature of the commentary, the reader will still find statements that need modification. It does not give a true idea of the situation to say that the *Theologumena Arithmetica*, which as it stands is a patchwork of passages from several hands, is "probably the work of Nicomachus" (p. 1), although it certainly is largely Nicomachean; nor does it mean much to say that Nicomachus, long before algebra was developed, was "among the first to attempt a systematic treatment of arithmetic distinct from algebra" (p. 2). The statement that the number contrasted by Nicomachus with his "scientific" number is the "ideal number" of Plato (p. 6) surely calls for proof, for the divine numbers which are discussed in the *Theologumena Arithmetica* ought to be brought into consideration, as well as the Platonic. Some of the formulas, also, might be criticized. $(2n - 1)/n$, for example, hardly comprehends all possible superpartials (defined in the treatise as the relation between two numbers "whenever a number contains in itself the whole compared and in addition more of its parts than one," I. 20. 1), and it is a bad slip to give $a : \sqrt{ab} = \sqrt{ab} : b$ as the formula of the "arithmetic proportion." The author may be misled by the free use of the term "proportion" among the ancients; but the "arithmetic proportion" is at any rate identical with our "arithmetic progression," wherein $a - b = b - c$.

The translation is after all the raison d'être of the dissertation. It is uneven in character, the chief fault being that Dr. Johnson follows Hoche's text too literally and thus is led into needlessly anacoluthic English (*e. g.*, Book II, Chap. 1, Section 1, and II. 27. 1) and, in a few cases, into translating a hopelessly corrupt text (*e. g.*, I. 13. 8, where ἑαυτοῦ is certainly preferable to ἑαυτῶν; II. 23. 5, where Ast's text is far better than the hopeless reading of Hoche). Somewhat more care in punctuation and in proof-reading would have saved the sense of several ambiguous passages (see, *e. g.*, I. 9. 1; and on p. 42, lines 15 ff.). But beyond this the version is not free from errors, of which the citation of a few of the most serious must suffice here. In I. 7. 4 Nicomachus states that even

numbers, when divided into two parts, show but "one of the two kinds" (τὸ ἕτερον εἶδος) of number in these parts; that is, both are odd or both even; Dr. Johnson says, "shows the first kind of number only (odd) without share in the rest." This is obviously false; because, for example, $8 = 5 + 3$ or $6 + 2$. Nicomachus tells in I. 12. 1–2 of the derivation of "composite numbers," using the word συντιθέναι in the sense of "to multiply" in the context; the translator gives the sense as "to add" and thereby creates confusion. Another serious error comes in the discussion of the "sieve of Eratosthenes" (I. 13). The Greek reads that the odd numbers "assume in turn the measuring function" (p. 33, 2, Hoche: ὥστε τὸ μὲν μετρεῖν διαδέξονται, κτλ.) in the process of discovering which terms are prime, but the translator has distorted the simple phrase into "their capacity of measuring will depend on the order in which they (*i. e.*, numbers to be measured) lie in the line" (p. 16). Still another slip gives the reader a fundamental misconception of Nicomachus's method of discovering "perfect numbers" (I. 16. 4). Dr. Johnson (p. 19) thus summarizes up the rule: "Sum up the powers of 2 and add unity to each sum; when the result is prime, multiply it by the last power of 2 in the series summed up; the product will be a perfect number." The translated section which follows is in conformity with this notion. But there is really no question of "adding unity to each sum"; εἶτα ἀεὶ κατὰ ἑνὸς πρόσθεσιν ἐπισωρεύειν (p. 41, 6, Hoche) means "then you must sum them up, as you take them one by one into the combination," as a comparison of the commentary by Iamblichus (*in Nic.*, p. 34, 2, edited by Pistelli) will amply confirm. The translator has probably failed to notice that unity is definitely included by Nicomachus in the series of (even-times even) numbers which he here uses, a fact which is made perfectly clear by the opening words of section 8 of the chapter. One leaves the reading of this dissertation with the feeling that the treatment accorded to Nicomachus is not as thorough-going nor as accurate as he deserves.

Frank Egleston Robbins.

The University of Michigan,
Department of Greek.

PROBLEMS FOR SOLUTION.

Send all communications about problems to B. F. Finkel, Springfield, Mo.

ALGEBRA.

477. Proposed by J. L. RILEY, Northeastern State Normal School, Tahlequah, Oklahoma.

Evaluate the product $(1 + r + r^2 + r^3)(1 + r^2 + r^4 + r^6) \cdots (1 + r^{2^{n-1}} + r^{2^n} + r^{3\cdot 2^{n-1}})$.

478. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

Solve the equations

$$\frac{l}{x} + \frac{y}{m} + \frac{z}{n} = 1, \qquad \frac{x}{l} + \frac{m}{y} + \frac{z}{n} = 1, \qquad \frac{x}{l} + \frac{y}{m} + \frac{n}{z} = 1.$$

GEOMETRY.

510. Proposed by JOSEPH E. ROWE, State College, Pa.

Show how to find the equation of a line parallel to a side of the triangle of reference and passing through a given point, in any system of homogeneous coördinates, using the condition that two lines are parallel in this system but not the condition that two lines are perpendicular. Illustrate the method by using it to find the trilinear coördinates of the points of contact of one escribed circle of the triangle.

511. Proposed by FRANK V. MORLEY, Student, Haverford College, Pa.

Let a_i $(i = 1, 2, 3, 4)$ be four points on a circle and let the in-center of the triangle formed by omitting a_i be c_i; prove that the four points c_i form a rectangle.

CALCULUS.

425. Proposed by O. S. ADAMS, U. S. Coast and Geodetic Survey, Washington, D. C.

Show that the infinite product

$$(1 - z)(1 + \tfrac{1}{2}z)(1 - \tfrac{1}{3}z)(1 + \tfrac{1}{4}z)\cdots = \frac{\sqrt{\pi}}{\Gamma(1 + \frac{1}{2}z)(\frac{1}{2} - \frac{1}{2}z)}.$$

426. Proposed by C. N. SCHMALL, New York City.

If A be the area of a plane triangle constructed with the sides a, b, c, such that

$$a^3 + b^3 + c^3 = 3k^3,$$

show that the maximum value of A is $\frac{1}{4}k^2$.

MECHANICS.

342. Proposed by WILLIAM HOOVER, Columbus, Ohio.

A uniform rod of length $2a$ is freely hinged at one end, at the other end a string of length b is attached which is fastened at its further end to a point on the surface of a homogeneous sphere of radius c. If the masses of the rod and sphere are equal, find the motion of the system when slightly disturbed from the vertical, and the cubic equation giving the corresponding small oscillations.

343. Proposed by J. ROSENBAUM, New Haven, Conn.

Two bodies of equal masses and coefficients of friction μ_1 and μ_2 are connected by a light spring of stiffness k and placed on an inclined plane. Discuss the motion of each body when the angle between the non-stretched spring and the plane is θ.

NUMBER THEORY.

261. Proposed by NORMAN ANNING, Chilliwack, B. C.

Show that for any positive integer n (excluding powers of 2) positive integers a_1, a_2, a_3, $\cdots a_k$, which are less than $n/2$ can be chosen in such a way that

$$2^k \cos(a_1\pi/n) \cos(a_2\pi/n) \cos(a_3\pi/n) \cdots \cos(a_k\pi/n) = 1.$$

262. Proposed by C. N. SCHMALL, New York City.

If x, y, z are three integers, consecutive among the integers prime to 3, show that

$$x(x - 2y) - z(z - 2y) = \pm 3.$$

SOLUTIONS OF PROBLEMS.

ALGEBRA.

J. A. BULLARD and J. W. BALDWIN solved 464. These solutions were received after selections for publication were made.

465. Proposed by CYRUS B. HALDEMAN, Ross, Ohio.

Having given $\tan^{-1} 1 = \tan^{-1} 1/2 + \tan^{-1} 1/3$, show that

$$\tan^{-1} 1 = 5 \tan^{-1} 1/8 + 2 \tan^{-1} 1/18 + 3 \tan^{-1} 1/57.$$

Solution by Horace Olson, Chicago, Illinois.

Put $\tan^{-1} 1/2 = \tan^{-1} 1/3 + \tan^{-1} x_1 = \tan^{-1}\dfrac{3x_1+1}{3-x_1}$. From this equation x_1 is found to be 1/7. Hence, $\tan^{-1} 1 = 2 \tan^{-1} 1/3 + \tan^{-1} 1/7$.

Now put

$$\tan^{-1} 1/3 = \tan^{-1} 1/7 + \tan^{-1} x_2 = \tan^{-1}\frac{7x_2+1}{7-x_2};$$

whence $x_2 = 2/11$, and $\tan^{-1} 1 = 3 \tan^{-1} 1/7 + 2 \tan^{-1} 2/11$.

Now put

$$\tan^{-1} 1/7 = \tan^{-1} x_3 + \tan^{-1} y_3 = \tan^{-1}\frac{x_3+y_3}{1-x_3y_3}.$$

This gives the indeterminate equation $x_3y_3 + 7(x_3 + y_3) = 1$, or $(x_3 + 7)(y_3 + 7) = 50$. One set of solutions of this equation is $x_3 = 1/8$, $y_3 = 1/57$. Hence, $\tan^{-1} 1 = 3 \tan^{-1} 1/8 + 3 \tan^{-1} 1/57 + 2 \tan^{-1} 2/11$. Put

$$\tan^{-1} 2/11 = \tan^{-1} 1/8 + \tan^{-1} x_4 = \tan^{-1}\frac{8x_4+1}{8-x_4};$$

whence $x_4 = 1/18$. Hence, $\tan^{-1} 1 = 5 \tan^{-1} 1/8 + 2 \tan^{-1} 1/18 + 3 \tan^{-1} 1/57$.

Also solved by E. B. Escott, W. J. Thome, M. T. Reed, G. W. Hartwell, and E. E. Whiteford.

466. Proposed by E. B. ESCOTT, Kansas City, Mo.

For what functions, f, are the following relations true:

When

$$\frac{f(x, y, z)}{X} = \frac{f(y, z, x)}{Y} = \frac{f(z, x, y)}{Z},$$

then

$$\frac{f(X, Y, Z)}{x} = \frac{f(Y, Z, X)}{y} = \frac{f(Z, X, Y)}{z}?$$

Solution by Albert A. Bennett, University of Texas.

We notice that $f(X, Y, Z)$ must be homogeneous, since if we replace X, Y, Z by cX, cY, cZ, in the first relation it is unaltered, and hence also the second. Thus x, y, z may be regarded as the homogeneous coördinates of the points in one plane, X, Y, Z of those in a second. The problem may therefore be expressed as follows: What are the involutoric plane transformations with triangular symmetry? Here "involutoric" is used in the restricted sense as of period two, and triangular symmetry is used in approximately the sense first suggested by Clifford, the point (1, 1, 1) being a center of triangular symmetry. There can be little doubt that there exist transformations of this form which are essentially transcendental and which may be regarded as limiting cases of algebraic birational transformations. To catalogue the explicit forms of even the algebraic cases is perhaps out of the question, since no explicit classification of algebraic forms of Cremona transformations has been attempted beyond the very simplest cases. Normal forms under the Cremona group are indeed known. Compare Pascal's *Repertorium* or other detailed articles on birational geometry.

It is furthermore obvious that a given geometric solution gives rise to an infinite number of analytic solutions. For example, if $g(x, y, z)$ be one solution, and $\varphi(x, y, z)$ be a constant or the equation of any triangularly symmetric self-corresponding curve whether the self-correspondence be singular or not, then $\varphi(x, y, z)g(x, y, z)$ is a solution. Apart from the trivial solutions $f(x, y, z) = 0$ and $f(x, y, z) = x$, the simplest case is probably the quadratic transformation given by $f(x, y, z) = 1/x$, and then taking $\varphi(x, y, z) = xyz$, or $\sqrt{xyz}$, or $xyz/\log(1 + xyz)$ or $(xy + yz + zx)(x + y + z)$, etc., we get other related solutions from $f(x, y, z) = 1/x$.

GEOMETRY.

An excellent solution of 485 by Analytical Geometry was received from O. S. ADAMS after selections for publication were made.

493. Proposed by FLORENCE P. LEWIS, Goucher College, Baltimore, Md.

Construct three circles each of which shall be tangent to the other two and to two sides of a given triangle.

SOLUTION BY C. F. GUMMER, Queen's University, Kingston, Ont.

If we impose the following conditions: (i) each circle is to touch a different pair of sides; (ii) the circles must be real; (iii) no two circles may coincide; we can get 68 solutions.

With the extra condition (iv) all points of contact must be distinct, we get only 32 solutions. We shall obtain formulae for these 32 cases, as being the most interesting.

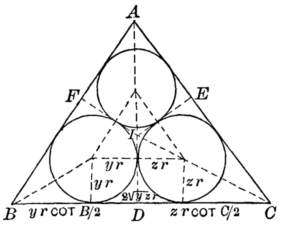


Figure for Equation 6.

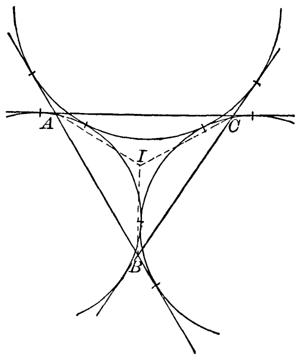


Figure for Equation 7.

Let ABC be the triangle, I, D, E, F the center and points of contact of the in-circle, I_1, D_1, $\cdots$ those of the ex-circles, r, r_1, r_2, r_3 the radii of these circles.

The above conditions show that the required circles must touch externally. Consider any three circles touching externally. If we draw a common tangent to each pair, observing (iv), we get one of eight possible triangles, and in every case the centers of the circles lie on three *concurrent* bisectors of the angles. Hence, our solutions will appear in four groups according as we place the centers on the set of bisectors meeting in I, I_1, I_2, or I_3.

Take the case where the centers are to lie on AI, BI, CI. Let the radii of the required circles be xr, yr, zr, positive in the direction of r. The conditions that the cricles touch one another externally are (since $a/r = \cot B/2 + \cot C/2$)

(1) $$y \cot B/2 + z \cot C/2 \pm 2\sqrt{yz} = \cot B/2 + \cot C/2,$$

(2) $$z \cot C/2 + x \cot A/2 \pm 2\sqrt{zx} = \cot C/2 + \cot A/2,$$

and

(3) $$x \cot A/2 + y \cot B/2 \pm 2\sqrt{zx} = \cot A/2 + \cot B/2,$$

from which $\sqrt{x}$, $\sqrt{y}$, $\sqrt{z}$ may be found. Since by reversing two of the ambiguities we only change the sign of the square root of one of the unknowns, they furnish not eight but two cases as regards x, y, and z, according as an odd or even number of them are minus signs. We may therefore take them all plus or all minus. To solve the equations, eliminate the constant terms between (1) and (2), and also between (1) and (3), obtaining

(4) $$(\sqrt{x} \cos A/2 \pm \sqrt{z} \sin A/2)^2 = (\sqrt{y} \cos B/2 \pm \sqrt{z} \sin B/2)^2$$

and

(5) $$(\sqrt{x} \cos A/2 \pm \sqrt{y} \sin A/2)^2 = (\sqrt{z} \cos C/2 \pm \sqrt{y} \sin C/2)^2,$$

which give the ratios of x, y, and z for the 8 cases corresponding to the set of internal bisectors. If we take the positive square roots in (4) and (5), we find, using the plus signs,

$$x : y : z = (1 + u)^{-2} : (1 + v)^{-2} : (1 + w)^{-2} \tag{6}$$

and, using the minus signs,

$$x : y : z = u^2(1 + u)^{-2} : v^2(1 + v)^{-2} : w^2(1 + w)^{-2} \tag{7}$$

where

$$u = \tan A/4, \qquad v = \tan B/4, \qquad w = \tan C/4;$$

and hence

$$1 - \Sigma u - \Sigma uv + uvw = 0. \tag{8}$$

By (1), the actual values of x, y, and z are $\dfrac{(1+v)(1+w)}{2(1+u)}$, etc., in the case of (6) and $\dfrac{u(1+v)(1+w)}{2vw(1+u)}$, etc., in the case of (7). From these two solutions we can get, not only the 8, but all the 32, by replacing A, B, C by $A + l\pi$, $B + m\pi$, $C + n\pi$, where $l + m + n$ is a multiple of 4, since these angles apply equally well to the triangle, and they leave the relation (8) unaltered.

To get a practical construction, let us denote by $\rho_1, \rho_{11}, \rho_{12}, \rho_{13}; \rho_2, \rho_{21}, \rho_{22}, \rho_{23}, \cdots$, the in- and ex-radii of AFI, BDI, CEI. Then

$$\rho_{11} = \frac{1 + u}{2} r, \qquad \rho_{12} = \frac{1 + u}{2u} r, \text{ etc.}$$

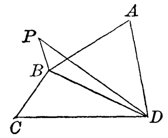
Hence in the case of (6), the radii of the required circles are the three fourth proportionals to ρ_{11}, ρ_{21}, ρ_{31} in different orders; and in the case of (7) the fourth proportionals to ρ_{12}, ρ_{22}, ρ_{32}. The remainder of the set of 8 cases may be solved by using other combinations of the ρ's, while all the 32 cases may be similarly treated by means of the triangles AF_1I_1, etc.

495. Proposed by N. P. PANDYA, Sojitra, India.

A point P moves so that the quadrilateral $PBCD$ is half of a given quadrilateral $ABCD$. Find the locus of P.

SOLUTION BY J. W. BALDWIN, University of Michigan.

In general the triangles ABD and BCD will not be equal. Let ABD be the larger of the two. Then we are to have triangle BCD + triangle PBD equal to half of the given quadrilateral $ABCD$ for all positions of P. That is, the triangle PBD must have a constant area; and having a fixed base BD must have a constant altitude, the distance from P to BD or BD produced. Hence the locus of P is a line parallel to BD. In case triangle ABD = triangle BCD the locus of P is the line of which BD is a segment and two sides PB and PD of the quadrilateral $PBCD$ fall in this line.



Also solved by W. J. THOME, G. W. HARTWELL, WILLIAM HOOVER, S. W. REAVES, W. R. RANSOM, J. W. CLAWSON, and the PROPOSER, some solvers using analytic methods and one using trilinear, perpendicular coördinates.

CALCULUS.

407. Proposed by PAUL CAPRON, Annapolis, Maryland.

A coffee pot in the form of a conical frustum, 10 inches high, with a lower base 8 inches in diameter and an upper base 6 inches in diameter, is held on a slant so that the lower base is barely covered by the coffee within, and the upper base is barely uncovered. How much coffee does the pot contain?

III. SOLUTION BY WILLIAM W. JOHNSON, Cleveland, Ohio.

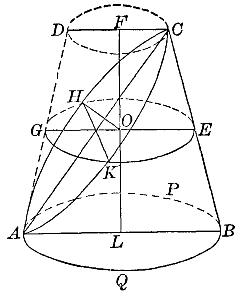
The quantity of coffee in the pot is equal to the volume of the conical ungula C-$APBQ$ formed by tipping the conical frustum on a slant, according to the conditions of the problem. In the figure let $AL = R$, $DF = r$, $FL = h$, $GO = x$, $FO = y$. Then,

$$\text{Volume } C\text{-}APBQ = V = \int_r^R S dy = \frac{h}{R-r}\int_r^R \left(x^2 \operatorname{arc}\cos\frac{2Rr-(R+r)x}{(R-r)x}\right.$$

$$\left. - \frac{1}{(R-r)^2}[2Rr-(R+r)x]\sqrt{4Rr(R+r)x-4Rrx^2-4R^2r^2}\right)dx,$$

in which, area $HKE = S$, and

$$dy = \left(\frac{h}{R-r}\right)dx;$$



since

$$y = \frac{h(x-r)}{R-r}.$$

Integrating, we obtain

$$V = \frac{\pi h R^{\frac{3}{2}}}{3(R-r)}(R^{\frac{3}{2}} - r^{\frac{3}{2}}).$$[1]

Putting $h = 10$, $R = 4$, and $r = 3$, we get

$$V = \frac{80}{3}(8 - 3\sqrt{3})\pi = 234.895 \text{ cu. in.}$$

409. Proposed by B. J. BROWN, Victor, Colorado.

Integrate the equation

$$\frac{\partial^2 z}{\partial x \partial y} + \frac{1}{x+y}\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) - \frac{2}{(x+y)^2}z = 0.$$

Solution by O. S. Adams, Coast and Geodetic Survey, Washington, D. C.

Let

(1) $$z = \frac{u}{(x+y)^2}.$$

Then, computing $\partial z/\partial x$, $\partial z/\partial y$ and $\partial^2 z/\partial x\partial y$, and substituting in the given equation, we may write the result as follows,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} - \frac{u}{x+y}\right) - \frac{1}{x+y}\left(\frac{\partial u}{\partial x} - \frac{u}{x+y}\right) - \frac{2u}{(x+y)^2} = 0.$$

Now let

(2) $$\frac{\partial u}{\partial x} - \frac{u}{x+y} = v.$$

Then

(3) $$u = \frac{(x+y)^2}{2}\frac{\partial v}{\partial y} - \frac{x+y}{2}v.$$

Computing $\partial u/\partial x$ and $u/(x+y)$, and substituting in (2), we have

$$\frac{\partial u}{\partial x} - \frac{u}{x+y} = \frac{(x+y)^2}{2}\frac{\partial^2 v}{\partial x\partial y} + \frac{x+y}{2}\frac{\partial v}{\partial y} - \frac{x+y}{2}\frac{\partial v}{\partial x} = v;$$

or

$$\frac{\partial^2 v}{\partial x\partial y} + \frac{1}{x+y}\frac{\partial v}{\partial y} - \frac{1}{x+y}\frac{\partial v}{\partial x} - \frac{2v}{(x+y)^2} = 0.$$

This equation may be written in the form,

$$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{v}{x+y}\right) - \frac{1}{x+y}\left(\frac{\partial v}{\partial x} + \frac{v}{x+y}\right) = 0.$$

Finally, let

(4) $$\frac{\partial v}{\partial x} + \frac{v}{x+y} = w.$$

Then

$$\frac{\partial w}{\partial y} - \frac{w}{x+y} = 0; \quad \text{or} \quad \frac{1}{x+y}\frac{\partial w}{\partial y} - \frac{w}{(x+y)^2} = 0,$$

[1] See Finkel's *Solution Book*, p. 319.

which may be written

$$\frac{\partial}{\partial y}\left(\frac{w}{x+y}\right) = 0, \text{ or on integrating, } \frac{w}{x+y} = \phi'''(x),$$

$\phi'''(x)$ being an arbitrary function of x.

Substituting the value of w thus given in (4), we find

$$(x+y)\frac{\partial v}{\partial x} + v = (x+y)^2\phi'''(x), \qquad \text{or} \qquad \frac{\partial}{\partial x}[(x+y)v] = (x+y)^2\phi'''(x).$$

Integrating with respect to x, the right-hand member being integrated twice by parts, we find

$$(5) \qquad (x+y)v = (x+y)^2\phi''(x) - 2(x+y)\phi'(x) + 2\phi(x) + 2\psi(y),$$

$\psi(y)$ being an arbitrary function of y with the factor 2 included for symmetry.

From (3), we have

$$(x+y)\frac{\partial v}{\partial y} - v = \frac{2u}{x+y}.$$

If we differentiate (5) with respect to y, we obtain

$$(x+y)\frac{\partial v}{\partial y} + v = 2(x+y)\phi''(x) - 2\phi'(x) + 2\psi'(y).$$

By substituting

$$(x+y)\frac{\partial v}{\partial y} = v + \frac{2u}{x+y},$$

we find

$$2v + \frac{2u}{x+y} = 2(x+y)\phi''(x) - 2\phi'(x) + 2\psi'(y);$$

or

$$(x+y)v + u = (x+y)^2\phi''(x) - (x+y)\phi'(x) + (x+y)\psi'(y).$$

Substituting the value of $(x+y)v$ from (5), we obtain

$$u = (x+y)[\phi'(x) + \psi'(y)] - 2\phi(x) - 2\psi(y).$$

Hence, from (1),

$$z = \frac{\phi'(x) + \psi'(y)}{x+y} - \frac{2\phi(x) + 2\psi(y)}{(x+y)^2};$$

or

$$z = (x+y)\left[\frac{\partial}{\partial x}\frac{\phi(x)}{(x+y)^2} + \frac{\partial}{\partial y}\frac{\psi(y)}{(x+y)^2}\right].$$

412. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

Given a triangular field of sides a, b, and c. Show how to divide the field into two equal parts by a straight fence so that the cost of the fence is the least.

Solution by W. R. Ransom, Tufts College.

Suppose $a > b > c$. Connect points on the sides a and b so as to form a new triangle with the same included angle and sides of length xa and yb. If this triangle has half the area of the given triangle, $y = 1/2x$. The square of length of the fence is then

$$\varphi(x) = a^2x^2 + \frac{b^2}{4x^2} - (a^2 + b^2 - c^2).$$

The minimum value of $\phi(x)$ is found to be $\frac{1}{2}[c^2 - (a-b)^2]$, where $x = \sqrt{b/2a}$ and $y = \sqrt{a/2b}$. From $a < b + c$ and $c < b$ we get $a < 2b$; hence $y < 1$. Similarly, $x < 1$ so that this fence lies wholly within the given field. Moreover, this fence is less than the minimum fence built across either of the other corners; for, multiply $a < b + c$ by $2(b - c)$ which is positive, and add $c^2 - b^2$, then, $c^2 - b^2 + 2ab - 2ac < b^2 - c^2$, or $c^2 - (b-a)^2 < b^2 - (c-a)^2$. Similarly, interchanging a and b, $c^2 - (b-a)^2 < a^2 - (c-b)^2$. This minimum triangle is isosceles, the sides along a and b being each $= \sqrt{ab/2}$.

Also solved by O. S. ADAMS, C. N. SCHMALL, PAUL CAPRON, and HORACE OLSON.

MECHANICS.

324. Proposed by H. S. UHLER, Yale University.

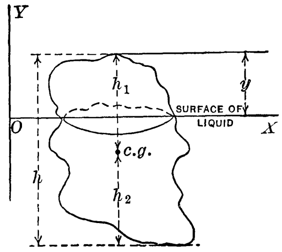
A rigid body of any shape is at rest in a neutral liquid which is also at rest and has an indefinitely great volume. The body is so situated that the free surface of the liquid is tangent to it at its highest point (or points). All the space above the liquid is filled with a neutral stagnant fluid whose density is not greater than the density of the liquid. Show that the work done in raising (pure translation) the body very slowly until the interface of the two fluids is tangent to it at its lowest point (or points) is expressible by the formula $mgh - gV(\rho_1 h_1 + \rho_2 h_2)$, where $m \equiv$ mass of body, $V \equiv$ volume of body, $\rho_1 \equiv$ mean density of lower medium, $\rho_2 \equiv$ density of upper medium, $h_1 \equiv$ distance of center of mass of the displaced liquid below the free surface in the initial position of the body, $h_2 \equiv$ elevation of center of mass of displaced fluid above the interface in final position of the body and $h = h_1 + h_2$. (Neglect surface-tension, etc.)

SOLUTION BY J. B. EPPES, Annapolis, Md.

Take the axes indicated in the figure. Let the highest point of the body at any time be a distance y above the surface of the liquid, and let A be area of the cross-section of the body made by the surface of the liquid.

Then the mass of the fluid displaced is $\rho_2 \int_0^y A dy$ and the mass of the liquid displaced is $\rho_1 \left(V - \int_0^y A dy \right)$.

Hence, the downward force is $g\left[m - \rho_2 \int_0^y A dy - \rho_1 \left(V - \int_0^y A dy \right)\right]$. Then, the total work in raising the body, is



$$\int_0^h \left[mg - \rho_2 g \int_0^y A dy - \rho_1 g \left(V - \int_0^y A dy \right) \right] dy = mgh$$

$$- \rho_2 g \int_0^h dy \int_0^y A dy - \rho_1 g V h + \rho_1 g \int_0^h dy \int_0^y A dy.$$

Now assume A as a function of y to be of the form

$$A \equiv B + Cy + Dy^2 + \cdots.$$

Then,

$$\int_0^h dy \int_0^y A dy = \int_0^h \left(By + \frac{Cy^2}{2} + \frac{Dy^3}{3} + \cdots \right) dy = \frac{Bh^2}{2} + \frac{Ch^3}{2\cdot 3} + \frac{Dh^4}{3\cdot 4} + \cdots$$

$$= \left[Bh^2 + \frac{Ch^3}{2} + \frac{Dh^4}{3} + \cdots \right] - \left[\frac{Bh^2}{2} + \frac{Ch^3}{3} + \frac{Dh^4}{4} + \cdots \right]$$

$$= h \int_0^h A dy - \int_0^h yA dy = hV - Vh_1 = Vh_2.$$

Hence, work $= mgh - \rho_2 g V h_2 - \rho_1 g V h + \rho_1 g V h_2 = mgh - gV(\rho_1 h_1 + \rho_2 h_2)$.

325. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

The lever of a testing machine is c feet long, and is poised on a knife edge a feet from one end and b feet from the other, and in a horizontal line above which the beam is symmetrical. The beam is m inches deep at the knife edge and tapers uniformly to a depth of n inches at each end; the width of the beam is the same throughout its length. Find the distance of the center of gravity of the beam from the knife edge.

SOLUTION BY W. J. THOME, University of Detroit.

It is a well known fact that if the parallel sides of a trapezoid are g and s, and the perpendicular distance between them is p, then the distance from the greater side g to the center of gravity of the trapezoid is

$$\frac{p}{3}\,\frac{(g+2s)}{(g+s)}.$$

Let t be the constant width of the lever, in inches;
d, the density of the material of the lever,
M, the total mass of the lever;
m_1, the mass of that part of the lever from the knife edge to the end distant a feet away;
m_2, the mass of that part of the lever from the knife edge to the end distant b feet away;
$\overline{X}$, the distance in feet from the knife edge to the center of gravity of the entire lever;
$\bar{x}_1$, the distance in feet from the knife edge to the center of gravity of the mass m_1; and
x_2, the distance in feet from the knife edge to the center of gravity of the mass m_2.
Then, using the foot as our unit of length, we have

$$M\overline{X} = m_2\bar{x} - m_1\bar{x}_1;$$

or

$$\left[\frac{1}{2}\left(\frac{m+n}{12}\right)a\frac{t}{12}d+\frac{1}{2}\left(\frac{m+n}{12}\right)b\frac{t}{12}d\right]\overline{X}$$

$$=\left[\frac{1}{2}\left(\frac{m+n}{12}\right)b\frac{t}{12}d\right]\left[\frac{b}{3}\frac{\left(\frac{m+2n}{12}\right)}{\left(\frac{m+n}{12}\right)}\right]-\left[\frac{1}{2}\left(\frac{m+n}{12}\right)a\frac{t}{12}d\right]\left[\frac{a}{3}\frac{\left(\frac{m+2n}{12}\right)}{\left(\frac{m+n}{12}\right)}\right].$$

Hence,

$$\overline{X} = \frac{1}{3}\,\frac{(m+2n)}{(m+n)}\,(b-a),$$

which is to be measured in the direction of whichever is the greater, b or a.

NUMBER THEORY.

240. Proposed by J. W. NICHOLSON, Louisiana State University.

If the roots of $x^3 - px + q = 0$ are rational, prove that $4p - 3x^2$ is a perfect square.

SOLUTION BY WILLIAM E. PATTEN, Government Institute of Technology, Shanghai, China.

Let the roots of $x^3 - px + q = 0$ be x_1, x_2, x_3. Then

$$x_1 + x_2 + x_3 = 0;$$

and

$$x_1x_2 + x_2x_3 + x_3x_1 = -p.$$

Therefore,

$$\begin{aligned} 4p - 3x_1^2 &= -4(x_1x_2 + x_2x_3 + x_3x_1) - 3x_1^2 \\ &= -4x_2x_3 - 4x_1(x_2+x_3) - 3x_1^2 \\ &= -4x_2x_3 - 4(-x_2-x_3)(x_2+x_3) - 3(-x_2-x_3)^2 \\ &= -4x_2x_3 + (x_2+x_3)^2 = (x_2-x_3)^2. \end{aligned}$$

Since the coefficient of the leading term of the given equation is unity, and the roots are rational, they are also integral.

Therefore, $x_2 - x_3$ is an integer, and $4p - 3x_1^2$ is a perfect square.

Similarly for the other roots:

$$4p - 3x_2^2 = (x_3 - x_1)^2, \qquad \text{and} \qquad 4p - 3x_3^2 = (x_1 - x_2)^2.$$

Also solved by H. N. CARLETON, J. E. ROWE, L. C. MATHEWSON, A. H. HOLMES, ELIJAH SWIFT, O. S. ADAMS, J. ROSENBAUM, LOUIS CLARK, E. E. WHITEFORD, and H. S. UHLER.

241. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

If $a^2 + b^2 = c^2$, where a, b, and c are integers, then prove that abc will be a multiple of 60.

SOLUTION BY ALBERT G. CARIS, Defiance College.

From the well-known theorem that any integral solution of $a^2 + b^2 = c^2$ may be put in the form $2xy$, $x^2 - y^2$, $x^2 + y^2$, where x and y are integers, it follows immediately that

$$abc = 2xy(x - y)(x + y)(x^2 + y^2).$$

Showing that this product is always a multiple of 3, 4, and 5 is sufficient to prove the proposed problem.

I. We may write $x = 2m - 1$, or $2m$ and $y = 2n - 1$, or $2n$, where m and n are integers. Whenever $x = 2m$, or $y = 2n$, abc is a multiple of 4. In all other cases $x = 2m - 1$ at the same time that $y = 2n - 1$ and consequently, $x - y$, $x + y$, and $x^2 + y^2$ are all multiples of 2. Therefore abc is always a multiple of 4.

II. We may write $x = 3r - 1$, $3r$, or $3r + 1$ and $y = 3s - 1$, $3s$, or $3s + 1$, where r and s are integers. Whenever $x = 3r$, or $y = 3s$, abc is a multiple of 3. The combinations resulting from all other cases may be arranged in the two groups below:

Group A		*Group B*	
x	y	x	y
$3r - 1$	$3s - 1$	$3r - 1$	$3s + 1$
$3r + 1$	$3s + 1$	$3r + 1$	$3s - 1$

From combinations of group A the factor $x - y = 3(r - s)$. From combinations of group B the factor $x + y = 3(r + s)$. Therefore abc is always a multiple of 3.

III. We may write

$$x = 5u - 2,\quad 5u - 1,\quad 5u,\quad 5u + 1,\quad \text{or}\quad 5u + 2$$

and

$$y = 5v - 2,\quad 5v - 1,\quad 5v,\quad 5v + 1,\quad \text{or}\quad 5v + 2,$$

where u and v are integers. Whenever $x = 5u$, or $y = 5v$, abc is a multiple of 5. The combinations resulting from all other cases may be arranged in the three groups below:

Group C		*Group D*		*Group E*	
x	y	x	y	x	y
$5u - 2$	$5v - 2$	$5u - 2$	$5v + 2$	$5u \pm 2$	$5v \pm 1$
$5u - 1$	$5v - 1$	$5u - 1$	$5v + 1$		
$5u + 1$	$5v + 1$	$5u + 1$	$5v - 1$	$5u \pm 1$	$5v \pm 2$
$5u + 2$	$5v + 2$	$5u + 2$	$5v + 2$		

All combinations of group C make $x - y = 5(u - v)$. All combinations of group D make $x + y = 5(u + v)$. All combinations of group E make $x^2 + y^2 = 5(5u^2 \pm 4u + 5v^2 \pm 2v + 1)$ or $5(5u^2 \pm 2u + 5v^2 \pm 4v + 1)$. Therefore, abc is always a multiple of 5. Hence, abc is always divisible by 60.

Also solved by S. A. COREY, A. H. HOLMES, HORACE OLSON, J. W. CLAWSON, J. ROSENBAUM, J. E. ROWE, H. C. FEEMSTER, H. N. CARLETON, W. J. THOME, ELIJAH SWIFT, and E. E. WHITEFORD.

243. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Determine rational values of x that will render $x^3 + px^2 + qx + r$ a perfect cube. Apply the result to $x^3 - 8x^2 + 12x - 6$.

SOLUTION BY E. B. ESCOTT, Kansas City, Mo.

Let $x^3 + px^2 + qx + r = (x + a)^3$. Expanding and collecting terms

$$(3a - p)x^2 + (3a^2 - q)x + (a^3 - r) = 0. \tag{1}$$

If x is rational, the discriminant must be a square, that is,

$$(3a^2 - q)^2 - 4(3a - p)(a^3 - r) = d^2. \tag{2}$$

One simple solution is $a = p/3$.

Then from (1), we get

$$x = -\frac{a^3 - r}{3a^2 - q} = -\frac{\frac{p^3}{27} - r}{\frac{p^2}{3} - q} = -\frac{p^3 - 27r}{9(p^2 - 3q)}. \tag{3}$$

Other solutions of (2) may be found by Euler's method. Expanding (2), we have

$$-3a^4 + 4pa^3 - 6qa^2 + 12ra + (q^2 - 4pr) = d^2.$$

If we know one solution, we can usually find as many as desired.

Applying the above results to $x^3 - 8x^2 + 12x - 6$, we have $p = -8$, $q = 12$, $r = -6$. By (3), we have $x = 25/18$. (2) becomes $(3a^2 - 12)^2 - 4(3a + 8)(a^3 + 6) = d^2$; or, expanded,

$$-3a^4 - 32a^3 - 72a^2 - 72a - 48 = d^2.$$

$a = -2$ is a solution, which gives $x = \pm 1$. Let $a = b - 2$. Then,

$$-3b^4 - 8b^3 + 48b^2 - 72b + 16 = d^2 = (kb^2 + lb + m)^2.$$

Expanding,

$$(k^2 + 3)b^4 + 2(kl + 4)b^3 + (2km + l^2 - 48)b^2 + 2(lm + 36)b + (m^2 - 16) = 0.$$

Let $m = 4$, $l = -9$, $k = -33/8$. Then

$$b = -\frac{2(kl + 4)}{k^2 + 3} = -\frac{752}{183} \quad \text{and} \quad a = -\frac{1118}{183}.$$

Substituting in (1), we have

$$x = \frac{51287}{8235} \quad \text{and} \quad \frac{1895}{549}.$$

Another value of a is -8, whence $x = 23/4$ and $11/2$. Also

$$a = -\frac{632}{361}, \quad -\frac{74}{13}, \quad -\frac{5738}{1381}, \quad \cdots.$$

The corresponding values of x are easily found.

Also solved by ELIJAH SWIFT, NORMAN ANNING, J. A. COLSON, and J. E. ROWE.

Editorial Note.—The problem in effect is to find rational points on the cubic

$$x^3 + px^2 + qx + r - y^3 = 0.$$

If the discriminant of $x^3 + px^2 + qx + r$ vanishes this cubic is rational and an infinity of rational points may be found by the method of section by a line through the double point. If not it is a cubic of one branch and genus 1. The theory of the rational points has been discussed by H. Poincaré, *Liouville Journal*, 1901, 161.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence.

NEW QUESTION.

34. Given the mixed integral and functional equation

$$\int_{x=0}^{x=x} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{x}{2}\right) + f(x)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

DISCUSSIONS.

I. CONCERNING AN ILLUSTRATION OF A CERTAIN NECESSARY CONDITION IN MINIMIZING A DEFINITE INTEGRAL WITH DISCONTINUOUS INTEGRAND.

By PAUL R. RIDER, Washington University.

The problem of minimizing a definite integral in which the integrand possesses a finite discontinuity along a plane curve has been considered by Bliss and Mason.[1] A certain necessary condition that they have discovered may be stated as follows:

If a curve C, of parameter t, which passes from the fixed point P_0 to the fixed point P_2 and crosses a curve D, whose equations are $x = x(\alpha)$, $y = y(\alpha)$, minimizes the sum of the two integrals

$$I = \int_{t_0}^{t_1} F(x, y, x', y')dt, \qquad i = \int_{t_1}^{t_2} f(x, y, x', y')dt,$$

the first integral to be taken from the point P_0 to the curve D and the second from the curve D to the point P_2, then at P_1, the point of intersection of C and D, the relation

$$x_\alpha(F_{x'} - f_{x'}) + y_\alpha(F_{y'} - f_{y'}) = 0$$

must hold. The argument of x_α, y_α is the value of α for the point P_1; the arguments of $F_{x'}$, $F_{y'}$ are the values of x, y, x', y' on the curve C_{01} at the point P_1, and those of $f_{x'}$, $f_{y'}$ are the values of the same variables on the curve C_{12} at the point P_1.

This condition is well illustrated in a problem (Calculus 389) which recently appeared in the MONTHLY:

> A man is at the southeast corner of a section of land and wishes to walk to the opposite corner in the least possible time. A circular track with a radius of $1/\pi$ miles is located in the section tangent to the west line at a point 120 rods from the south line. Conditions are such that he can walk at the rate of 4 miles an hour inside the track and 3 miles an hour outside the track. What course should he choose and how long is it? See the figure.

Solutions of the problem were given in the MONTHLY, Vol. 23, No. 24 (April, 1916), p. 125 by H. S. Uhler, and p. 127 by A. H. Holmes.

[1] BLISS and MASON, "A Problem of the Calculus of Variations in which the Integrand is Discontinuous," *Transactions of the American Mathematical Society*, Vol. 7 (1906), pp. 325–336.

Since the time is to be the least possible, the sum of the three integrals

$$I_{01} = \int_{t_0}^{t_1} \frac{1}{3} \sqrt{x'^2 + y'^2}dt, \qquad i = \int_{t_1}^{t_2} \frac{1}{4} \sqrt{x'^2 + y'^2}dt, \qquad I_{23} = \int_{t_2}^{t_3} \frac{1}{3} \sqrt{x'^2 + y'^2}dt$$

must be minimized.

Here the curve D is the circle, the joint P_0 being the origin,

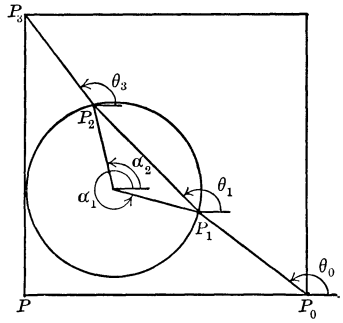
$$x = \frac{1}{\pi} \cos \alpha - 1 + \frac{1}{\pi}, \qquad y = \frac{1}{\pi} \sin \alpha + \tfrac{3}{8},$$

and the curve C is the broken line $P_0P_1P_2P_3$. (See figure.) There are two points of discontinuity, P_1 and P_2. For simplicity let us set

$$F = \tfrac{1}{3} \sqrt{x'^2 + y'^2}, \qquad f = \tfrac{1}{4} \sqrt{x'^2 + y'^2}.$$

It follows that the equations

$$(1) \qquad \begin{aligned} x_\alpha(F_{x'} - f_{x'}) + y_\alpha(F_{y'} - f_{y'}) &= 0, \\ x_\alpha(f_{x'} - F_{x'}) + y_\alpha(f_{y'} - F_{y'}) &= 0 \end{aligned}$$



must be satisfied at P_1 and P_2 respectively. But

$$x_\alpha = -\frac{1}{\pi} \sin \alpha, \qquad y_\alpha = \frac{1}{\pi} \cos \alpha,$$

$$(2) \qquad F_{x'} = \frac{x'}{3\sqrt{x'^2 + y'^2}}, \qquad F_{y'} = \frac{y'}{3\sqrt{x'^2 + y'^2}}.$$

$$f_{x'} = \frac{x'}{4\sqrt{x'^2 + y'^2}}, \qquad f_{y'} = \frac{y'}{4\sqrt{x'^2 + y'^2}}.$$

If θ is the angle that a given line makes with the positive x-axis (taking PP_0 as the positive x-direction), then

$$F_{x'} = \tfrac{1}{3} \cos \theta, \qquad F_{y'} = \tfrac{1}{3} \sin \theta, \qquad f_{x'} = \tfrac{1}{4} \cos \theta, \qquad f_{y'} = \tfrac{1}{4} \sin \theta.$$

If we use these values, and those of x_α, y_α taken from (2), equations (1) assume the form

$$(3) \qquad \begin{aligned} -\sin \alpha_1(\tfrac{1}{3} \cos \theta_0 - \tfrac{1}{4} \cos \theta_1) + \cos \alpha_1(\tfrac{1}{3} \sin \theta_0 - \tfrac{1}{4} \sin \theta_1) &= 0, \\ -\sin \alpha_2(\tfrac{1}{4} \cos \theta_1 - \tfrac{1}{3} \cos \theta_2) + \cos \alpha_2(\tfrac{1}{4} \sin \theta_1 - \tfrac{1}{3} \sin \theta_2) &= 0. \end{aligned}$$

The meanings of α_1, α_2, θ_0, θ_1, θ_2 are evident from the accompanying figure.

From the solutions of the problem by Professor Uhler and Mr. Holmes are obtained the values

$$\alpha_1 = 360^\circ - 15^\circ\, 45'\, 25'', \qquad \theta_0 = 90^\circ + 52^\circ\, 45'\, 48'',$$
$$\alpha_2 = 180^\circ - 75^\circ\;\; 6'\, 19'', \qquad \theta_1 = 90^\circ + 44^\circ\;\; 5'\, 13'',$$
$$\theta_2 = 90^\circ + 37^\circ\;\; 1'\, 51''.$$

Whence,

$$\begin{array}{llll} \sin\alpha_1 = -.271557, & \cos\alpha_1 = .962422, & \sin\theta_0 = .605109, & \cos\theta_0 = -.796143, \\ \sin\alpha_2 = .966400, & \cos\alpha_2 = -.257044, & \sin\theta_1 = .718285, & \cos\theta_1 = -.695749, \\ & & \sin\theta_2 = .798312, & \cos\theta_2 = -.602245. \end{array} \tag{4}$$

If we substitute these numerical values in the first equation of (3) the left side becomes

$$.271557(-\tfrac{1}{3}\times.796143+\tfrac{1}{4}\times.695749)+.962422(\tfrac{1}{3}\times.605109-\tfrac{1}{4}\times.718285)=.003532.$$

Substituting the values (4) in the second equation of (3), we get for its left side

$$-.966400(-\tfrac{1}{4}\times.695750+\tfrac{1}{3}\times.602245)-.257044(\tfrac{1}{4}\times.718285-\tfrac{1}{3}\times.798312)=-.003668.$$

Thus it is seen that the values (4) approximately satisfy conditions (1).

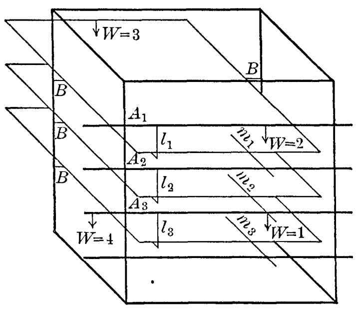
It is to be noted that in Professor Uhler's solution use is made of the law of refraction. Since this law is a special case of the Bliss-Mason condition, it follows that if Professor Uhler's results are used, the foregoing serves as a check on his solution.

II. Relating to an Equation Balance.

By E. L. Rees, University of Kentucky.

The equation balance described below, though less simple in design and construction than some others, offers, in the opinion of the writer, many compensating advantages.

This machine consists of a frame supporting two sets of levers. The levers of one set, which we shall call the scale levers, have their fulcrums at the points marked A in the figure. Just back of, and on a level with, each scale lever, except the top one, is a platform lever supported at its center on the knife edges marked B. These two sets of levers are connected in the following manner: A point one unit from the fulcrum on the first scale lever is joined to the first platform lever by a connecting rod l_1; similarly the second scale lever is connected to the second platform lever, etc. On each platform lever is a movable arm (m_1, m_2, m_3) resting on the corresponding scale lever. This arm passes through a slot in the scale lever so that all vertical forces may be transmitted through the arm to the scale lever.



The following example will serve to illustrate both the principle and the operation of the machine.

Let us solve the cubic equation $2x^3 - 3x^2 + x - 4 = 0$. Having balanced all the levers, we place on the first scale lever a weight 2, representing the first coefficient, in the equation. We then place the movable arms (m_1, m_2, m_3) in the same vertical line with this weight. Calling

its lever arm x, we see from the moments that a force of $2x$ is transmitted through the connecting rod l_1, and the arm m_1, to the point immediately below on the second scale lever. We now place a weight 3, representing the second coefficient, on the rear end of the first platform lever so that it will exert a negative or upward force on the scale lever. We now have a force $2x - 3$ applied to the second scale lever at a distance x from the fulcrum. This in turn is transferred to the third scale lever as a force of $(2x - 3)x$ to which we add a force of one unit (third coefficient) by attaching a weight 1 to the arm m_2 at this point. We also place a weight 4 (constant in equation) on the third scale lever one unit to the left of the fulcrum since the constant is negative. The resulting moment will then be $[(2x - 3)x + 1]x - 4 = 2x^3 - 3x^2 + x - 4$. We now move the weights and arms back and forth on the scale levers, keeping them in the same vertical line, until the levers are balanced. The moment will then be zero and the length of the lever arm, which is seen to be the root of the equation, may be read from the scale on the first lever. This of course gives us only positive roots. Negative roots may be found by changing the signs of the roots of the equation.

Obviously the number of scale levers corresponds to the degree of the equation of highest degree that can be solved. The figure represents a balance which may be used for solving quadratics, cubics, and quartics. In this machine the unit was taken as two inches and the working interval from zero to ten inches, thus giving roots up to five units. As the mechanical details have not yet been perfected only the general plan of construction has been given.

The satisfactory results obtained from this crudely constructed balance with which the writer experimented give assurance that a carefully constructed machine, with proper lengths of levers and unit determined experimentally, will yield results of considerable precision.

III. Relating to the Real Locus Defined by the Equation $x^y = y^x$.

By Philip Franklin, College of the City of New York.

In the September number of the Monthly (Vol. XXIII, pp. 233–237) in a discussion of the real locus defined by the equation $x^y = y^x$, E. J. Moulton raised the question of the continuity of the locus in the third quadrant.

Having met this function in another connection, I was led to consider it in the following manner:

Putting $y = mx$ in $x^y = y^x$, $x^m = mx$, from which

$$x = m^{1/(m-1)} \qquad \text{and} \qquad y = m^{m/(m-1)}.$$

The curve is easily plotted from these equations or from the equation derived from them, $r = \sec\theta \tan \theta^{1/\tan(\theta-1)}$.

In the first quadrant the curve is continuous since we obtain points on the curve for all positive values of m.

In the second and fourth quadrants the curve is discontinuous, since, in general, the $(m - 1)$th root of a negative number does not have a real value.

In the third quadrant points are obtained only for those values of m for which $m^{1/(m-1)}$ has a real negative value, which is not true in all cases; *e. g.*, when m is an even integer. The curve is, therefore, discontinuous in the third quadrant.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Indiana.

Dr. H. A. SAYRE, professor of mathematics at the University of Alabama, died on Dec. 2, 1916.

Dr. HENRY GUNDER, formerly professor of mathematics at Findlay College, died on Nov. 25 at the age of seventy-nine years.

The National Education Association, under the presidency of Dr. R. J. ALEY, president of the University of Maine, will meet this year at Portland, Oregon, July 7–14. The Department of Superintendence met in Kansas City on February 26 to March 3, 1917.

"On the rational, integral invariants of nilpotent algebras" is the title of a paper by Dr. OLIVE C. HAZLETT, in the December, 1916, number of the *Annals of Mathematics*.

The November number of *Genetics* contains an application of mathematics to some problems of animal husbandry under the title "Some breeding properties of the generalized Mendelian population," by E. N. WENTWORTH and B. L. REMICK, of the Kansas State Agricultural College.

The annual meeting of the Pittsburgh Section of Mathematics Teachers of the Middle States and Maryland Association was held at the University of Pittsburgh, Saturday, January 27, under the presidency of Professor C. S. ATCHISON, of Washington and Jefferson College. Papers were presented on the following subjects: "Prominent deceased mathematicians," by J. A. SILVER; "Recreations in geometry," by J. W. MORRISON; and abstracts from KEYSER's "The Human Worth of Rigorous Thinking," by W. F. LONG.

Volume 2, No. 11, *Proceedings of the National Academy of Science*, contains mathematical contributions by Professor E. V. HUNTINGTON on "A set of independent postulates of cyclic order," and by Professor HENRY BLUMBERG on "Certain general properties of functions."

The call has been issued for the second annual meeting of the Ohio Section for the first week in April. The Executive Committee consists of Professor T. M. Focke, Case School of Applied Science, chairman, Professor C. C. Morris, Ohio State University, and Professor G. N. Armstrong, Ohio Wesleyan University, secretary.

"Calculation of the first thirty-two Eulerian numbers from central differences of zero" is the title of a paper in *The Quarterly Journal of Pure and Applied Mathematics*, Vol. 47, 1916, pp. 103–126, by S. A. JOFFE, New York City. The

Eulerian number E_n is the coefficient of $x^{2n}/(2n)!$ in the development of sec x in a power series of x. The first twenty-seven Eulerian numbers were computed by Dr. GLAZIER, *Quarterly Journal*, Vol. 45. Mr. JOFFE has verified the calculations of Dr. GLAZIER for the first twenty-seven Eulerian numbers, and has added five more numbers of the series.

The Annual Register of the American Mathematical Society for the year 1916 shows a membership of 732; during the year the attendance at general and sectional meetings numbered 490; and 205 papers were presented. The treasurer's report shows a balance of $10,198.38. The library contains 5,377 volumes.

School and Society has collected data showing that the universities of the United States granted 607 doctorates during the academic year 1915–1916. Of this number 332 degrees were granted in the sciences, 34 being in mathematics. The number receiving the doctorate in mathematics is not large, being only slightly more than 5 per cent. of the total, and is evidently not nearly sufficient to supply the demand for high-class instructors in the colleges and universities of America. If distributed geographically over the United States, but two doctors in mathematics could be supplied to every three states of the Union. The following table taken from the *Bulletin* of the American Mathematical Society shows the fourteen institutions granting the doctorates in mathematics with the names of those receiving the degree. BRYN MAWR: Mary G. Haseman; CALIFORNIA: T. A. Pierce, A. R. Williams; CHICAGO: A. F. Carpenter, A. M. Harding, W. L. Hart, J. O. Hassler, A. Henderson, A. L. Nelson, S. W. Reaves, A. R. Schweitzer, Pauline Sperry, Mary E. Wells, C. H. Yeaton; CLARK: P. Leyzerah; COLUMBIA: P. H. Linehan, F. J. McMackin; CORNELL: L. C. Cox, J. V. DePorte; HARVARD: P. M. Batchelder, R. W. Brink, A. L. Miller, N. Miller; JOHNS HOPKINS: F. D. Murnaghan, J. R. Musselman, C. H. Rawlins; MICHIGAN: W. Van N. Garretson; PENNSYLVANIA: J. R. Kline, J. H. Weaver; PRINCETON: J. W. Alexander, R. E. Gilman; VIRGINIA: E. S. Smith; WISCONSIN: T. M. Simpson; YALE: G. H. Light.

The following appointments and promotions have been announced: Dr. D. F. BARROW, of the University of Georgia, has been appointed instructor in mathematics in the Sheffield Scientific School; Dr. A. L. MILLER and Mr. R. B. BARNARD have been appointed instructors in mathematics at the University of Michigan; Dr. E. T. BELL has been promoted to an assistant professorship of mathematics at the University of Washington; Mr. C. W. WESTER has been appointed assistant professor of mathematics at the Iowa State Teachers College; Dr. P. H. LINEHAN has been promoted to an assistant professorship of mathematics at the College of the City of New York; Assistant Professor J. F. REILLY has been promoted to an associate professorship of mathematics at the University of Iowa; Dr. R. E. ROOT has been promoted to a professorship of mechanics and engineering mathematics at the U. S. Naval Academy; Assistant Professor T. FORT, of the University of Michigan, has accepted the professorship of mathematics and head of the department at the University of Alabama.

Summer Sessions.

In the March and April issues of the Monthly it is proposed to give a synopsis of graduate and undergraduate courses in mathematics offered by various universities and colleges during the coming summer. The importance of providing serious summer work, continuing from six to twelve weeks, is being recognized by a very large number of institutions. In so far as known to the writer, the first summer session of any university in this country was organized in 1890; since that date a large number of the institutions have taken on this broader field of usefulness, and many are providing summer curricula as complete as those offered during the traditional college year. Below are enumerated the mathematics courses so far as announcements reached the Monthly in time for this issue:

Cornell University. Summer session, July 9–Aug. 17. By Professor V. Snyder: Foundations of elementary mathematics, five hours; Solutions by ruler and compass, five hours.—By Professor C. F. Craig: Advanced calculus, six hours.—By Professor F. W. Owens: Projective geometry, six hours. Courses in algebra, trigonometry, analytic geometry and elementary calculus will be given.

The University of Colorado. Summer session, June 25–Aug. 4. By Professor A. Cohen (Johns Hopkins University): Differential Equations; Introductory course in analysis.—By Professor B. F. Finkel (Drury College): The teaching of mathematics; Fundamental concepts of mathematics; Least squares; Fourier's series.—By Dr. G. H. Light: Algebra; Trigonometry; Differential equations; Differential geometry.—Instructors have not yet been provided for courses in theory of equations, definite integrals, and theory of functions of a complex variable.

Indiana University. Summer session, June 14–August 10. By Professor S. C. Davisson: Theory of surfaces, five hours; Advanced calculus, five hours; Teaching of mathematics, two hours.—By Professor D. A. Rothrock: Calculus of variations, five hours; Solid analytic geometry, three hours.—By Professor U. S. Hanna: Differential equations, ten hours; Theory of functions, five hours. Courses in algebra, trigonometry, analytic geometry and elementary calculus are also offered.

University of Michigan. Summer session, July 2–Aug. 24. By Professor W. W. Beman: Differential equations; Teachers' course in algebra and geometry. —By Professor J. L. Markley: Functions of a complex variable; Advanced algebra.—By Professor W. B. Ford: Advanced calculus; Theory of potential.—By Professor L. C. Karpinski: History of mathematics.—By Professor J. W. Bradshaw: Advanced analytics; Projective geometry.—By Dr. H. C. Carver: Mathematical theory of finance, insurance and statistics. Courses are offered in elementary and college algebra, trigonometry, analytic geometry, and elementary calculus. Courses in modern physical and practical astronomy are offered by Professors Hussey, Curtiss and Dr. Kiess, of the department of astronomy.

UNIVERSITY OF KANSAS: Summer session, first term, June 7–July 18. By Professor U. G. MITCHELL: Higher algebra, three hours; Teachers' course, three hours.—By Professor E. B. STOUFFER: Differential equations, three hours; Differential calculus, three hours: Analytics of the line and circle, two hours.—By Professor H. E. JORDAN: Mathematical analysis, five hours. Second term, July 19–Aug. 15. By Professor J. N. VAN DER VRIES: Mathematical theory of investment, two hours. Courses are also announced in solid geometry, algebra and trigonometry.

THE UNIVERSITY OF CHICAGO. Summer terms, June 18–July 25 and July 26–Aug. 31. The courses offered by Professor MOORE continue during the first term, all others continue until Aug. 31. All advanced courses are given four hours per week. By Professor E. H. MOORE: The spectrum of limited infinite hermitian matrix; Series.—By Professor H. E. SLAUGHT: Differential equations. —By Professor E. J. WILCZYNSKI: Projective differential geometry.—By Professor J. W. A. YOUNG: Solid analytics.—By Professor A. C. LUNN: Relativity; Functions of a complex variable.—By Professor D. N. LEHMER (University of California): Synthetic projective geometry.—By Professor G. D. BIRKHOFF (Harvard University): Ordinary differential equations of the second order. Elementary courses are also offered in college algebra, trigonometry, analytic geometry and differential and integral calculus, and courses in the teaching of mathematics.

NORTHWESTERN UNIVERSITY. Summer session, June 25–Aug. 4. By Professor E. J. MOULTON: Teachers' course; College algebra.—By Dr. C. E. WILDER: Trigonometry; Analytic geometry; Differential calculus.

KANSAS STATE AGRICULTURAL COLLEGE. Summer session, June 23–Aug. 3. By Professor W. H. ANDREWS: Teachers' course; Integral calculus; Trigonometry.—By Professor A. E. WHITE: Differential calculus; Analytic geometry; Solid geometry. Courses are also offered in elementary algebra and plane geometry.

SYRACUSE UNIVERSITY. Summer session, July 8–Aug. 16. By Professor W. H. METZLER: Teachers' course; Solid analytic geometry.—By Professor W. G. BULLARD: Advanced algebra; Modern algebra; Solid geometry.—By Professor F. F. DECKER: Teachers' course in algebra and geometry. Courses are announced in elementary algebra and geometry, trigonometry, analytic geometry, calculus, descriptive geometry, and mechanics.

UNIVERSITY OF TEXAS. First term, June 13–July 25. By Professor M. B. PORTER: Subject matter and teaching of high school mathematics, five hours.—By Professor C. D. RICE and Dr. GOLDIE HORTON: Calculus, 15 hours.—By Professor E. L. DODD: Advanced calculus, 10 hours.—By Professor E. P. R. DUVAL: Differential equations, 5 hours. Courses in solid geometry, college algebra, trigonometry and analytic geometry are also announced. Second term, July 26–Sept. 4. By Dean H. Y. BENEDICT and Mr. H. J. ETTLINGER: 5 hours advanced work and elementary courses.

Notes on the Association.

Arrangements are in progress for the second summer meeting of the Association to be held in Cleveland, Ohio, September 6 and 7. The summer meeting of the Society will immediately precede that of the Association. A joint Committee of Arrangements has been appointed consisting of Professors F. N. Cole, W. D. Cairns, E. V. Huntington, T. M. Focke, A. D. Pitcher, and D. T. Wilson. The Program Committee for the Association consists of Professors C. S. Slichter, L. S. Hulburt, and E. J. Wilczynski.

The only part of the program thus far determined is the retiring address of past President Hedrick, in accordance with the action of the Council in New York, that each retiring president shall give an address at the next following summer meeting of the Association. Suggestions as to the program will be welcomed by the committee, and these may be sent to the chairman, Professor C. S. Slichter, University of Wisconsin, Madison, Wis.

In response to numerous questions, it will doubtless be of general interest to give some data with reference to the distribution of the votes for the officers of the Association at the last annual meeting. There were 405 votes cast, but of these 28 were unsigned, so that the location of 377 voters is definitely known. It is thought that probably most of the unsigned ballots were cast by those who voted in person at New York, but this gives no clue to the location of these voters, except that doubtless a number of them should be credited to New York, thus raising the percentage of this state as listed below.

Of the states having a membership of 40 or more, the percentages of those voting are as follows: Illinois 45%, Massachusetts 44%, Ohio 41%, California 36%, New York 34%, Missouri 33%, Pennsylvania 31%.

Of the states having a membership between 20 and 40, the percentage of those voting are as follows: Colorado 80%, Kansas 50%, Michigan 40%, Indiana 90%, New Jersey 28%, Texas 25%, Maryland 23%, Iowa 22%, Wisconsin 21%.

Of the states having a membership between 10 and 20, the percentages of those voting are as follows: Maine 43%, Virginia 41%, North Carolina 36%, Rhode Island 36%, New Hampshire 33%, Minnesota 31%, Nebraska 31%, Alabama 30%, District of Columbia 29%, Washington 21%, Georgia 21%, Canada 20%, Connecticut 16%, South Dakota 10%.

Of the states having a membership less than 10 the percentages of those voting are as follows: North Dakota 100% (one member), Utah 75%, New Mexico 66%, Arizona 50%, South Carolina 50%, Idaho 50%, Oklahoma 37%, Wyoming 33%, Arkansas, Delaware, Florida, Vermont and West Virginia each 25%, Kentucky 16%, Oregon 14%, Louisiana, Mississippi, Montana, Nevada, and Tennessee each 0%.

These figures show that the voters (and hence also the non-voters) were quite evenly distributed throughout the country. Reference to the charter

membership list will show from these percentages the actual number of voters in each state. The percentage figures, when not exact, are given to the nearest integer.

The distribution of ballots for the presidential candidates is of special interest since the result was so close. These figures show, in general, a very even distribution as between the two candidates. For instance, the presidential votes were equally (or very nearly equally) divided in the following states: Illinois, Massachusetts, Minnesota, Ohio, Pennsylvania, and New Jersey. In some states there was a clear preponderance in favor of Huntington, as in New York and Michigan, while in others there was a like preponderance for Cajori, as in California, Colorado and Missouri, but it would be difficult to discover any general geographical cleavage.

It seems probable that the full significance of the franchise was not generally understood and that the next election will bring out a much larger vote. It is hoped that the above analysis of the figures may contribute toward this end.

Applications for membership in the Association are steadily coming in. The following fourteen, received since the New York meeting, were acted on by the Council by mail vote on February 10, 1917. Ten other applications are now in hand (March 10, 1917).

R. J. Aley, President, University of Maine, Orono, Me.

J. N. Broadlick, Teacher, High School, Pittsburg, Kan.

J. W. Campbell, Lecturer in Mathematics, Wesley College, Winnipeg, Can.

W. E. Etzel, Professor of College Mathematics, College of St. Thomas, St. Paul, Minn.

P. E. Kretzmann, Professor of Mathematics and Science, Concordia (Junior) College, St. Paul, Minn.

Clarence McCormick, Instructor in Mathematics, University of Minnesota, Minneapolis, Minn.

Sister Mary Magna, Head of the Science Department, St. Benedict's College, St. Joseph, Minn.

R. H. Marshall, Assistant Professor of Mathematics, State Normal Manual Training School, Pittsburg Kan.

Flora Porter, Teacher, Nashville, Tenn.

L. W. Reid, Professor of Mathematics, Haverford College, Haverford, Pa.

Evan Thomas, Professor of Mechanics and Mathematics, University of Vermont, Burlington, Vt.

Willis Whited, Engineer of Bridges, Pennsylvania State Highway Department, Harrisburg, Pa.

T. C. Wollan, Professor of Mathematics, Park Region Luther College, Fergus Falls, Minn.

Haverford College, Haverford, Pa., to institutional membership.

Communications.

I. *To the Editors of the* Monthly:

Professor J. W. Young—as chairman of the National Committee on Mathematical Requirements—has requested Professor D. E. Smith and myself to coöperate in the preparation of a report on the criticisms of mathematics, making a critical examination of the grounds of the more prominent and more responsible attacks on mathematics, with a view to determining the criticisms which are clearly not valid, those which are clearly justifiable, and those concerning the validity of which there is reasonable doubt, with a view in the latter case to resolving the doubt if possible.

We desire, with your permission, to bring this matter to the attention of readers of the American Mathematical Monthly, with the hope that they may assist us by bringing to our attention all possible material of value, particularly such as might otherwise escape attention. Communications on the subject may be addressed either to D. E. Smith, Teachers College, Columbia University, or to H. W. Tyler, Massachusetts Institute of Technology, Cambridge.

H. W. Tyler.

II. *To the Editors of the* Monthly:

In the February issue of the Monthly you remark (page 73) concerning "A problem in probability" by C. S. Jackson that "The proof sheets of this article never reached us from the author, having probably been lost in ocean transit." That Mr. Jackson died very suddenly on October 18, 1916, does not seem to have been earlier remarked in this country. An appeal (which has been recently circulated) on behalf of the widow and nine children left by the deceased in very straitened circumstances contains the following sentence: "So passed away one who was universally respected and esteemed, and who was held in great regard by all who knew him, both for his refined and gentle character and high intellectual attainments." The appeal is signed by officers in the Royal Military Academy at Woolwich where Mr. Jackson was mathematical master for more than a quarter of a century.

Charles Samuel Jackson took his B.A. at Cambridge, in 1889 and his M.A. in 1896. In 1894 he became a barrister at Lincoln's Inn. He was the author of a dozen short articles in *Revista de la Sociedad Española, Messenger of Mathematics* and *Mathematical Gazette,* and of the pamphlet entitled "The Calculus as a school subject" in a volume prepared for the International Commission on the Teaching of Mathematics. He was also joint author of five books: 1, with R. M. Milne, "A first statics" (1903); 2, with W. M. Roberts, "A first dynamics" (1909); 3, with H. C. Dunlop, "Slide rule notes" (1913); 4, with W. M. Roberts, "A book of elementary mechanics" (1914); 5, with F. J. W. Whipple and L. Roberts, "A twentieth century arithmetic" (1915).

Professor Jackson was a charter member of the Association.

R. C. Archibald.

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DISCUSSION OF FLUXIONS: FROM BERKELEY TO WOODHOUSE.[1]

By FLORIAN CAJORI, Colorado College.

The first direct statement of Newton's method and notation of fluxions was printed in 1693 in Wallis's *Algebra*. Here and in the *Principia* of 1687 Newton made use of infinitely small quantities, but in his *Quadrature of Curves* of 1704 he declared that "in the method of fluxions there is no necessity of introducing figures infinitely small." No other publication of Newton, printed either before 1704 or after, equalled the *Quadrature of Curves* in mathematical rigor. Here Newton reached his high water mark of rigidity in the exposition of fluxions. By a fluxion, Newton always meant a finite velocity. With one exception, all British writers on the new calculus before the appearance of Berkeley's *Analyst* in 1734 used the Newtonian notation consisting of dots or "prick'd letters," and also Newton's word "fluxion." But strange to say, most of these writers did not use Newton's concepts. They applied the term "fluxion" to the infinitely small quantities of Leibniz,—thus using a home label on goods of foreign manufacture. Of sixteen or more writers in Great Britain during the period of 1693–1734, nine or more call a fluxion an infinitely small quantity; three writers do not define their terms, while only four follow Newton's exposition of 1687 or 1693, involving fluxions as finite velocities and "moments" as infinitely small quantities, or else follow Newton's exposition of 1704, involving fluxions as finite velocities and avoiding infinitely small quantities almost entirely. The nine or more who used fluxions in the sense of infinitely small quantities had no hesitation in dropping quantities from an equation when they were very small in comparison with the other quantities. Altogether these writings contained a medley of philosophical doctrine which presented a great opportunity

[1] A paper read before the Mathematical Association of America at the second annual meeting, New York, December 30, 1916.

for destructive criticism on the part of such a close reasoner and skilful debater as Bishop Berkeley. Before this no mathematical subject, except Zeno's paradoxes on motion, had ever offered itself as a topic for picturesque dialectics; before Berkeley only once was such expert and splendid dialectical energy brought to bear on a fundamental topic in mathematics.

Berkeley's *Analyst* marks a turning point in the history of mathematical thought in Great Britain. His criticisms were not openly accepted by mathematicians of his day; nevertheless such effort was put forth to avoid his objections that in eight years the logical exposition of fluxions was immensely improved.

In the library of Trinity College, Cambridge, there is a marble bust of James Jurin, a noted physician, at one time a student at Trinity. He undertook a defence of Newton and the calculus. Under the pseudonym of "Philalethes Cantabrigiensis," Jurin wrote two long replies to Berkeley, full of noisy rhetoric and giving little that was truly substantial. Berkeley found a second antagonist in John Walton, a professor of mathematics in Dublin, who had a good intuitive grasp of fluxions, but lacked deep philosophical insight and showed himself inexperienced in the conduct of controversies. Walton wrote two replies to Berkeley and an augmentation of his second reply. Altogether this discussion involved eight articles, three by Berkeley, two by Jurin and three by Walton.

Berkeley made some mistakes. One was his failure to see or admit that the Newton of 1704 was not the Newton of 1687 or 1693. Berkeley's contention that no geometrical quantity can be exhausted by division is in consonance with the claim made by Zeno in his *Dichotomy* or his *Achilles*.

In the *Analyst* Berkeley does not refer to Zeno, but according to Berkeley's argument, Achilles could not catch the tortoise. Nor can the modern reader agree with Berkeley in the claim that second or third fluxions are more mysterious than the first fluxion.

Berkeley experienced difficulty in conceiving fluxions as being proportional to the nascent increments or to the evanescent increments. Newton, himself, in his *Principia*, gave expression to the philosophical weakness of this explanation, for, strictly speaking, there is no first or prime ratio, nor is there a last or ultimate ratio. In his second reply to Berkeley Jurin defines a "nascent increment" in the Newtonian fashion as "less than any finite magnitude," also as "an increment just beginning to exist from nothing . . . but not yet arrived at any assignable magnitude how small so ever." Lagrange in a letter to Euler of November 24, 1759, said that he experienced trouble with Newton's exposition, since it considered the ratio of two quantities at the moment when they ceased to be quantities. Lagrange seems to have been convinced, says Jourdain, that the use of infinitesimals was rigorous and used both the infinitesimal method and the method of derived functions side by side, during his whole life. The question arises did Berkeley believe that the calculus of fluxions was capable of rational exposition or not? Two noted mathematicians have vouchsafed opposite opinions on this point. Sir William Rowan Hamilton of quaternion fame says: "On the whole, I think that Berkeley persuaded himself that he was in earnest against

fluxions, especially of orders higher than the first, as well as against matter." To this De Morgan replied: "I have no doubt Berkeley knew that fluxions were sound enough." Berkeley himself said: "I have no controversy about your conclusions, but only about your logic and method." In view of the further fact that Berkeley in the *Analyst* advanced the theory of "Compensation of Errors" we incline to the opinion of De Morgan. The theory of "Compensation of Errors," we may add, was advanced independently by Lagrange and L. N. M. Carnot. According to Philip E. B. Jourdain this theory is found also in Maclaurin's *Fluxions.*

There are four other points in Berkeley's *Analyst* to which we desire to direct attention. First his protestation against dropping quantities because they are comparatively very small. Jurin in his first reply argues in favor of the rejection of infinitesimals. In his second reply, after having received a castigation from Berkeley, Jurin says that this part of his argument was intended for popular consumption, for men such as one meets in London, who, when told that if Sir Isaac Newton were to measure the height of St. Paul's Church by fluxions he would be out not more than one tenth of a hair's breatdh, and when further told that two books had been written in this controversy, would fly into a passion, would make reflections about "somebody's being overpaid," and would use expletives not fit for print.

Secondly, Berkeley's denial of the existence of infinitely small quantities is in conformity with the tenets of the recent school of Weierstrass and Georg Cantor.

Interesting is Berkeley's attack upon Newton's derivation of the *moment* or increment of a rectangle AB, as it is given in the *Principia.* Newton derives this moment by the difference $(A + \frac{1}{2}a)\ (B + \frac{1}{2}b) - (A - \frac{1}{2}a)\ (B - \frac{1}{2}b) = Ab + Ba$, where a and b are assumed to be the increments of the sides. Berkeley argues with conviction that the increment of the rectangle AB is $bA + aB + ab$. Jurin takes the arithmetical mean of the increment $bA + aB + ab$ of the rectangle AB and of the decrement $bA + aB - ab$ of AB and obtains the desired true increment or "moment" as $aB + bA$. Sir William Rowan Hamilton sided with Berkeley against Newton on this point but no eighteenth century mathematician in England admitted the validity of Berkeley's criticism.

Lastly we come to the most fundamental of Berkeley's criticisms of Newton which centers upon what is called Berkeley's *lemma*: If in a demonstration an assumption is made, by virtue of which certain conclusions follow, and if afterward that assumption is destroyed or rejected, then all the conclusions that had been reached by the first assumption must also be destroyed or rejected. Berkeley applied this *lemma* to Newton's mode of deriving the fluxion of x^n as given in the *Quadrature of Curves* of 1704. Newton gives x a finite increment o, expands $(x + o)^n$ by the binomial formula, subtracts x^n and divides the remainder by o. He then lets o be zero and obtains the fluxion nx^{n-1}. Berkeley says that this reasoning is not fair or conclusive. "For when it is said, let the increment be nothing, the former supposition that the increment be something is

destroyed and yet the expression got by that former supposition is retained. By Berkeley's *lemma,* this is a false way of reasoning, "such as would not be allowed of in Divinity."

It is interesting to observe that no British mathematician of the eighteenth century acknowledged the soundness of Berkeley's *lemma* and its application.

Jurin, in his second reply to Berkeley, argues against the *lemma* thus: "You say that if one supposition be made, and be afterwards destroyed by a contrary supposition, then everything that followed from the first supposition is destroyed with it." Not so, when the supposition and its contradiction are made at different times. "Let us imagine yourself and me to be debating this matter in an open field, . . . a sudden violent rain falls . . . we are all wet to the skin . . . it clears up . . . you endeavor to persuade me I am not wet. The shower, you say, is vanished and gone and consequently your wetness must have vanished with it." The first recognition in England of the soundness of Berkeley's *lemma* came in 1803 from Robert Woodhouse, who, in his *Principles of Analytical Calculation,* says that the methods of treating the calculus "all are equally liable to the objection of Berkeley, concerning the *fallacia suppositionis,* or the *shifting of the hypothesis.*" In finding the fluxion of x^n, the binomial expansion is effected "on the express supposition, that o is some quantity, if you take o equal to zero, the hypothesis is, as Berkeley says, shifted and there is a manifest sophism in the process."

After Berkeley terminated his debate with the mathematicians, two mathematicians started a quarrel among themselves. Thus arose the second controversy on fluxions, which is comparatively little known.

Benjamin Robins, a self-educated mathematician, felt that Jurin had not entered a satisfactory defence of Newton, so Robins himself in 1735, published a tract entitled *A Discourse Concerning the Nature and Certainty of Sir Isaac Newton's Method of Fluxions and of Prime and Ultimate Ratios.* Robins makes no reference to Berkeley or Jurin, or to their controversy. He lays the foundation of the calculus upon the concept of a *limit.* He speaks of a *limit* as a magnitude "to which a varying magnitude can approach within any degree of nearness whatever, though it can never be made absolutely equal to it." Here for the first time is the stand taken openly, clearly, explicitly, that a variable can never reach its limit. From the standpoint of debating, this stand is a decided gain, but it is a gain made at the expense of generality. He descends to a very special type of variation which is not the variation encountered in ordinary mechanics; it is an artificial variation which does not permit Achilles to catch the tortoise. But this narrow concept of a limit nevertheless answers very well the needs of ordinary geometry. Robins's tract is remarkable for clearness and soundness of exposition; it is a marked advance in that respect. The use of infinitely small quantities is rigidly excluded. The objections raised by Berkeley against Fluxions did not apply to Robins's exposition. A long account of Robins's *Discourse,* prepared by Robins himself, was published in a London monthly called *The Present State of the Republick of Letters.* In the next number of this monthly appeared an article

by Jurin, under the pseudonym "Philalethes Cantabrigiensis," in which he says that in his debate with Berkeley he adhered strictly to Newton's language, but that some other defenders of Newton (meaning Robins) were guilty of departing from it. Jurin argues that the words *fiunt ultimo æquales* used by Newton in Lemma I of Book I in the *Principia,* mean that the quantities (the inscribed and circumscribed polygons) "do at last become actually, perfectly, and absolutely equal"; in modern phraseology, the limit is reached. Several passages in the writings of Newton are examined and many illustrations are given. Robins prepared a reply to Jurin, and thus a controversy had gotten under way which threatened at one time to become endless. For two years there was a steady stream of articles in the *Republick of Letters* and its successor *The Works of the Learned.* Pemberton entered the controversy during the second year on the side of Robins, but contributed nothing of value. About twenty articles were written, one of which filled one hundred and thirty-six pages. All articles taken together covered over seven hundred printed pages. They were attempts to ascertain what Newton's ideas of *fluxions* and *moments* were, and whether Newton meant that a variable can reach its limit or cannot. And a good part of this material has escaped the attention of mathematical historians until now. The first few articles displayed care and ability, the later articles suffered in scientific value from the excessive heat of controversy. Jurin's articles against Robins are superior to his articles against Berkeley. The debate is the most thorough discussion of the theory of limits carried on in England during the eighteenth century. It constitutes a refinement of previous conceptions. In my judgment Jurin's interpretation of Newton was more nearly correct than that of Robins. The two disputants examined and reëxamined every passage of Newton's printed papers bearing on fluxions. Robins saw in Newton's condensed writings only variables which do not reach their limits; Jurin insisted that Newton permitted variables to reach their limits. Jurin admitted the calculus could be consistently founded upon Robins's idea of a limit, but he also insisted that Robins misrepresented Newton. Jurin's conceptions were quite broad for his time. He said: "Now whether a quantity or a ratio shall arrive at its limit or shall not arrive at it, depends entirely upon the supposition we make of the time during which the quantity or ratio is conceived constantly to tend or approach towards its limit." In other words, whether a variable reaches its limit or not is a matter of choice. We may impose conditions, so that the variable reaches its limit, or conditions under which it does not reach its limit. Thus Jurin was perhaps the first consciously to modify and generalize the limit concept. Modifications and generalizations of this have been going on ever since and are still in progress. A serious difficulty in permitting variables of the kind ordinarily arising in geometry to reach their limits lay in the fact that the imagination is not able to follow the variable through an infinity of steps that lead into the limit. The imagination exhausts itself in the effort. It is right here that Robins's variables which do not reach their limits had a great advantage. Jurin took great pains to devise illustrations of limit-reaching variables, intended to

aid the imagination, though, as he admits, incapable of exhibiting the process "all the way." In one place Jurin says: "Since Mr. Robins is pleased to talk so much about straining our imagination, . . . let us see if we cannot find some plain and easy way of representing to the imagination that actual equality, at which the inscribed and circumscribed figures will arrive with each other, and with the curvilineal figure, at the expiration of the finite time." His procedure amounts to expressing the inscribed and circumscribed polygons as functions of the time, such that the limit is reached in a finite time.

It is interesting that, toward the end of his long debate with Robins, Jurin begins to disavow infinitely small quantities. He brings out the difference between infinitesimals as variables, and infinitesimals as constants. He rejects all quantity "fixed, determinate, invariable, indivisible, less than any finite quantity whatsoever," but he usually admits somewhat hazily a quantity "variable, divisible, that, by a constant diminution, is conceived to become less than any finite quantity whatever, and at last to vanish into nothing."

Soon after the Berkeley onslaught, there appeared nine British texts on fluxions, only one of which was of decidedly inferior type. None of these texts refer to the Jurin-Robins dispute. The latter was not widely noticed. The two thought-compelling publications that were widely read were Newton's *Quadrature of Curves* of 1704, and Berkeley's *Analyst*. The latter tract was always criticized by the mathematicians, yet always held in awe. These two tracts, together with Robins's *Discourse*, and Maclaurin's celebrated work on *Fluxions*, which appeared in 1742, mark the highest point of logical precision reached in England during the eighteenth century. All three of the great sections of the British Isles had contributed to this end: England through Newton and Robins, Ireland through Berkeley, and Scotland through Maclaurin. Maclaurin was familiar with the writings of the other three. He took the Greek demonstrative rigor as a model. In a biography of Maclaurin it is stated that several years before the publication of his fluxions, his demonstrations had been communicated to Berkeley and "Mr. Maclaurin had treated him with the greatest personal respect and civility; notwithstanding which, in his pamphlet on tar-water, he (Berkeley) renews the charge, as if nothing had been done" to remove the logical difficulties. Maclaurin avoided the use of infinitely small quantities, "an infinitely little magnitude being," as he expressed himself, "too bold a postulatum for such a science as geometry." He laid less stress upon the concept of a limit than did Robins and Jurin, and followed more closely the kinematical concepts of Newton. The term velocity had been the subject of dispute between Berkeley and Walton. Maclaurin perceived the difficulty of arguing that variable velocity is a physical fact. He defined the velocity of a variable motion as the space that *would be described* if the motion had continued uniform. He also quotes Barrow: Velocity is the "power by which a certain space may be described in a certain time" and then explains "power" by the consideration of "cause" and "effect" in a way that sounds odd in a work on fluxions. However, when we think of the Thomson-Dirichlet Principle we must acknowledge that the eighteenth century was not

the only time when physical concepts were brought to the aid of mathematical theory. Apparently following Robins, Maclaurin's explanations imply that he does not encourage variables actually to reach their limits. Maclaurin secured his rigor of demonstration at a tremendous sacrifice. His work on *Fluxions* consists of seven hundred sixty-three pages; the first five hundred ninety pages do not contain the notation of fluxions at all; the mode of exposition is rhetorical. This part deals with the derivation of the fluxions of different geometric figures, of logarithms, of trigonometric functions, also with the discussion of maxima and minima, asymptotes, curvature and mechanics, in a manner that the ancients might have adopted and with a verbosity of which the ancients are guilty. The consequence was that the work was not attractive reading. It was much praised and much neglected. Fifty-nine years elapsed before a second edition appeared. As we shall see, the book did not stop disputes on fluxions.

The middle and latter part of the eighteenth century were periods of mediocrity. There appeared a dozen books on fluxions, of which those of WIlliam Emerson and Thomas Simpson were the most noted. Both Emerson and Simpson were self-educated mathematicians, possessing the strength and the weakness usual with such preparation. Emerson returned to the use of infinitely small quantities, but a fluxion was defined as a velocity. This return to the use of infinitely small quantities is noticeable in several English texts of the second half of the century. An old lady once defended Calvinism by saying that if you took away her total depravity you took away her religion. There were mathematicians who believed that if you took away infinitely small quantities you took away all their mathematics. Simpson, in his text of 1750, which is a thorough revision of his text of 1737, avoids the use of infinitely small quantities. His definition of fluxion is as follows: "The magnitude by which any flowing quantity *would be* uniformly increased in a given time, with the generating celerity at any proposed position, or instant (was it from thence to continue invariable), is the fluxion of the said quantity at that position or instant." Substantially this definition of a fluxion was adopted later by Charles Hutton. Simpson dodges the word velocity, and remarks: "If motion in (or at) a point be so difficult to conceive that *some* have gone even so far as to dispute the very existence of motion, how much more perplexing must it be to form a conception, not only of the velocity of a motion, but also infinite changes and affections of it, in one and the same point, where all the orders of fluxions have to be considered." Simpson's definition and treatment of fluxions avoided the fictitious infinitesimals, as well as the perplexing term "velocity." Nevertheless, it did not enjoy security against attack, but was fiercely criticized in the London *Monthly Review*. The critic claimed that it is objectionable to define fluxion as the "magnitude by which any flowing quantity would be uniformly increased," for it was argued, that "in quantities *uniformly* generated, the *fluxion* must be the *fluent* itself, or else a part of it." It was claimed that Simpson's endeavor to exclude velocity "cannot be made intelligible without introducing velocity into it." "Again he

mistakes the *effect* for the *cause;* for the thing generated must owe its existence to something, and this can only be the velocity of its motion, but it can never be the *cause itself,* as his definition would erroneously suggest." This obscure criticism of obscure points in Simpson's exposition initiated a third debate on fluxions which was carried on in the *Ladies Diary* and in ephemeral journals called the *Palladium,* the *Lady's Philosopher,* the *Mathematical Exercises* (edited by John Turner). The debate was carried on between friends of Emerson on one side and friends of Simpson on the other. Emerson and Simpson do not themselves appear in the controversy. The friends of Emerson published in 1752 in London an anonymous pamphlet, entitled *Truth Triumphant or Fluxions for the Ladies, Showing the Cause to be Before the Effect, etc.,* which was criticized by the friends of Simpson as a "scurrilous pamphlet." It contains much that is foolish, a few passages eulogizing the works of Emerson, but also critical considerations which are of some interest and disclose the need of a more satisfactory arithmetical continuum. All in all this debate was carried on upon a much lower scientific plane than the former debates. The debaters represented the rank and file of mathematicians.

In the second half of the century several abortive attempts at arithmetization of the calculus were made. The most worthy of these attempts is due to John Landen, but his analysis is so complicated as to be prohibitive. Towards the latter part of the eighteenth century the efforts at rigorous exposition, which were so conspicuous in the years 1735–1742, slackened more and more. Colin Maclaurin was seldom read and John Robins was altogether forgotten. William Hales's discovery of Robins's *Discourse* in 1804 astonished him as would the discovery of a new work of Archimedes. The first edition of the *Encyclopedia Britannica,* 1771, permitted a "fluxion" to degenerate into an "increment" acquired in "less than any assigned time." The same article on fluxions appeared in the second edition (1779) and in the third edition (1797). In 1801 there was published in London *Agnesi's Analytical Institutions,* which many years earlier had been translated by John Colson from the Italian into English. How Colson's conscience must have troubled him, when a fluxion stood out in his translation as something "infinitely small," may be judged by the consideration that in 1736 he brought out an English translation of Newton's *Method of Fluxions.* With Newton a fluxion always meant a finite velocity. We wonder what Robins and Maclaurin would have thought had they been alive in 1897 and 1901 and read these definitions. What horrible visions would these ghosts of departed quantities have brought to Bishop Berkeley had he been alive!

As we look back over the century we see that the eight years immediately following Berkeley's *Analyst* were *eight great years,* during which Jurin, and especially Robins and Maclaurin made wonderful strides in the banishment of infinitely small quantities and the development of the concept of a limit. Both before and after that period of eight years, there existed in most writings of the eighteenth century in Great Britain, a mixture of Continental and British conceptions of the new calculus, a superposition of British symbols and phraseology

upon the older Continental concepts. The result was a system, destitute of scientific interest. Newton's notation was poor and Leibniz's philosophy of the calculus was poor. That result represents the temporary survival of the least fit of both systems. The more recent international course of events has been in a diametrically opposite direction, namely, not to superpose Newtonian symbols and phraseology upon Leibnizian concepts, but, on the contrary, to superpose the Leibnizian notation and phraseology upon the limit-concept, as developed by Newton, Jurin, Robins, Maclaurin, D'Alembert and later writers.

About the opening of the nineteenth century more recent continental authors began to attract the attention of the English. Extensive accounts appeared in the London *Monthly Review* of Lagrange's *Theory of Functions,* Lacroix's *Differential Calculus,* Carnot's *Reflexions on the Metaphysics of the Infinitesimal Calculus.* These texts were compared with English publications in a way not altogether favoring the English. Finally in 1805 Robert Woodhouse of Caius College, Cambridge, brought out his *Principles of Analytical Calculation* which contained many keen criticisms of both Continental and British mathematicians. Woodhouse is the first English mathematician who had a good word for Berkeley. He said: "I cannot quit this part of my subject without commenting on the *Analyst* and the subsequent pieces, as forming the most satisfactory controversial discussion in pure science that ever yet appeared: into what perfection of perspicuity and logical precision the doctrine of fluxions may be advanced, is no subject of consideration; but view the doctrine as Berkeley found it, and its defects in metaphysics and logic are clearly made out. If for the purpose of habituating the mind to just reasoning . . . I were to recommend a book, it would be the *Analyst.*"

Woodhouse is the forerunner in Cambridge of Babbage, Peacock, and the younger Herschel, in the promotion of the principles of pure D-ism in opposition to the dot-age of the university.

As usually happens in reformations so here there was discarded and lost not only what was antiquated, but also what was meritorious. Robins's *Discourse* of 1735, with its full and complete disavowal of infinitesimals and clear-cut, though narrow, conception of a limit was quite forgotten and D'Alembert's definition was recommended and widely used in England. Now Robins and D'Alembert had the same conception of a limit. Both held the view that variables cannot reach their limits. However, there was one difference; Robins embodied this restriction in his definition of a limit; D'Alembert omitted it from his definition, but referred to it in his explanatory remarks.

Some of the eighteenth century British conceptions possessed great merit. Perhaps no intuitional conceptions available in the study of the calculus are clearer and sharper than *motion* and *velocity.* These ideas offer even now great help in approaching the first study of the calculus. A second point of merit lay in the abandonment of the use of infinitely small quantities. Not all English authors of the eighteenth century broke away from infinitesimals, but those who did were among the leaders: Robins, Maclaurin, Simpson, Vince, and a few others.

From the standpoint of rigor, the treatment of the calculus by these men was far in advance of the Continental. In Great Britain there was achieved in the eighteenth century in the geometrical treatment of fluxions that which was not achieved in the algebraical treatment until the nineteenth century. It was not until after the time of Weierstrass that infinitesimals were cast aside by mathematical writers on the Continent.

Judged by modern standards all eighteenth century expositions of the calculus, even the best British expositions, are defective. As pointed out by Landen and Woodhouse, there was an unnaturalness in founding the calculus upon *motion* and *velocity*. These notions apply in a real way only to dynamics. Moreover, not all continuous curves can be conceived as traceable by the motion of a point. The notion of variable velocity is encumbered with difficulties. Then again, in all discussion of limits during the eighteenth century, the question of the *existence of a limit* of a given sequence was never raised. The word "quantity" was not defined; quantities were added, subtracted, multiplied and divided. Were these quantities numbers, or were they considered without reference to number? Both methods are possible. Which did British authors follow? No explicit answer to this was given. Our understanding of authors like Maclaurin, Rowe and others, is that in initial discussions such phrases as "fluxion of a curvilinear figure" are used in a non-arithmetical sense; the idea is purely geometrical. When later the finding of the fluxions of terms in the equations of curves is taken up, the arithmetical or algebraical conception is predominant. Rarely does a writer speak of the difference between the two. Perhaps

> "His notions fitted things so well
> That which was which he could not tell."

The theory of irrational number caused no great anxiety to eighteenth century workers. Operations applicable to rational numbers were extended without scruple to a domain of numbers which embraced both rational and irrational. There was no careful exposition of the number system used. The modern theories of irrational number have brought about the last stages of what is called the *arithmetization* of mathematics. As now developed in books which aim at rigor the notion of a limit makes no reference to quantity and is a purely ordinal notion. Of this mode of treatment the eighteenth century had never dreamed.

NOTE ON SOME APPLICATIONS OF A GEOMETRICAL TRANSFORMATION TO CERTAIN SYSTEMS OF SPHERES.[1]

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In a paper in Volume VI of the *Proceedings of the Edinburgh Mathematical Society*, Professor Allardice considers the transformation *in plano:*

[1] Presented to the San Francisco Section of the American Mathematical Society, April 12, 1913.

Let l be a fixed straight line, C any plane curve, t a tangent to C meeting l in X and making an angle θ with l; then C', the transformed curve of C, is the envelope of a straight line t' through X making an angle ϕ with l, where θ and ϕ are connected by the relation

$$\tan \tfrac{1}{2}\phi = k \tan \tfrac{1}{2}\theta.$$

The length of the tangent from any point in l to C remains unaltered by the transformation. A similar method is applicable in space and the following paper gives some of the results in its applications to certain systems of spheres. The method of transformation is as follows:

Let α be a fixed plane, S any surface, and β a plane tangent to S intersecting α in the line i and making with α an angle θ. If β' be a plane through i making with α an angle ϕ, determined by the relation

(1) $$\tan \tfrac{1}{2}\phi = k \tan \tfrac{1}{2}\theta,$$

then the envelope of β' will be defined as the transformed surface of S, or, more simply, the "*transform of S.*"

From the given relation, $\tan \frac{1}{2}\phi = k \tan \frac{1}{2}\theta$, and the identity,

$$\tan \phi = \frac{2 \tan \phi/2}{1 - \tan^2 \phi/2},$$

we find that

(2) $$\tan \phi = \frac{2k \tan \theta}{\pm (1 - k^2)\sqrt{1 + \tan^2 \theta} + 1 + k^2}.$$

It is evident from the nature of this relation that for every value of k there are two values of $\tan \phi$, according as we consider the angle $\alpha\beta$ as θ or $(\theta + \pi)$.

For simplicity, we will use Cartesian coördinates to find the equation of the transform of a given sphere from its equation, and we will take the xy-plane as the plane of transformation, α. Let θ be the angle formed by the xy-plane and the plane $ux + vy + wz - 1 = 0$, and ϕ be the angle formed by the xy-plane and the plane $ux + vy + w_1z - 1 = 0$. (Since the intercepts of both planes on the x-axis and on the y-axis are equal, respectively $1/u$ and $1/v$, they intersect the xy-plane in the same line, as required by the transformation.)

From the formula for the angle between two planes,

$$\tan^2 \delta = \frac{(BC' - B'C)^2 + (CA' - C'A)^2 + (AB' - A'B)^2}{(AA' + BB' + CC')^2},$$

we have

$$\tan^2 \theta = \frac{u^2 + v^2}{w^2} \qquad \text{and} \qquad \tan^2 \phi = \frac{u^2 + v^2}{w_1^2}.$$

Substituting these values in equation (2) and solving for w, we obtain

(3) $$w = \frac{(1 + k^2)w_1 \pm (1 - k^2)\sqrt{u^2 + v^2 + w_1^2}}{2k},$$

or,

$$(4)\quad w^2 = \frac{2(1 + k^4)w_1^2 + (1 - k^2)^2(u^2 + v^2) \pm 2w_1(1 - k^4)\sqrt{u^2 + v^2 + w_1^2}}{4k^2}.$$

Now, let the center of the sphere lie in the z-axis and its equation will be

$$(5)\qquad x^2 + y^2 + z^2 + 2nz + d = 0.$$

Expressing the condition that the plane $ux + vy + wz - 1 = 0$ be tangent to this sphere, we have

$$(6)\qquad (d - n^2)u^2 + (d - n^2)v^2 + dw^2 + 2nw + 1 = 0,$$

which is the tangential equation of the sphere.

Substituting the values of w and w^2 found in (3) and (4) and then simplifying and factoring the result, we have the equation of the transform of the given sphere

$$(7)\qquad \begin{aligned} &\left[\{(k^2 + 1)^2d - 2n^2(k^4 + 1) + 2n(k^4 - 1)\sqrt{n^2 - d}\}(u^2 + v^2) + 4dk^2w_1^2\right. \\ &\quad \left.+ 4k^2 + 4k\{n(k^2 + 1) + (k^2 - 1)\sqrt{n^2 - d}\}w_1\right] \times \left[\{(k^2 + 1)^2d\right. \\ &\quad - 2n^2(k^4 + 1) - 2n(k^4 - 1)\sqrt{n^2 - d}\}(u^2 + v^2) + 4dk^2w_1^2 + 4k^2 \\ &\quad \left.+ 4k\{n(k^2 + 1) - (k^2 - 1)\sqrt{n^2 - d}\}w_1\right] = 0. \end{aligned}$$

This equation breaks up into two equations which are the tangential equations of two spheres, and may be written in the form

$$(8)\qquad A_1(u^2 + v^2) + Cw_1^2 + 2N_1w_1 + D = 0;$$

$$(9)\qquad A_2(u^2 + v^2) + Cw_1^2 + 2N_2w_1 + D = 0.$$

The Cartesian equations of these forms are

$$(10)\qquad (CD - N_1^2)x^2 + (CD - N_1^2)y^2 + A_1Dz^2 + A_1C + 2A_1N_1z = 0;$$

$$(11)\qquad (CD - N_2^2)x^2 + (CD - N_2^2)y^2 + A_2Dz^2 + A_2C + 2A_2N_2z = 0,$$

where, from the actual values of the coefficients involved,

$$(CD - N_1^2) = A_1D, \qquad \text{and} \qquad (CD - N_2^2) = A_2D.$$

Simplifying and substituting, we have finally

$$(12)\quad 4k^2(x^2 + y^2 + z^2) + 4k\{n(k^2 + 1) + (k^2 - 1)\sqrt{n^2 - d}\}z + 4dk^2 = 0;$$

$$(13)\quad 4k^2(x^2 + y^2 + z^2) + 4k\{n(k^2 + 1) - (k^2 - 1)\sqrt{n^2 - d}\}z + 4dk^2 = 0:$$

the equations of two spheres whose centers lie in the z-axis. Hence, in general, a sphere is transformed into two spheres whose centers lie in a line perpendicular to the plane of transformation and passing through the center of the given sphere.

Now, let ρ represent the radius of the given sphere and δ the distance of its

center from the plane of transformation; also, let r' and r'' and d' and d'' represent respectively the radii and distances from the plane of transformation of the centers of the transformed spheres. The equations of the latter may then be written

$$x^2 + y^2 + z^2 + [-\delta(k + 1/k) + (k - 1/k)\rho]z + d = 0; \tag{14}$$

$$x^2 + y^2 + z^2 + [-\delta(k + 1/k) - (k - 1/k)\rho]z + d = 0: \tag{15}$$

whence the resulting relations:

$$\begin{aligned} d' + r' &= (\delta - \rho)k; \qquad & d'' + r'' &= (\delta + \rho)k; \\ d' - r' &= (\delta + \rho)/k; \qquad & d'' - r'' &= (\delta - \rho)/k. \end{aligned} \tag{16}$$

We will next consider for what values of k the transformed spheres will degenerate into points. It is evident that we must consider each case separately.

In the first case we have

$$r' = \left| -\frac{1}{2k}\{(\delta + \rho) - k^2(\delta - \rho)\} \right| = 0,$$

whence

$$k^2 = (\delta + \rho)/(\delta - \rho). \tag{17}$$

Here we have two real values for k, provided the given sphere does not cut the plane of transformation; and a zero or infinite value for k if the given sphere is tangent to that plane. The corresponding values of the distance are

$$d' = \mp \sqrt{\delta^2 - \rho^2},$$

or in terms of the original coefficients,

$$d' = \mp \sqrt{d}. \tag{18}$$

In the second case, by similar methods, we find that

$$k^2 = (\delta - \rho)/(\delta + \rho), \tag{19}$$

and

$$d'' = \mp \sqrt{\delta^2 - \rho^2}, \qquad \text{or,} \qquad d'' = \mp \sqrt{d}. \tag{20}$$

It follows that, in general, a sphere may be transformed into a point for any one of four values of k, the points being the same in pairs, and all four being equidistant from the plane of transformation, one pair on each side. It is to be especially noted that while each of the two transformed spheres may degenerate into points, both spheres cannot become points at the same time, unless either $\rho = 0$, or $\delta = 0$; *i. e.*, when the given sphere is itself a point and this point may be transformed into its image with respect to the plane of transformation, or when the center of the given sphere lies in the plane of transformation and the points are imaginary. We may also note that a point can be transformed into only one sphere.

We have already seen, equations (12) and (13), that for the same value of k a given sphere may be transformed into two distinct spheres. These two spheres evidently result from the two values of ϕ, according as we consider the angle $\alpha\beta$ as θ or $(\theta + \pi)$. In case the angle is θ we will say that the transformed sphere is traced out by a *direct* movement, and where the angle is $(\theta + \pi)$ we will say that it is traced out by an *indirect*, or *inverse*, movement. It is evident that if we transform two spheres, *both* directly, or *both* inversely, the *direct* common tangent planes to the two spheres become common tangent planes to the transformed spheres, while if one sphere is transformed directly and the other inversely, the *transverse* common tangent planes become common tangent planes to the transformed spheres.

Forming the equation of the radical plane of the given sphere and either of its transforms, we have

$$z = 0,$$

which is also the plane of transformation. It follows that the plane of transformation is the radical plane of the given sphere and its transforms, and that the length of a tangent line from any point in the plane of transformation to the sphere is unaltered by the transformation. Hence the distance between the points of contact of common tangent planes to two spheres remains unaltered. This is the important property of the transformation. By transforming several spheres into points at the same time, relations and properties of spheres may be obtained directly from known relations between points.

For two or more spheres to be transformed simultaneously into points, it is evidently sufficient that the values of k^2 be equal. Considering the case of two spheres with radii ρ' and ρ'' and with the distances of their centers from the plane of transformation, δ' and δ'', respectively, we have

$$k^2 = (\delta' - \rho')/(\delta' + \rho') \quad \text{and} \quad k^2 = (\delta'' - \rho'')/(\delta'' + \rho'');$$

whence

$$(\delta' - \rho')/(\delta' + \rho') = (\delta'' - \rho'')/(\delta'' + \rho''), \quad \text{or} \quad \delta'/\delta'' = \rho'/\rho''.$$

This is the condition that the plane of transformation pass through the direct center of similitude of the two spheres. Using the reciprocal values for k^2 we obtain the same result. Finally, let

$$k^2 = (\delta' - \rho')/(\delta' + \rho') \quad \text{and} \quad k^2 = (\delta'' + \rho'')/(\delta'' - \rho'');$$

whence

$$\delta'/\delta'' = - \rho'/\rho'':$$

the condition that the plane of transformation pass through the inverse center of similitude of the two spheres. We conclude that two spheres may be transformed simultaneously into points if the plane of transformation contain a center of similitude of the two spheres. Likewise, if the plane of transformation contain any one of the four axes of similitude of three spheres, these spheres will all be transformed into points by the same transformation. Further, if the plane

of transformation be any one of the eight planes of similitude of four spheres,[1] the four spheres may be transformed into points simultaneously; and, finally, any number of spheres which have a common plane of similitude may be so transformed. Two spheres on the same side of the plane of transformation will be transformed, either both directly, or both inversely; while of two on opposite sides, one will be transformed directly, the other inversely.

By giving k every possible value a given sphere may be transformed into an infinite series of spheres which, by virtue of the properties of the plane of transformation as the radical plane of a sphere and its transforms, is cut orthogonally by a system of spheres. Then, since the given sphere is one of the series, if we transform the whole system for the same value of k, it will transform into itself.

It may be noted at this point that the results here obtained by analytic methods may readily be obtained by synthetic methods, and in the sequel they will be interpreted in accordance with the particular problem under consideration. In all problems which follow it will be necessary to associate only those common tangent planes and distances between points of contact of common tangent planes to two spheres which determine the centers of similitude in the planes of similitude used as planes of transformation. Thus we will associate with the plane of similitude passing through the six direct centers of similitude of four spheres the external common tangent planes of each pair of spheres, and only the external common tangent planes of these spheres.

Some Applications of the Method.

1. The condition that four planes passing through a point should be tangent to the same sphere is given by the equation

$$\tan \tfrac{1}{2}\phi' \tan \tfrac{1}{2}\phi'' = \tan \tfrac{1}{2}\theta' \tan \tfrac{1}{2}\theta'',$$

where ϕ', ϕ'', θ', θ'' are the angles formed by the given planes and the plane determined by the lines of intersection of the two pairs of opposite planes.

Transform the sphere into a point with the external diagonal plane as the plane of transformation, and let the angles of the transformed planes with this plane of transformation be ϕ and θ, corresponding respectively to the angles ϕ', ϕ'' and θ', θ''. We then have the following relations:

$$\tan \tfrac{1}{2}\phi = k \tan \tfrac{1}{2}\phi'; \qquad \tan \tfrac{1}{2}\theta = k \tan \tfrac{1}{2}\theta';$$

$$\tan \tfrac{1}{2}(\phi + \pi) = k \tan \tfrac{1}{2}\phi''; \qquad \tan \tfrac{1}{2}(\theta + \pi) = k \tan \tfrac{1}{2}\theta''.$$

Eliminating ϕ, θ, k from these equations, we have the given condition.

[1] The twelve centers of similitude of four spheres lie in sets of six in a plane: the six direct centers lie in a plane; the three direct centers of any three spheres lie in a plane with the three inverse centers not paired with them; and any two direct centers, using each sphere only once, lie in a plane with the four inverse centers not paired with them. Thus there are eight planes of similitude, one containing all the direct centers; four containing three direct and three inverse centers; and three containing two direct and four inverse centers.

2. If five spheres have a common plane of similitude they may be transformed into five points by the same transformation. Therefore the distances between the points of contact of the common tangent planes of each pair of spheres satisfy the relation connecting the ten straight lines joining five points in space.

Let d_1, d_2, d_3, $\cdots$ d_{10} be the distances. Then the relation is given by the determinant[1]

$$\begin{vmatrix} 0 & d_1^2 & d_2^2 & d_3^2 & d_4^2 & 1 \\ d_1^2 & 0 & d_5^2 & d_6^2 & d_7^2 & 1 \\ d_2^2 & d_5^2 & 0 & d_8^2 & d_9^2 & 1 \\ d_3^2 & d_6^2 & d_8^2 & 0 & d_{10}^2 & 1 \\ d_4^2 & d_7^2 & d_9^2 & d_{10}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0.$$

3. If, in addition to the conditions of No. 2, the five spheres also touch a sixth sphere, the ten distances will satisfy the additional relation connecting five points lying on the surface of a sphere.

Transforming the five spheres into points as above, they must lie on the surface of the transform of the sixth sphere and will therefore satisfy the additional relation given by the determinant[1]

$$\begin{vmatrix} 0 & d_1^2 & d_2^2 & d_3^2 & d_4^2 \\ d_1^2 & 0 & d_5^2 & d_6^2 & d_7^2 \\ d_2^2 & d_5^2 & 0 & d_8^2 & d_9^2 \\ d_3^2 & d_6^2 & d_8^2 & 0 & d_{10}^2 \\ d_4^2 & d_7^2 & d_9^2 & d_{10}^2 & 0 \end{vmatrix} = 0.$$

4. The radius R of a given sphere S tangent to four given spheres may be found in terms of R, the distance δ of the center of S from the plane containing the six direct centers of similitude of the four spheres, the distances between the points of contact of common tangent planes to pairs of spheres, and the radius and distance from its center to the plane of similitude of any one of the given spheres.

If we transform the four spheres into points by the same transformation, these four points will lie on the transform of S. Hence

$$R' = \frac{1}{2}\sqrt{\frac{(\Sigma aa')[\Pi(-aa' + bb' + cc')]}{\Sigma[a^2a'^2(b^2 + c^2 - a^2 + b'^2 + c'^2 - a'^2)] - (a^2b^2c^2 - \Sigma a^2b'^2c'^2)}},$$

where a, a', etc., are the distances between the points of contact of the common

[1] Cayley, "Collected Mathematical Papers," Volume I, p. 1 et seq.

tangent planes of two pairs of spheres with no sphere in common, and where Σ and π represent cyclo-symmetrical functions only. But

$$R' = \left| \frac{1}{2k} \{\delta(k^2 - 1) \mp (k^2 + 1)R\} \right|,$$

where

$$k^2 = (d_1 - r_1)/(d_1 + r_1), \qquad \text{or} \qquad k^2 = (d_1 + r_1)/(d_1 - r_1),$$

from equations (17), and (19).

Hence

$$R = | - \delta(r_1/d_1) \pm 2\{(\sqrt{d_1^2 - r_1^2})/d\}R' |.$$

5. The locus of the center of a sphere S such that the distances between the points of contact of the common tangent planes to S and four fixed spheres are equal is a straight line perpendicular to a plane of similitude of the four spheres.

Transforming the fixed spheres into points, the transformed spheres S' will form a system of spheres with a common center; namely, the center of the sphere determined by the four points. Moreover, all the spheres S' with real tangents to the four points lie within this sphere; consequently the distance between the points of contact cannot exceed the radius of this sphere, and hence it is a maximum when S' becomes a point; *i. e.*, when the four fixed spheres and S have a common plane of similitude. The theorem follows directly from the fact that the system S' is a system of concentric spheres. There are eight cases for consideration corresponding to the eight planes of similitude of the four fixed spheres.

6. The centers of the spheres tangent to four given spheres lie in pairs on eight straight lines passing through the radical center and perpendicular to one of the planes of similitude of the four spheres.

This is only a special case of the preceding. It is evident that the radical plane of any pair of the tangent spheres is the plane of similitude, which is perpendicular to the line joining the centers of the two spheres. The results of this theorem can be employed in the construction of the spheres tangent to four given spheres.

7. If S is a sphere tangent to four given spheres, which are tangent to one another, the five spheres have common planes of similitude.

Transforming the four spheres into points, the points will coincide, since each sphere touches one of the others. But the four points must lie on S', the transform of S, and therefore S' must also reduce to a point simultaneously with the other four; hence the result.

8. The locus of the centers of all spheres which touch a given sphere and have a common plane of similitude with it is an ellipsoid of revolution.

Let r be the radius and d be the distance from the common plane of similitude of the center of the given sphere, and let ρ and δ be the radius and distance respectively of the variable sphere. Transform the spheres into points for the same value of k and we have

$$d' = \frac{d}{2}(k + 1/k) - \frac{r}{2}(k - 1/k); \qquad \delta' = \frac{\delta}{2}(k + 1/k) - \frac{\rho}{2}(k - 1/k).$$

But, since all the spheres are tangent to the given sphere, the points into which they transform are coincident, and therefore, $d'' = \delta'$. Hence

$$(k - 1/k)(r + \rho) = \left(d + \delta - \frac{4d'}{k + 1/k}\right)(k + 1/k),$$

or, $(r + \rho)/(\delta + c) = e$, where e and c are constants.

Therefore, the locus of the centers of the spheres is the locus of a point, the ratio of whose distances from the center of the given sphere to its distance from a plane parallel to and at the distance c from the common plane of similitude is constant; *i. e.*, an ellipsoid of revolution.

THE ASTROLABE.

By MARCIA LATHAM, Hunter College, New York City.

Among ancient civilized peoples for many centuries the most important instrument used by astronomers, astrologers and surveyors was the astrolabe, or, more properly, the astrolabe planisphere.

The word is derived from two Greek words meaning "to follow the stars," and is therefore applicable in general to any astronomical or astrological instrument. Indeed, the name has been applied to at least three distinct forms. The first of these, better known as the armillary sphere, or *instrumentum armillarum* (instrument of rings), consists of two, three, or more brass circles, representing the circles of the celestial sphere, hinged together in the proper relations, and bearing tubes for sighting the heavenly bodies. An engraving of this may be found on the title page of many an old book on mathematics or astronomy. It was doubtless invented by Hipparchus, and was used by Ptolemy, who described it in the *Almagest*, Book V, Proposition 1. A large wooden ring, used by mariners in the days of Columbus and Vasco da Gama to take the altitude of the sun, was also known as an astrolabe. The most important form, however, was the astrolabe planisphere, or, simply, the planisphere.

The theory of the planisphere was given in the second century of our era by Ptolemy in a treatise on the planisphere, which is not extant in the Greek but survived in the Arabic, from which it was translated by Commandinus.

The Arabs, probably applying the theory of Ptolemy, constructed exceedingly accurate astrolabes. Remains of astrolabes have been found in Assyrian excavations, and the instrument is still of practical value in India and Persia. From Arabia it passed again into Europe, and was used by astronomers and surveyors in Italy, England and other countries until the eighteenth century. Bion wrote a treatise on the astrolabe in 1702, and Boileau mentioned it in a satire written in 1693:

> "Une astrolabe en main, elle a dans sa gouttière
> À suivre Jupiter passé la nuit entière."

The simplest and most available English work on the description and use of the astrolabe is the *Tractatus de Conclusionibus Astrolabii* (*Bred and Mylke for Childeren*) written by Geoffrey Chaucer in 1391 and 1392, in the form of a letter to his "lytle sonne Lowis," then ten years old, who was already familiar with the celestial globe and had expressed a desire for information concerning the astrolabe. Chaucer's work, according to Professor Skeat, was based upon the "Compositio et Operatio Astrolabii," a Latin treatise by Messahallah, an Arabian astronomer of the ninth century.

More careful descriptions are to be found in the geometrical and astronomical treatises by Clavius (1612), Metius (1626 and 1633), Danti (1569), Bruni (1625), Reisch (1503), Gemma Frisius and Deschâles; but the standard authority is Johannes Stoefler (or Stoflerinus), whose *Elucidatio fabricæ ususque astrolabii* was published in 1510. (The copy consulted in preparing this paper was published in Oppenheim in 1524.)

The astrolabe planisphere is essentially the projection of the celestial sphere upon a plane. In most forms the projection is stereographic, that is, the point of sight is at one of the poles of the circle upon whose plane the sphere is projected. Three primitive planes have been used:

1. The plane of the equator, the point of sight being at the south pole. This form is called the "equinoctial astrolabe," and is the one suggested by Ptolemy. It is also known as the "septentrional astrolabe."
2. The plane of the equinoctial colure.
3. The plane of the solstitial colure.

The objection to the equinoctial astrolabe is that the nature of the projection varies with the latitude. To meet this difficulty Gemma Frisius used the second form, which, being independent of the location of the observer, is called the "universal astrolabe," or "Catholicon."

Johannes de Rojas, a Spaniard, projected the sphere orthographically on the plane of the solstitial colure, the result being the "Analemma," which is also a universal astrolabe. There was still the objection that in general the projections of equal arcs of a circle are not equal. To obviate this difficulty de la Hire (1640–1718) used the method of globular projection, on the plane of the meridian. In this method the point of sight is on the axis of the primitive circle, outside the sphere and at a distance from its surface equal to the product of its radius by the sine of forty-five degrees. This was further modified by Parent, but Clavius returned to the form suggested by Ptolemy, and most of his successors have followed his example.

Summarizing, then, we may say that the most important form is the equinoctial or septentrional astrolabe planisphere, suggested by Ptolemy and revived by Clavius, the principle of which is stereographic projection upon the plane of the equinoctial, the point of sight being at the south pole.

The astrolabe was usually made of brass or copper, circular in shape, and from four to seven inches in diameter (some of the Eastern instruments are much larger), and very carefully and beautifully engraved. Chardin tells us that in

Persia the astronomers themselves made them, not entrusting them to ordinary artisans.

The fundamental part is called the *mater* or mother, and is a heavy circular piece with a ring which permits it to assume a vertical position when suspended from the right thumb. One side is the back or *dorsum*, on which rotates the *rule*, a flat bar bearing sights similar to those of a gun. The other side, called the *front* or face, is hollowed out so that any one of several plates, or *tables*, may be fitted into it. Above such a plate lies the *rete* (net), and the *label*, which is simply a pointer; and the whole is fastened in place by a pin or wedge (see Fig. 1).[1]

Fig. 1.

Examining it more closely we find around the rim of the "back" several rings containing respectively the signs of the zodiac, divided into degrees; the months of the year, divided into days; and sometimes the golden letters of the church calendar, or the names of the winds. Fig. 2 is a diagram of the back. *AB* represents the meridian, and *CD* the east and west line, but it should be noted that the observer's right indicates the west and his left the east. Above the line

[1] This astrolabe is in the collection of Professor David Eugene Smith, of Columbia University.

CD are often placed the "curves of the unequal hours." These are circles, each passing through the center of the astrolabe and one of the six points of division of the quadrant, and having its center on the meridian line.

Below *CD* is the "geometric square" (*Quadratus geometricus*), or "square of the shadows," *EGHKL*. *EG, GH, HK* and *KL* are each divided into a number of equal parts. The vertical sides constitute the "*umbra versa*," which corresponds to the shadows cast on a wall by a horizontal staff fastened to the side of the wall, the altitude of the sun being not more than forty-five degrees. The horizontal part corresponds to shadows cast on the ground by a vertical staff when the sun's altitude is not less than forty-five degrees, and is called the "*umbra recta*." The geometric square is in itself a complete instrument, and was so used.

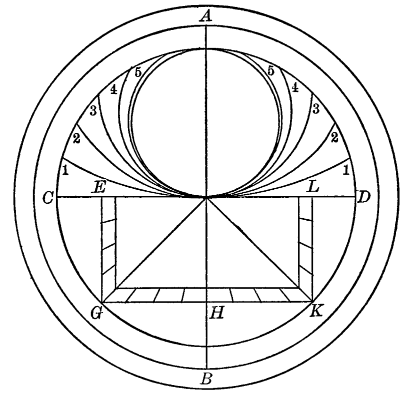


FIG. 2.

Turning the instrument over, we find that the concave part of the face bears three concentric circles, which are the projections of the equator and the tropics, the hole in the center representing the north pole.

The thin plates that fit into this depression constitute the essential part of the astrolabe, and will be described below. The rete, called by the French *l'araigne* (spider) is a plate of filigree metal, consisting of a circle representing one half of the zodiac, and many small tongues of metal, each of which serves to locate some important fixed star. Those within the zodiac have, of course, north latitude, and those without, south latitude. A point projecting from the outer rim of the rete is called the *denticle*. The wedge holding the different parts in place was often in the shape of a horse, and was named the "*equus restringens*."

With the aid of figures 3 and 4 the method of constructing the plates can be easily understood by any one familiar with the principles of stereographic projection.

In figure 3, the largest circle is the projection of the Tropic of Capricorn, *AB* that of the meridian, and *CD* of the east and west line. Arc *AH* is equal to the maximum declination of the sun. *GHM* is the projection of the equator, and *LM* that of the Tropic of Cancer. *AHMN* is therefore that of the ecliptic.

In figure 4 arc *HL* is the complement of the latitude of the place for which the plate is constructed, and *Z* is the projection of the zenith and *Y* that of the nadir. Arc *TK* is equal to arc *HL* and *M* is a point on the oblique horizon. *GMH* is the projection of the principal almucantar, or circle of altitude. Arc *LW* is equal to arc *LX* and circle *EF* is an almucantar. *GHZ* is the projection of the prime vertical. *NP* contains the centers (as *S*) of all azimuths (as *UV*), which must also pass through the zenith.

Below the horizon the plates bear certain "curves of the hours," constructed

as follows: Divide those portions of the tropics and the equator lying below the horizon each into twelve equal arcs, and pass circles through the corresponding points of division.

Lastly, since the instrument was used also for astrological purposes, the plates bear lines dividing the celestial sphere into the twelve astrological "houses." To obtain these, divide the Tropic of Capricorn into twelve equal arcs. Construct a circle through the north point of the horizon, and the points of division respectively nearest the meridian to the right above and to the left below. A similar procedure will determine all the lines separating the twelve houses.

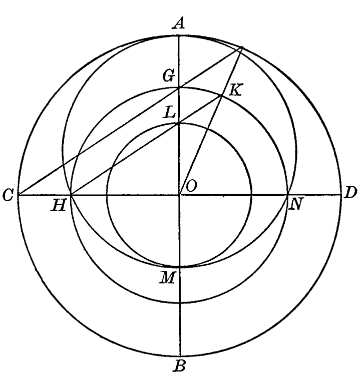


FIG. 3.

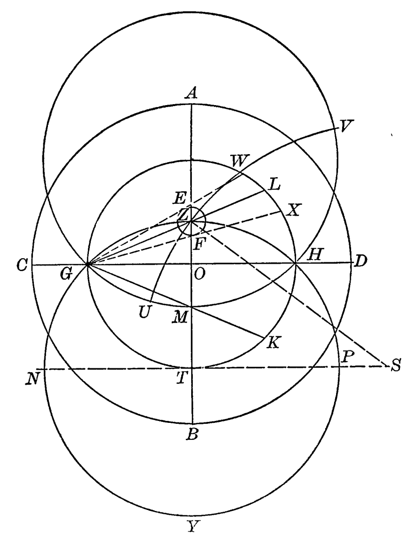


FIG. 4.

Having already explained the construction of the zodiac on the rete, it remains to determine the positions of the tongues locating the fixed stars. The position of a star can easily be fixed when its latitude and longitude are known.

In some cases, particularly among the Orientals, the instrument was constructed for only one latitude; in such instruments the sphere is projected directly upon the mother, and there are no extra plates.

A much longer article than this is to be could well be devoted to an enumeration of the uses of the astrolabe. We shall confine ourselves to mentioning a few of these.

To determine the sign and degree of the zodiac for any day of the year, use the back of the instrument, and lay the label over the day. Its extremity will indicate the corresponding degree.

The back is also used to find the altitude of the sun. Suspend the instrument on the right thumb, and turn the rule until the rays of the sun shine through both sights. The rule will then point out the number of degrees in the sun's altitude. By a similar procedure, the altitude of a star can be found.

The time of the day may be found as follows: First get the sun's altitude; turn the astrolabe over, and revolve the rete (westward or eastward according as the observation is made in the morning or afternoon) until the degree of the zodiac corresponding to that day falls upon the proper circle of altitude. Lay the label across this degree, and it will point out a number of degrees on the margin, from which the time can be calculated.

Other matters as easily determined are the time of dawn and of sunrise for a given day; the meridian altitude of the sun, given the degree of the sun; and, conversely, the degree, when the meridian altitude is known; the latitude of any place; and the cardinal points of the compass.

The curves of the unequal hours on the back may also be used to find the time of the day. Determine the meridian altitude of the sun. Place the rule on the corresponding degree and mark with ink on the rule the point where it cuts the arc of the twelfth hour. Then turn the rule until the sun shines through the sights, and the point marked on the rule will fall on or near the circle corresponding to the hour of the day.

The geometric square was invaluable to the surveyor before the invention of the telescope. The instrument known as "Jacob's Staff" was its only rival in usefulness. By means of it the surveyor could find the height of an accessible or inaccessible tower, by one or two observations; the height of a tower on an inaccessible hill; the distance between two inaccessible points; the distance between two towers not standing on the same plane; the distance of a tower whose height is known; the breadth of a stream; the depth of a pit, or of a valley with sloping walls.

Let it be required, for example, to find the height of a tower when the observer may take any position in front of it. Holding the astrolabe suspended from his right thumb, with the rule fixed in position along the diagonal of the square, the observer will walk backward or forward until he sights the top of the tower along the rule. Then the height of the tower is equal to the distance of the observer from its foot, increased by the height of the observer.

If, however, the observer must not change his position, he may move the rule around until the top of the tower is sighted, and note the point where the rule cuts the side of the square (see Fig. 5). If each side is divided into one hundred equal parts, and the rule cuts off sixty parts on the umbra versa, the height of the tower is equal to sixty hundredths of the distance of the observer from its foot (always increased by the height of the observer). If it cuts off sixty parts on the umbra recta the height is one hundred sixtieths of the distance to its foot. These statements are easily proved by means of similar triangles.

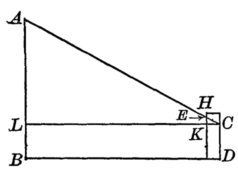


FIG. 5.

If the tower is inaccessible, two observations may be made and the ratio of the height to the distance between the two positions determined. These are some of the simplest applications, but the mathematical principles are the same throughout.

While this article does not give an exhaustive description of the construction and uses of the astrolabe, it is hoped that it may at least serve to show by what means the sciences of astronomy and surveying attained so great a degree of development before the days of Tycho Brahe and Galileo.

A USEFUL PRINCIPLE IN CURVE TRACING.

By ARNOLD EMCH, University of Illinois.

1. The usual elementary methods of curve tracing in rectangular coördinates consist:

(1) In plotting points of the curve by assigning arbitrary values to one of the variables and finding the corresponding values of the other variable;
(2) In determining the tangents at these points by means of the derivatives;
(3) In finding the intersections of the curve with definite straight lines, or other conveniently chosen curves;
(4) In establishing possible properties of symmetry;
(5) In determining concavity and convexity, maxima and minima;
(6) In determining the asymptotes;
(7) In determining possible singular and inflexional points.

This list of tests and different steps to be taken in the investigation of a curve may, of course, be extended according to the difficulties and the nature of the problem.[1]

In ordinary curve tracing little use is made of other algebraic methods than those mentioned above. It is the purpose of this note to show the effectiveness of a certain algebra-geometric method in elementary curve tracing.

2. Restricting ourselves to algebraic curves, let such a curve be represented by the equation

$$F(x, y) = 0 \tag{1}$$

in which $F(x, y)$ is an irreducible polynomial. It is always possible to write $F(x, y) = 0$ in the identical form

$$F(x, y) \equiv \phi_1(x, y)\psi_2(x, y) - \phi_2(x, y)\psi_1(x, y) = 0, \tag{2}$$

where ϕ_1, ϕ_2, ψ_1, ψ_2 are also polynomials in x and y, of which some may reduce to constants. If it should prove to be convenient, we might use

$$F(x, y) - G(x, y) + G(x, y)$$

in place of $F(x, y)$. Thus, as $G(x, y)$ may be any polynomial, we may resolve $F(x, y)$ in an infinite number of ways into the form (2). Equation (2) is evidently the result of the elimination of λ between the simultaneous equations

[1] See PERCIVAL FROST: *An Elementary Treatise on Curve Tracing*, pp. 177–187 (1911).

$$\phi_1(x, y) + \lambda\phi_2(x, y) = 0, \tag{3}$$

$$\psi_1(x, y) + \lambda\psi_2(x, y) = 0. \tag{4}$$

All sets of points (x, y) that satisfy (3) and (4) simultaneously also satisfy (2). Conversely, any set (x_1, y_1) that satisfies (2), with the possible exception of a limited number of sets that satisfy (2) for some particular value of λ, is a simultaneous solution of (3) and (4). As a matter of fact

$$\frac{\phi_1(x_1, y_1)}{\phi_2(x_1, y_1)} = \frac{\psi_1(x_1, y_1)}{\psi_2(x_1, y_1)} = -\lambda_1,$$

for all sets (x_1, y_1) of (2), for which these ratios exist. For a variable parameter λ, (3) and (4) represent two projective pencils of curves, whose product is the given curve (2). Hence the well-known

THEOREM. *An algebraic curve may always be generated by two projective pencils of algebraic curves.*

This theorem forms the basis of Steiner's method of investigation of algebraic curves and is closely connected with Noether's famous theorem concerning the possibility of representing $F(x, y)$ in the form given in (2), when ψ_1 and ψ_2 are given.[1]

3. The possibility of representing a polynomial in the form given in (2) and by Steiner's theorem may be utilized with great advantage in cases where the explicit representation of one variable in terms of the other involves the solution of an equation of higher than the second or third degree. But the method may also be used in any other case and may be embodied in the following principle:

Resolve $F(x, y)$ into the identical form $\phi_1(x, y)\psi_2(x, y) - \phi_2(x, y)\psi_1(x, y)$, in such a manner that the equations

$$\phi_1(x, y) + \lambda\phi_2(x, y) = 0,$$

$$\psi_1(x, y) + \lambda\psi_2(x, y) = 0,$$

assume as simple forms as possible (geometrically). Construct these curves for as many values of λ as seem necessary; then for every value of λ the two curves intersect in points of the given curve.

4. Examples:

Conics. The application of the principle to equations of the second degree leads to the well-known projective theory of conics. ϕ_1, ϕ_2, ψ_1, ψ_2 are linear in x and y, and (3) and (4) become ordinary projective pencils of straight lines.

A quartic: $x^4 - y^4 - x^2y^2 + x^3y - x^3 + y^2 = 0$. This may be written in the form

$$x^3(x + y - 1) - y^2(x^2 + y^2 - 1) = 0, \tag{5}$$

[1] See references to this subject in Pascal's *Repertorium der höheren Mathematik*, Vol. II, first part, 2d ed. (1910), pp. 287–289 and pp. 306–307.

so that the quartic may be generated by the two projective pencils,

$$x^2 + y^2 - 1 - 2\lambda(x + y - 1) = 0, \tag{6}$$

$$- x^3 + 2\lambda y^2 = 0. \tag{7}$$

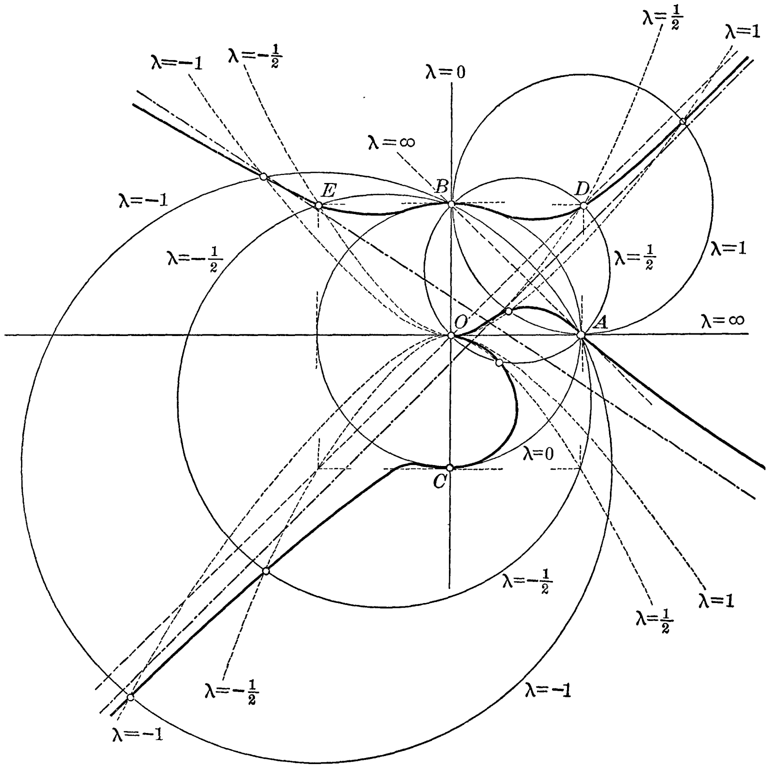
Equation (6) may be written in the form

$$(x - \lambda)^2 + (y - \lambda)^2 = (\lambda - 1)^2 + \lambda^2$$

and represents a circle whose center has the coördinates (λ, λ), and whose radius is $\sqrt{(\lambda - 1)^2 + \lambda^2}$, and which passes through the points $A(1, 0)$ and $B(0, 1)$ (see figure.) Hence, (6) represents a pencil of circles through A and B, with their centers on the bisector of the first and third quadrants. Writing (7) in the form

$$y^2 = \frac{x^3}{2\lambda}, \tag{9}$$

a semi-cubical parabola is obtained, which passes through the points $(2\lambda, \pm 2\lambda)$, which has a cusp at the origin with the x-axis as a tangent, and which may easily



be plotted for a given value of λ. In fact, when μ is any real number, all points $(2\lambda\mu^2, \pm 2\lambda\mu^3)$ lie on (9). For $\lambda > 0$ the cubics lie on the right side of the y-axis;

for $\lambda < 0$ on the left side. For $\lambda = \frac{1}{2}$ the circle (6) passes through the origin and cuts the corresponding cubic $y^2 = x^3$ in two coincident points. Hence, as for this value of λ the circle is not tangent to the cubic at the origin, the quartic must have a singular point at the origin. For $\lambda = \frac{1}{2} - \epsilon$ (ϵ = arbitrarily small positive quantity) the circle does not cut the cubic in real points in the neighborhood of the origin. For $\lambda = \frac{1}{2} + \epsilon$ there are two real points of intersection arbitrarily close to the origin on the right side of the y-axis, one below, the other above the x-axis. From this we conclude that the singularity of the quartic at the origin is a cusp. When $\lambda \doteq 0$, (6) and (7) approach $x^2 + y^2 - 1 = 0$ and $x^3 = 0$, which shows that at B and C (see figure) the circle has three points in common with the quartic. In other words, the unit-circle osculates the quartic at B and C. But at B we have an additional point of intersection of the unit-circle with the line $x + y - 1 = 0$, so that at B the unit-circle has a contact of the third degree with the quartic.

When $\lambda \doteq \infty$, (6) and (7) approach $x + y - 1 = 0$ and $y^2 = 0$, which shows that the quartic touches the line $x + y - 1 = 0$ at A. But another point of intersection, that of the same line with $x^2 + y^2 - 1 = 0$, at A, together with the other two, makes $x + y - 1 = 0$ an inflexional tangent at A. It is easily seen that the quartic also passes through $D(1, 1)$ and $E(-1, 1)$. Any number of other points of the quartic may be obtained as intersections of corresponding curves of the two projective pencils (6) and (7). In the figure, such intersections are shown for $\lambda = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \infty$. The two real asymptotes were determined by the method of substitution.

Lemniscate.[1] The equation of this bicircular rational quartic is

$$(x^2 + y^2)^2 - 2a^2(x^2 - y^2) = 0, \tag{10}$$

and may be written in the form

$$(x^2 + y^2)(x^2 + y^2) - 2a^2(x + y)(x - y) = 0. \tag{11}$$

It is the product of the two projective pencils of circles

$$(x^2 + y^2) + \lambda 2a(x + y) = 0, \tag{12}$$

$$a(x - y) + \lambda(x^2 + y^2) = 0. \tag{13}$$

These equations may be written in the form

$$(x + \lambda a)^2 + (y + \lambda a)^2 = 2\lambda^2 a^2, \tag{14}$$

$$\left(x + \frac{a}{2\lambda}\right)^2 + \left(y - \frac{a}{2\lambda}\right)^2 = \frac{a^2}{2\lambda^2}. \tag{15}$$

They represent two projective pencils of circles which all pass through the origin. All circles of (14) are tangent to the line $x - y = 0$; all circles of (15) are tangent

[1] The name of this curve is derived from the Greek λημνίσχος, which means a loop in shape of the figure 8, and dates back to Jac. Bernoulli who discovered this curve. See his works (Genevae, 1744), Vol. 1, p. 609.

to the line $x + y = 0$. The centers of the circles (14) lie on $x + y = 0$, those of (15) on $x - y = 0$. The construction of the circles (14) and (15) for a given λ is therefore extremely simple, and consequently also the construction of points of the lemniscate as intersections of (14) and (15). It is obvious that the circular points at infinity (isotropic points) and the origin are singular points (inflexional double points) of the quartic.

In an equally simple manner the theorem can be established, by the same principles, that all bicircular quartics may be generated by projective pencils of circles.

It is hoped, however, that the few examples explained above will suffice to show the great value of the principle for actual constructive purposes.

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

A Short Course in Elementary Mechanics for Engineers. By CLIFFORD NEWTON MILLS, B.S., A.M., Assistant Professor of Mathematics, South Dakota State College of Agriculture and Mechanic Arts. The D. Van Nostrand Company, New York, 1916. xi + 127 pages. $1.00.

This little book contains in a condensed form a discussion of the simpler subjects in kinematics, kinetics and statics. As only uniform motion, uniformly accelerated motion, and the forces which produce such motions are considered, the author has willed to treat them without the use of mathematics beyond trigonometry. The course is written from the definition viewpoint, much detailed discussion being omitted. For this reason and for the further reason that the clean-cut demonstrations of the calculus are not used the reasoning may not be always as rigid as the old-time mathematicians would like.

The engineering unit, the pound, is used throughout as the unit of force; it is defined as "that force which will accelerate a mass of one pound g feet per second per second." The pound of mass is not definitely defined but these relations are given,

$$\text{Weight} = \text{mass} \times g$$

$$\text{Mass} = W/g$$

thus avoiding hair-splitting distinctions.

Problems involving center of gravity and moment of inertia have been ingeniously worked out without the calculus; true, the summation sign, Σ, has been employed, but the summations are evaluated by algebraic processes.

Friction which might have been treated in an elementary manner, and which is of great importance to machinists and others who would have use for this book, seems to have been omitted. There is a large supply of well-graded problems.

The reviewer does not wish to give his full endorsement to a work in mechanics

for "engineers" that does not use the calculus. It was well enough in the early days to "work up" in the profession, to apply from a hand-book formulas about the derivation of which he was ignorant. But at the present time engineering is supposed to be a learned profession, and the engineer who is unacquainted with calculus can hardly be said to be up to date. But on the whole this book ought to prove an excellent work for beginning classes in mechanics, for extension courses, for trade and manual training schools, and for those who do not understand the calculus or prefer not to use it.

The following correction is noted at the suggestion of the author. The line near the middle of page 55 should read as follows: "Now, if the values of W, a, b, and c are known, and $a = b$, then".

GEORGE R. CHATBURN.

UNIVERSITY OF NEBRASKA.

Elements of Analytic Geometry. By A. ZIWET, Professor of Mathematics, University of Michigan, and A. D. HOPKINS, Instructor in Mathematics, University of Michigan. New York, The Macmillan Company, 1916. vi + 280 pages.

The present edition of this Analytic Geometry is a reproduction of the earlier book by the same authors[1] with the omission of those parts concerned chiefly with college algebra. The reason given for the abridged edition is that college algebra is usually taught previously to, and independent of analytic geometry.

Thus, in discussing the intersection of two straight lines, the determinant form of the solution is retained, but the development of the determinant is omitted.

The algebraic treatment of simultaneous equations, determinants, permutations and combinations, complex numbers and theory of algebraic equations, including numerical equations is omitted.

The part on the plane now occupies 180 pages. The part on three dimensions is almost exactly as before, and occupies 75 pages. As the parts retained have not been altered, the comments made on the previous book apply to this one.

VIRGIL SNYDER.

CORNELL UNIVERSITY,
ITHACA, N. Y.

Differential and Integral Calculus. By CLYDE E. LOVE. The Macmillan Co., New York, 1916. xviii + 343 pages. $2.10.

The first impression made by a book is physical; with this text that impression is most agreeable. The 339 5 x 8 inch pages of excellent typography give the impression of skillful brevity. A careful examination shows that none of the topics of the traditional American text is omitted and that after all the book is of about the usual length. In fact the preface lays no claim to brevity or any other sort of novelty. It is clearly a book well tested before publication, bearing

[1] *Analytic Geometry and Principles of Algebra*, 1913. Reviewed in this MONTHLY, Vol. xxi, pp. 85–89.

no marks of an author's idiosyncrasies. There is no suggestion of anything but a standard treatment of a well-delimited mathematical subject. No applications outside of geometry and mechanics appear, and since in these fields the problems deal extensively with interesting but unnatural curves and with material points having mass but no volume it is questionable whether an ordinary student will discover any applications at all.

The topics are well arranged. Differentiation has numerous applications before transcendental functions are taken up. A moderate treatment of integration (culminating in integration by parts and rational fractions) is followed by all sorts of problems in single integration except fluid pressure. There is then a return to the differential calculus for Taylor's Series and partial differentiation. Then come multiple integrals, with fluid pressure in a chapter by itself. A good introduction to differential equations, some kinematical applications, and an index close the book.

Under maxima and minima (p. 41) problems 26–32 seem to be new, and there is a bit of graphical differentiation on p. 43 which is out of the ordinary. But in the main the standard problems and the standard methods of varying them have been used.

Some features which the reviewer enjoyed were: the arrow notation for limits, the avoidance of the too explicit formulas for summation (for instance $\pi\int r^2\cdot dh$ is given for a volume of revolution rather than $\pi\int y^2\cdot dx$, exceptional cases given among the problems on the too obvious Rolle's Theorem, the use of the steel spring instead of a sinusoidal formula in illustrating a type of limit, and the honesty of the admission of the reason for taking up triple integrals. The use of the phrases "arbitrarily close approximation" (p. 151), "expand about the point $x = a$" (p. 223), "cross derivatives" (p. 238) is to be commended. The reviewer hopes that Osgood's adjective "respectable" as applied to functions, and Kowaleski's "almost all" ("fast alle") may next be admitted to the dignified currency which such a textbook confirms.

De minimis non curat lex, but trifles furnish much of his opportunity to the reviewer.

Fault may be found with the statement (p. 51) about the meaning of arc sin x, according to which we must infer that the area under $y^2(1 - x^2)$ from $x = 0$ to $x = 1$ is the *angle* $\pi/2$ radians. Surely in the calculus arc sin means a *number* (of radians in the angle), not the *angle* itself (however measured).

On page 59 the important fact that $e = 2.718 +$ is predicted but it is justified only in an exercise to be solved by the student later (on page 230). On page 228 the limiting behavior of the important function $x^n/n!$ is obscurely disposed of by reference to an exercise seven pages earlier.

No reason is given for adopting the notation dy/dx (p. 14) nor is the student warned against the false interpretation of dy and dx which he inevitably adopts here. Again no reason is assigned for introducing differentials (p. 68) in chapter 6; no use is made of them until chapter 11 and then only on the plea (p. 116) that reasons will appear later. As a matter of fact the reasons do appear on

pages 122 and 150, and the reviewer thinks that there should be this definite reference forward on pages 68 and 116.

All the expansions are to infinite series; the familiar approximations for reciprocals, logarithms, roots, etc., so useful in computing, are nowhere given.

In dealing with variables, limits, and infinitesimals, there is nowhere shown an actual variable in the modern sense, a succession of numerical values; the ordinary student thinks that constants enjoy a monopoly of the Arabic notation, variables requiring always to be *spelled* in letters.

Perry says, in his *Calculus for Engineers*, that the only integrals needed outside of pure theory are $\int x^n \cdot dx$, $\int dx/x$, and $\int e^x \cdot \sin x \cdot dx$. The latter does not appear in this text. Hyperbolic functions appear on page 64 and disappear on page 65, leaving no applications behind.

The proof (p. 123) that we may make a substitution before integrating will seem unnecessary to students just as it does later (p. 267 and p. 272) to the author himself.

"A feature of the book is its insistence on the importance of checking the results of exercises." In every respect but this the text seems to bear out the claims set up in the preface. The reviewer found no attention given to the general question of checking, and except under double integration (where more than one order of work is often possible) very few problems call for solutions in two ways. Moreover, the book omits a great many answers in places where a printed answer is the only reliable check a student can get. No rough methods of checking are suggested, such as sketching derivative or integral curves or other graphical devices, nor is there any hint of checking limits by computing neighboring values, nor of checking differentials by computing small increments, nor of checking integrals by Simpson's rule.

A few criticisms which might equally well be made of other texts beside the one under review are: the absence of any but abstract exercises in indeterminate forms, the failure to illustrate the important formula

$$\frac{df(x, y)}{dv} = \frac{\partial f}{\partial x}\frac{dx}{dv} + \frac{\partial f}{\partial y}\frac{dy}{dv},$$

so that the student can realize that it is good for anything or even recognize it when he meets it in physics, the retention of the phrase "total pressure" for the force on a plane area, and the pretense that *infinite series* are used in computation. The reviewer contends that it is approximation formulas that are used in computation, infinite series being used only in analysis.

W. R. Ransom.

Tufts College, Mass.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

ALGEBRA.

479. Proposed by S. A. COREY, Albia, Iowa.

Prove or disprove

$$\left\{\begin{vmatrix} x & -v & -z \\ -y & -z & v \\ -z & y & -x \end{vmatrix}^2 + \begin{vmatrix} y & -v & -z \\ x & -z & v \\ v & y & -x \end{vmatrix}^2 + \begin{vmatrix} x & y & -z \\ -y & x & v \\ -z & v & -x \end{vmatrix}^2 + \begin{vmatrix} x & -v & y \\ -y & -z & x \\ -z & y & v \end{vmatrix}^2\right\} \div \begin{vmatrix} x & -y & -z & v \\ y & x & -v & -z \\ z & v & x & y \\ v & -z & y & -x \end{vmatrix}^2$$

$$= (x^2 + y^2 + z^2 + v^2)^{-1}.$$

480. Proposed by FRANK IRWIN, University of California.

Solve the equation

$$(x-1) - 2\left(1-\frac{1}{x}\right) - 3\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right) - 4\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)\left(1-\frac{3}{x}\right) - \cdots$$
$$- n\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)\cdots\left(1-\frac{n-1}{x}\right) = 0.$$

Also the equation

$$(x-a_1) - a_2\left(1-\frac{a_1}{x}\right) - a_3\left(1-\frac{a_1}{x}\right)\left(1-\frac{a_2}{x}\right) - \cdots$$
$$- a_n\left(1-\frac{a_1}{x}\right)\left(1-\frac{a_2}{x}\right)\cdots\left(1-\frac{a_{n-1}}{x}\right) = 0.$$

[Adapted from a formula of Tait's.]

GEOMETRY.

512. Proposed by J. L. RILEY, Northeastern State Normal School, Tahlequah, Okla.

Determine, geometrically, where the circle of curvature at any point of an ellipse again meets the ellipse.

513. Proposed by ALBERT A. BENNETT, University of Texas.

The following construction for angle-trisection was given some years ago in a non-mathematical journal. Let ABC be a right triangle with AB as hypothenuse. Let BD be a ray drawn parallel to AC and extending in the same direction. Let AEF be a variable ray meeting the segment BC in E, and the ray BD in F. Show, by elementary methods, that when the variable ray is so adjusted that $EF = 2AD$, then $\angle\ EAC = \frac{1}{3}\ \angle\ BAC$.

CALCULUS.

427. Proposed by ROGER S. JOHNSON, Adelbert College, Cleveland, Ohio.

Of all ellipses circumscribed about a given parallelogram, the maximum, with regard to area, has as conjugate diameters the diagonals of the parallelogram.

428. Proposed by J. L. RILEY, Northeastern State Normal School, Tahlequah, Okla.

A loop of a lemniscate rolls in contact with the axis of x. Prove that the locus of the node is given by the equation

$$1 + \frac{dy}{dx} = \left(\frac{a}{y}\right)^{\frac{4}{3}}$$

and that $2\rho\rho' = a^2$, if ρ, ρ' be corresponding radii of curvature of this locus and the lemniscate.

MECHANICS.

344. Proposed by J. ROSENBAUM, New Haven, Conn.

Two bodies of equal masses, and coefficients of friction μ_1 and μ_2 are connected by a light, flexible string, and placed on an inclined plane. What is the angle, θ, between the string and the plane if the inclination, α, of the plane is a minimum when the bodies are on the point of motion?

345. Proposed by J. L. RILEY, Northeastern State Normal School, Tahlequah, Okla.

Two particles A and B are together in a smooth circular tube. A attracts B with a force whose acceleration is ω^2 and moves along the tube with uniform angular velocity 2ω, B being initially at rest: prove that the angle Φ subtended by AB at the center after a time t is given by the equation

$$\log\tan\frac{\pi+\Phi}{4}=\omega t.$$

NUMBER THEORY.

263. Proposed by J. L. RILEY, Northeastern State Normal School, Tahlequah, Okla.

To find values, positive integral, which verify the equation

$$X^3+2=Y^2 \qquad \text{(Gerono).}$$

264. Proposed by C. F. GUMMER, Kingston, Ontario.

Find a general formula for three squares in arithmetical progression. Is it possible for the common difference to be a perfect square?

SOLUTIONS OF PROBLEMS.

ALGEBRA.

A solution of 464, by Mrs. Elizabeth Brown Davis, was received after selections had been made for publication.

455. Proposed by JOS. B. REYNOLDS, Lehigh University.

Solve for x_n (not in determinant form) the simultaneous equations,

$$\tfrac{4}{3}x_n+2x_{n-1}+2x_{n-2}\cdots 2x_4+2x_3+2x_2+2x_1=g,$$

$$\tfrac{10}{3}x_n+\tfrac{22}{3}x_{n-1}+8x_{n-2}\cdots 8x_4+8x_3+8x_2+8x_1=4g,$$

$$\tfrac{16}{3}x_n+\tfrac{40}{3}x_{n-1}+\tfrac{52}{3}x_{n-2}\cdots 18x_4+18x_3+18x_2+18x_1=9g,$$

$$\tfrac{22}{3}x_n+\tfrac{58}{3}x_{n-1}+\tfrac{82}{3}x_{n-2}+\tfrac{94}{3}x_{n-3}\cdots 32x_4+32x_3+32x_2+32x_1=16g,$$

$$\tfrac{28}{3}x_n+\tfrac{76}{3}x_{n-1}+\tfrac{112}{3}x_{n-2}+\tfrac{136}{3}x_{n-3}+\tfrac{148}{3}x_{n-4}\cdots=25g,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(\tfrac{1}{3}+2n-1)x_n+(\tfrac{1}{3}+6n-5)x_{n-1}+(\tfrac{1}{3}+10n-13)x_{n-2}+(\tfrac{1}{3}+14n-25)x_{n-3}\cdots(\tfrac{1}{3}+2n^2-1)x_1=n^2g.$$

Solution by C. F. Gummer, Kingston, Ont.

The equations may be written

$$2x_n+3x_{n-1}+3x_{n-2}+\cdots+3x_1=\frac{3g}{2},$$

$$5x_n+11x_{n-1}+12x_{n-2}+\cdots+12x_1=\frac{12g}{2},$$

$$8x_n+20x_{n-1}+26x_{n-2}+27x_{n-3}+\cdots+27x_1=\frac{27g}{2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(3n-1)x_n+(9n-7)x_{n-1}+\cdots+(3n^2-1)x_1=\frac{3n^2g}{2},$$

the coefficients in each column on the left being in arithmetic progression from the principal diagonal downwards, and the differences of the progressions in adjacent columns forming the progression 3, 9, 15, $\cdots$. Let us write the determinant solution for x_n in the form $x_n = d_n'/d_n$.

Then

$$d_n = \begin{vmatrix} 2 & 3 & 3 & \cdots & 3 \\ 5 & 11 & 12 & \cdots & 12 \\ 8 & 20 & 26 & \cdots & 27 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

which becomes, if we subtract adjacent rows, leaving the first unchanged.

$$\begin{vmatrix} 2 & 3 & 3 & 3 & \cdots & 3 \\ 3 & 8 & 9 & 9 & \cdots & 9 \\ 3 & 9 & 14 & 15 & \cdots & 15 \\ 3 & 9 & 15 & 20 & \cdots & 21 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

and on repeating the process

$$\begin{vmatrix} 2 & 3 & 3 & 3 & \cdots & 3 \\ 1 & 5 & 6 & 6 & \cdots & 6 \\ 0 & 1 & 5 & 6 & \cdots & 6 \\ 0 & 0 & 1 & 5 & \cdots & 6 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

If now we perform the same operation once with columns, we get

$$d_n = \begin{vmatrix} 2 & 1 & 0 & 0 & \cdot & 0 & 0 \\ 1 & 4 & 1 & 0 & \cdot & 0 & 0 \\ 0 & 1 & 4 & 1 & \cdot & 0 & 0 \\ 0 & 0 & 1 & 4 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & 4 & 1 \\ 0 & 0 & 0 & 0 & \cdot & 1 & 4 \end{vmatrix}.$$

Let d_{n-1}, d_{n-2} be corresponding determinants of orders $n - 1$ and $n - 2$. Developing d_n by means of the last column and row, we obtain $d_n = 4d_{n-1} - d_{n-2}$. Hence, by the usual method for such difference equations,

$$d_n = A(2 + \sqrt{3})^n + B(2 - \sqrt{3})^n.$$

But $d_1 = 2$ and $d_2 = 7$. Hence $A = B = \frac{1}{2}$, and d_n is given.

Again,

$$d_n' = \frac{g}{2}\begin{vmatrix} 3 & 3 & 3 & 3 & \cdot\cdot & 3 \\ 12 & 11 & 12 & 12 & \cdot\cdot & 12 \\ 27 & 20 & 26 & 27 & \cdot\cdot & 27 \\ 48 & 29 & 41 & 47 & \cdot\cdot & 48 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 3n^2 & 9n - 7 & \cdot\cdot & \cdot\cdot & \cdots & 3n^2 - 1 \end{vmatrix}.$$

Subtracting the first column from each of the others, we have

$$d_n' = \frac{g}{2}\begin{vmatrix} 3 & 0 & 0 & 0 & \cdot\cdot & 0 \\ 12 & -1 & 0 & 0 & \cdot\cdot & 0 \\ 27 & -7 & -1 & 0 & \cdot\cdot & 0 \\ 48 & -19 & -7 & -1 & \cdot\cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 3n^2 & \cdot & \cdot & \cdot & \cdot\cdot & -1 \end{vmatrix} = (-1)^{n-1}\frac{3g}{2}.$$

Hence,

$$x_n = \frac{(-1)^{n-1}3g}{(2 + \sqrt{3})^n + (2 - \sqrt{3})^n}.$$

467. Proposed by IRA M. DE LONG, University of Colorado.

Determine the function f, from the functional relation $f(x + y) = f(x) + f(y) + 2xy$.

I. Solution by A. Cohen, Johns Hopkins University.

From $f(x+y) = f(x) + f(y) + 2xy$ we have, when $y = 0$, $f(x) = f(x) + f(0)$; hence,

$$f(0) = 0. \tag{I}$$

Similarly, letting $y = x$, $2x$, $3x$ in turn, we have respectively,

$$f(2x) = 2f(x) + 2x^2; \quad f(3x) = 3f(x) + 6x^2; \quad f(4x) = 4f(x) + 12x^2.$$

This suggests the law,

$$f(nx) = nf(x) + n(n-1)x^2. \tag{II}$$

Assuming this and letting $y = nx$, we have $f[(n+1)x] = (n+1)f(x) + n(n+1)x^2$
Thus the law is established for n a positive integer.
That it also holds for m/n, a rational fraction, may be seen as follows:
Letting $y = -x$ and using (I)

$$f(-x) = -f(x) + 2x^2. \tag{III}$$

From this readily follows

$$f(-nx) = -nf(x) + n(n+1)x^2 = -nf(x) + (-n)(-n-1)x^2. \tag{III$'$}$$

If $\quad y = -\dfrac{(n-1)}{n}x, \quad f\left(\dfrac{x}{n}\right) = f(x) + f\left(-\dfrac{(n-1)}{n}x\right) - \dfrac{2(n-1)}{n}x^2.$

Using (III$'$), $\quad f\left(-\dfrac{(n-1)}{n}x\right) = (1-n)f\left(\dfrac{x}{n}\right) + \dfrac{n(n-1)}{n^2}x^2.$

Hence $\quad f\left(\dfrac{x}{n}\right) = f(x) + (1-n)f\left(\dfrac{x}{n}\right) + \dfrac{1-n}{n}x^2.$

From this follows at once $\quad f\left(\dfrac{x}{n}\right) = \dfrac{1}{n}f(x) + \dfrac{1}{n}\left(\dfrac{1}{n} - 1\right)x^2.$

Using (II) and then (IV),

$$\begin{aligned} f\left(\frac{m}{n}x\right) &= mf\left(\frac{x}{n}\right) + \frac{m(m-1)}{n^2}x^2 \\ &= \frac{m}{n}f(x) + \frac{m}{n}\frac{1-n}{n}x^2 + \frac{m}{n}\frac{m-1}{n}x^2 = \frac{m}{n}f(x) + \frac{m}{n}\left(\frac{m}{n} - 1\right)x^2 \end{aligned}$$

which proves the law (II) for n any positive rational fraction.

Making use of (III$'$), the law is readily seen to hold for n a negative rational fraction.

Assuming $f(x)$ to be a continuous function of x, the law will hold for n any real number, as can be shown by using a sequence of rational numbers whose limit is n.

Once this is established, the rest follows immediately.

Assuming that $f(x)$ has a derivative, we have on differentiating (II) with respect to n, $x\dfrac{df(nx)}{d(nx)} = f(x) + (2n-1)x^2$, which holds for all real values of n. In particular, for $n = 1$, $x\dfrac{df(x)}{dx} = f(x) + x^2$, or $xdf - fdx = x^2dx$. An obvious integrating factor is $1/x^2$. Introducing this and integrating we have $f/x = x + c$; whence

$$f(x) = x^2 + cx, \text{ where } c \text{ is any constant.} \tag{V}$$

II. Solution by Nathan Altshiller, University of Oklahoma.

Let us determine f from the relation

$$f(x+y) = f(x) + f(y) + 2cxy, \tag{f}$$

where c is a given constant.

If in (f) we put $y = x$, we have

$$f(2x) = 2f(x) + 2cx^2 \tag{1}$$

and if we put $y = 2x$, we have

$$f(x+2x) = f(x) + f(2x) + 2c\cdot x\cdot 2x,$$

which may be written, taking (1) into account,

(2) $$f(3x) = 3f(x) + 3\cdot 2cx^2.$$

In order to prove that this last formula is general, *i. e.*, that we have, for any positive integral value of n,

(*F*) $$f(nx) = nf(x) + n(n-1)cx^2,$$

we assume that we have for $n = m$

(3) $$f(mx) = mf(x) + m(m-1)cx^2$$

and shall prove that (*F*) holds then for $n = m + 1$. On the strength of (*f*) we have

$$f[(m+1)x] = f(mx) + f(x) + 2cmx^2;$$

whence, substituting for $f(mx)$ its development from (3) and simplifying,

$$f[(m+1)x] = (m+1)f(x) + (m+1)mcx^2.$$

If in (*f*) we put $x = y = 0$, we have

$$f(0+0) = f(0) + f(0) - 2c\cdot 0\cdot 0$$

or $f(0) = 2f(0)$, hence $f(0) = 0$. Now for $y = -x$, (*f*) gives

$$f(x-x) = f(x) + f(-x) - 2cx^2$$

and since $f(0) = 0$, we have, solving for $f(-x)$,

(4) $$f(-x) = -f(x) + 2cx^2.$$

This formula shows that (*F*) holds for $n = -1$. Hence in order to prove that (*F*) is valid for all negative integral values of n, it is sufficient to prove that it holds for $n = -(m+1)$, if it holds for $n = -m$. We assume therefore that

(5) $$f(-mx) = -mf(x) + (-m)(-m-1)cx^2.$$

We have from (*f*)

$$f[-(m+1)x] = f(-mx) + f(-x) + 2c(-mx)(-x).$$

Replacing $f(-mx)$ and $f(-x)$ by their developments from (5) and (4), respectively, and simplifying, we obtain

$$f[-(m+1)x] = -(m+1)f(x) + [-(m+1)][-(m+1)-1]cx^2.$$

Since (*F*) is valid for all integral values of n, positive or negative, we have, k being any integer,

(6) $$f(x) = f\left(k\cdot\frac{x}{k}\right) = kf\left(\frac{x}{k}\right) + k(k-1)c\left(\frac{x}{k}\right)^2,$$

which, when solved for $f(x/k)$, may be written in the form

(7) $$f\left(\frac{1}{k}\cdot x\right) = \frac{1}{k}fx + \frac{1}{k}\left(\frac{1}{k}-1\right)cx^2.$$

Hence, (*F*) is valid for $n = 1/k$. Now let p, q be any two integers, positive or negative. We may write, since p is an integer,

$$f\left(p\cdot\frac{x}{q}\right) = pf\left(\frac{x}{q}\right) + p(p-1)c\frac{x^2}{q^2}$$

or, taking (7) into consideration and simplifying,

$$f\left(\frac{p}{q}\cdot x\right) = \frac{p}{q}fx + \frac{p}{q}\left(\frac{p}{q}-1\right)cx^2.$$

We have thus proved that (*F*) is valid for any rational value of n. Any irrational quantity s may be considered as the limit of a variable rational quantity, and since (*F*) holds for all these rational values, it also holds for s, if the function f is continuous, which we suppose. Hence, (*F*)

is valid for any real value of n. Since in the above considerations no restrictions were put upon x, (F) holds for any value of x, real or complex, if we assume, as we may, that (f) holds for such values of the variables. Thus we may write:

$$f(z) = f(u + vi) = f(u) + f(vi) + 2cuvi. \tag{8}$$

But

$$f(u) = f(u \cdot 1) = uf(1) + u(u - 1)c(1)^2,$$

$$f(vi) = f(v \cdot i) = vf(i) + v(v - 1)c(i)^2;$$

hence, (8) may be written:

$$f(z) = f(u + vi) = uf(1) + vf(i) + c(u + vi)^2 - (u - v)c. \tag{9}$$

In order to determine the value of $f(1)$ we put $u = 1$, and $v = 0$ and we have from (9) $f(1) = f(1)$, which shows that the value of $f(1)$ is arbitrary. By putting $u = 0$, $v = 1$, we may show in the same way that $f(i)$ is arbitrary. Hence denoting $f(1) - c$ by α and $f(i) + c$ by β, where α and β are two arbitrary quantities, (9) may be written

$$f(u + vi) = c(u + vi)^2 + \alpha u + \beta v.$$

If $\alpha = \beta = 0$, *i. e.*, if $f(1) = 1$ and $f(i) = -1$, and $c = 1$ we have

$$f(u + vi) = (u + vi)^2.$$

Also variously solved by T. M. SIMPSON, O. S. ADAMS, ELIJAH SWIFT, E. R. SMITH, HORACE OLSON, A. A. BENNETT, C. F. GUMMER, and J. L. WALSH.

468. Proposed by H. C. FEEMSTER, York College, Neb.

In each of the following series find the nth term and sum:

(a) $$2 + 5 + 9 + 15 + 24 + \cdots,$$

(b) $$1 + 6 + 10 + 20 + 35 + \cdots,$$

(c) $$1 + 5 + 15 + 35 + 70 + \cdots,$$

SOLUTION BY J. L. RILEY, Tahlequah, Okla.

(a) Using the method of differences we have

$$\begin{array}{ccccccc} 3 & 4 & 6 & 9 \cdots \\ & 1 & 2 & 3 \cdot \cdots \\ & & 1 & 1 \cdot \cdot \cdots \\ & & & 0 \cdot \cdot \cdot \cdots \end{array}$$

$$U_n = 2 + 3(n-1) + \frac{(n-1)(n-2)}{\underline{|2}} + \frac{(n-1)(n-2)(n-3)}{\underline{|3}}$$

$$= \frac{n^3 - 3n^2 + 20n - 6}{6}, \text{ the } n\text{th term},$$

$$S_n = 2n + \frac{3n(n-1)}{\underline{|2}} + \frac{n(n-1)(n-2)}{\underline{|3}} + \frac{n(n-1)(n-2)(n-3)}{\underline{|4}}$$

$$= \frac{n}{24}(n^3 - 2n^2 + 35n + 14), \text{ the sum.}$$

(b) In the series $1 + 6 + 10 + 20 + 35 + \cdots$, let $U_n = A + Bn + Cn^2 + Dn^3 + En^4$. Then

$$\begin{cases} A + B + C + D + E = 1, \\ A + 2B + 4C + 8D + 16E = 6, \\ A + 3B + 9C + 27D + 81E = 10, \\ A + 4B + 16C + 64D + 256E = 20, \\ A + 5B + 25C + 125D + 625E = 35. \end{cases}$$

From these equations we find the values of A, B, C, D, E; whence,

$$U_n = -20 + 36n - \frac{115n^2}{6} + \frac{9n^3}{2} - \frac{n^4}{3}.$$

From this we get the 6th term to be 46.

Thus, the series is

$$1 + 6 + 10 + 20 + 35 + 46 + \cdots.$$

Using the method of differences we get

$$\begin{array}{cccccc} 5 & 4 & 10 & 15 & 11 & \cdots \\ -1 & 6 & 5 & -4 & \cdot & \cdots \\ & 7 & -1 & -9 & \cdot & \cdot\cdots. \\ & -8 & -8 & \cdot & \cdot & \cdots \\ & & 0 & \cdot & \cdot & \cdots \end{array}$$

Then

$$S_n = n + \frac{5n(n-1)}{\underline{|2}} - \frac{n(n-1)(n-2)}{\underline{|3}} + \frac{7n(n-1)(n-2)(n-3)}{\underline{|4}} - \frac{8n(n-1)(n-2)(n-3)(n-4)}{\underline{|5}}.$$

$$= -\frac{n}{120}(8n^4 - 115n^3 + 510n^2 - 1145n + 622).$$

(*c*) In the series $1 + 5 + 15 + 35 + 70 + \cdots$

$$U_n = A + Bn + Cn^2 + Dn^3 + En^4 + \cdots.$$

$$\left\{\begin{array}{l} A + B + C + D + E = 1, \\ A + 2B + 4C + 8D + 16E = 5, \\ A + 3B + 9C + 27D + 81E = 15, \\ A + 4B + 16C + 64D + 256E = 35, \\ A + 5B + 25C + 125D + 625E = 70, \end{array}\right\} \begin{array}{l} A = 0, \\ B = 1/4, \\ C = 11/24, \\ D = 1/4, \\ E = 1/24. \end{array}$$

Hence,

$$U_n = \frac{n}{4} + \frac{11n^2}{24} + \frac{n^3}{4} + \frac{n^4}{24}.$$

Then

$$24S_n = 6\Sigma n + 11\Sigma n^2 + 6\Sigma n^3 + \Sigma n^4,$$

$$24S_n = 3n(n+1) + \frac{11n(n+1)(2n+1)}{6} + \frac{3n^2(n+1)^2}{2} + \frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30}.$$

$$S_n = \frac{n}{120}(n^4 + 10n^3 + 35n^2 + 50n + 24) = \frac{n}{120}(n+1)(n+2)(n+3)(n+4).$$

Also solved by O. S. ADAMS, H. H. CONWELL, PAUL CAPRON, and WILLIAM TIER.

GEOMETRY.

496. Proposed by NATHAN ALTSHILLER, University of Oklahoma.

Find all the lines such that the pairs of tangent planes to a given sphere (ellipsoid) passing through them, shall be orthogonal.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

One obvious solution is given when one of the tangent planes is fixed in position, for then this plane is the locus of lines common to it and a second tangent plane orthogonal to the first.

It is not troublesome to show that the equation of a first plane embracing a line

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}, \tag{1}$$

$$\frac{(x-a)\lambda_1}{l} + \frac{(y-b)\mu_1}{m} + \frac{(z-c)\nu_1}{n} = 0, \tag{2}$$

with the condition

$$\lambda_1 + \mu_1 + \nu_1 = 0, \tag{3}$$

and that of a second plane embracing (1) is

$$\frac{(x-a)\lambda_2}{l} + \frac{(y-b)\mu_2}{m} + \frac{(z-c)\nu_2}{n} = 0, \tag{4}$$

with the condition

$$\lambda_2 + \mu_2 + \nu_2 = 0. \tag{5}$$

If (2) and (4) are orthogonal,

$$\frac{\lambda_1}{l} \cdot \frac{\lambda_2}{l} + \frac{\mu_1}{m} \cdot \frac{\mu_2}{m} + \frac{\nu_1}{n} \cdot \frac{\nu_2}{n} = 0, \tag{6}$$

and if (2) and (4) touch

$$a_1x^2 + b_1y^2 + c_1z^2 = 1, \tag{7}$$

$$\left(\frac{\lambda_1}{l}\right)^2\Big/a_1 + \left(\frac{\mu_1}{m}\right)^2\Big/b_1 + \left(\frac{\nu_1}{n}\right)^2\Big/c_1 = \left(\frac{a\lambda_1}{l} + \frac{b\mu_1}{m} + \frac{c\nu_1}{n}\right)^2, \tag{8}$$

and

$$\left(\frac{\lambda_2}{l}\right)^2\Big/a_1 + \left(\frac{\mu_2}{m}\right)^2\Big/b_1 + \left(\frac{\nu_2}{n}\right)^2\Big/c_1 = \left(\frac{a\lambda_2}{l} + \frac{b\mu_2}{m} + \frac{c\nu_2}{n}\right)^2. \tag{9}$$

Now (3), (5), (6), (8), (9) is a system of *five* equations for the determination of the *six* ratios λ_1/ν_1, μ_1/ν_1; λ_2/ν_2, μ_2/ν_2; and l/n, m/n, giving an indeterminate solution. The values of l/n, m/n are the only ones needed in (1).

There seems to be a missing condition in the statement of the problem.

If the direction of the line (1) were constant, or $l : m : n$, constant, the *locus* of the line (1) would be a right circular cylinder, or, in other words, the locus of the line of intersection of constant direction of pairs of orthogonal tangent planes to a central conicoid is a right circular cylinder.

Note.—The single condition on the line restricts it to be a member of a line complex whose order apparently may be as high as 8. It would be desirable to determine this explicitly. EDITORS.

497. Proposed by NATHAN ALTSHILLER, University of Oklahoma.

Find the locus of the mid-point of the segment determined by two spheres on any line passing through any point common to the two spheres.

SOLUTION BY S. W. REAVES, University of Oklahoma.

Let the plane of the common circle of the spheres be chosen for yz-plane, and the line of centers for x-axis. Let the radius of the common circle be k, and let $(a, 0, 0)$ and $(b, 0, 0)$ be the centers of the two spheres. Then the equation of one sphere is

$$(x-a)^2 + y^2 + z^2 = a^2 + k^2,$$

or

$$x^2 + y^2 + z^2 - 2ax = k^2; \tag{1}$$

and, similarly, the equation of the other is

$$x^2 + y^2 + z^2 - 2bx = k^2. \tag{2}$$

Let l, m, n be the direction cosines of an arbitrary line through the point M $(0, 0, k)$. Then the equation of the line may be written

$$\frac{x}{l} = \frac{y}{m} = \frac{z-k}{n} = r, \tag{3}$$

where r is the length of the segment joining $(0, 0, k)$ and (x, y, z).

To find the length of the segment MP cut from this line by the sphere (1), we substitute $x = lr$, $y = mr$, $z = nr + k$ in equation (1) and solve for that value of r which is not zero. We thus find at once

$$r = MP = 2al - 2kn.$$

Substituting the same values in (2), we find likewise for the segment MQ intercepted by the

second sphere,

$$MQ = 2bl - 2kn.$$

If R be the mid-point of PQ, then

$$MR = \frac{MP + MQ}{2} = (a + b)l - 2kn.$$

Hence, if in (3) we set

$$r = (a + b)l - 2kn, \tag{4}$$

we readily obtain expressions for the coördinates x, y, z of the middle point R of the segment determined by the two spheres on the line through $(0, 0, k)$ and having the direction (l, m, n). To find the equation of the locus of R for all directions of the line we must eliminate l, m, n, which we do as follows:

From (3), we have,

$$x^2 + y^2 + (z - k)^2 = r^2; \tag{5}$$

and from (4) and (3),

$$r = (a + b)\left(\frac{x}{r}\right) - 2k\left(\frac{z - k}{r}\right),$$

or

$$r^2 = (a + b)x - 2k(z - k). \tag{6}$$

Equating the values of r^2 given by (5) and (6), we have for the equation of the locus of R,

$$x^2 + y^2 + z^2 - (a + b)x = k^2. \tag{7}$$

The locus is, therefore, a sphere whose center is midway between the centers of the given spheres and which contains all points common to these spheres. It is evident then that the choice of any point other than M common to the two spheres would lead to the same locus.

Also solved by the PROPOSER.

CALCULUS.

413. Proposed by OSCAR S. ADAMS, Washington, D. C.

Determine a function of x independent of b, such that

$$\int_b^{b+1} f(x)dx = \frac{1}{b+1},$$

the real part of b being positive.

SOLUTION BY C. F. GUMMER, Kingston, Ontario.

There is one solution only satisfying the conditions:
(i) $f(x)$ is one-valued and continuous in the right half plane of complex numbers, and
(ii) $f(x + n)$ approaches 0 as n approaches $+\infty$ through integral values for every x.
Let $f(x)$ be such a solution. Differentiating the equation

$$\int_b^{b+1} f(x)dx = \frac{1}{b+1}, \tag{1}$$

we get, after substituting x for b,

$$f(x + 1) - f(x) = -\frac{1}{(x + 1)^2}.$$

Similarly,

$$\begin{aligned} f(x + 2) - f(x + 1) &= -\frac{1}{(x + 2)^2}, \\ &\vdots \\ f(x + n) - f(x + n - 1) &= -\frac{1}{(x + n)^2}. \end{aligned}$$

Adding these equations, we have

$$f(x) = \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \cdots + \frac{1}{(x+n)^2} + f(x+n).$$

Letting n approach $+\infty$, we get, on account of (ii),

$$f(x) = \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \cdots \text{ to infinity.} \tag{2}$$

To show that (2) is a solution of (1), we observe that the series converges uniformly with respect to x over the right half plane including the axis of imaginaries, since $|x+n| \geqq n$.

Hence,

$$\int_b^{b+1} f(x)dx = \left(\frac{1}{b+1} - \frac{1}{b+2}\right) + \left(\frac{1}{b+2} - \frac{1}{b+3}\right) + \cdots = \frac{1}{b+1},$$

for any path of integration lying in the above region.

The series (2) may be summed by means of the gamma function. Thus integrating between the limits from 0 to x, we have

$$\int_0^x f(x)dx = \left(\frac{1}{1} - \frac{1}{x+1}\right) + \left(\frac{1}{2} - \frac{1}{x+2}\right) + \cdots = S, \text{ say.}$$

Since S is uniformly convergent,

$$\begin{aligned}\int_0^x S dx &= \left\{\frac{x}{1} - \log\left(1 + \frac{x}{1}\right)\right\} + \left\{\frac{x}{2} - \log\left(1 + \frac{x}{2}\right)\right\} + \cdots \\ &= -\log\left[\left\{e^{-(x/1)}\left(1 + \frac{x}{1}\right)\right\}\left\{e^{-(x/2)}\left(1 + \frac{x}{2}\right)\right\}\cdots\right] \\ &= -\log\left(\frac{1}{\Gamma(x)xe^{\gamma x}}\right) = \log\Gamma(x+1) + \gamma x.\end{aligned}$$

Hence,

$$f(x) = \frac{dS}{dx} = \frac{d^2}{dx^2}[\log\Gamma(x+1)].$$

The general solution of (1) subject to (i) but not to (ii) is now seen to be

$$f(x) = \frac{d^2}{dx^2}\log\Gamma(x+1) + \frac{d}{dx}\phi(x),$$

where $\phi(b+1) = \phi(b)$, and, therefore, $\phi(x)$ is any analytic function admitting the period 1.

Also solved by W. R. RANSOM and the PROPOSER.

414. Proposed by C. N. SCHMALL, New York City.

Among spherical triangles having the same base and equal altitudes, show that the isosceles triangle has the greatest vertical angle. Show that this is also true for plane triangles.

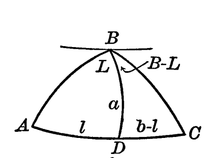
SOLUTION BY W. J. THOME, University of Detroit.

ABC is any spherical triangle. BD is a great circle arc perpendicular to AC. The angle B is broken up into two parts, L and $(B - L)$, and the corresponding opposite sides are l and $(b - l)$. Suppose B moves around on a small circle concentric with AC. Then BD remains constant in value.

In the $\triangle ABD$, $\cot L = \sin a \cot l$, and in the $\triangle BDC$, $\cot(B - L) = \sin a \cot(b - l)$. Now $B = L + (B - L) = \cot^{-1}(\sin a \cot l) + \cot^{-1}(\sin a \cot(b - l))$.

Differentiating,

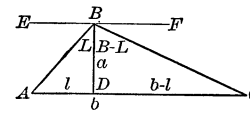
$$\frac{dB}{dl} = \sin a\left[\frac{-\csc^2 l}{1 + [\sin a \cot l]^2} + \frac{\csc^2(b - l)}{1 + [\sin a \cot(b - l)]^2}\right].$$



which is 0 if $b = 2l$. This evidently gives a maximum value of B. But if $b = 2l$, $AB = BC$ and the $\triangle$ is isosceles.

For the plane triangle, let ABC be any plane triangle. BD is a $\perp$ to AC from B. The vertex B may move anywhere on the straight line EF which is parallel to AC, thus keeping the altitude BD constant in value. The angle B is broken up into the two parts L and $(B - L)$ and the corresponding opposite sides are l and $(b - l)$.

In the $\triangle ABD$, $\tan L = l/a$ and in the $\triangle BDC$,



$$\tan (B - L) = \frac{b - l}{a}.$$

Now

$$B = L + (B - L) = \tan^{-1} (l/a) + \tan^{-1} \frac{b - l}{a}.$$

Differentiating,

$$\frac{dB}{dl} = a \left[\frac{1}{a^2 + l^2} - \frac{1}{a^2 + (b - l)^2} \right]$$

which is 0 if $b = 2l$, giving a maximum value of B. But if $b = 2l$, $AB = BC$ and the triangle is isosceles.

Also solved by M. R. Gaffet.

MECHANICS.

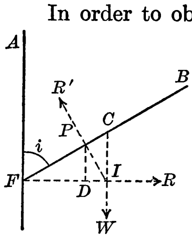
326. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

A uniform beam, of length $2l$, rests in equilibrium against a smooth vertical wall and upon a peg at a distance a from the wall, show that the inclination of the beam to the vertical is

$$\sin^{-1} \left(\frac{a}{l} \right)^{\frac{1}{3}}.$$

Solution by H. S. Uhler, Yale University.

In order to obtain the given inclination it is necessary to assume that the reaction of the peg is normal to the beam, for, if this be not the case, the beam may be in equilibrium on various slants depending upon the value of the coefficient of friction.



The necessary and sufficient condition for static equilibrium is that the lines of action of the three forces acting on the beam pass through a single point, I. These forces are obviously the weight of the beam, W, acting vertically downward, the horizontal reaction of the *smooth* wall, R, and the normal reaction of the peg, R'. $\overline{BC} = \overline{CF} = l$. $\overline{DF} = a$. $P \equiv$ peg. Let $\overline{FP} \equiv m$ and $\overline{FI} \equiv n$. Since $\overline{IP}$ and $\overline{PD}$ are, respectively, the perpendiculars dropped from the vertices of the right angles upon the hypotenuses of the triangles CIF and FPI we have $n^2 = lm$ and $m^2 = an$, so that $a^3/m^3 = a/l$ or $a/m = (a/l)^{1/3}$. But $i = \sin^{-1} (a/m) = \sin^{-1} (a/l)^{1/3}$, as required.

Also solved by A. M. Harding, L. A. Warren, G. Paaswell, Paul Capron, O. S. Adams, and Horace Olson.

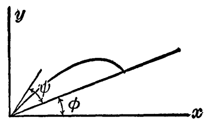
327. Proposed by C. N. SCHMALL, New York City.

An inclined plane makes an angle φ with the horizontal plane, and from its foot a body is projected upward at an angle ψ to the plane, and with a velocity v. Show that it will strike the plane perpendicularly if $\tan \psi = \frac{1}{2} \cot \varphi$ and that its range up the plane in this case will be

$$\frac{2v^2 \sin \varphi}{g(1 + 3 \sin^2 \varphi)}.$$

Solution by A. M. Harding, University of Arkansas.

The position of the projectile at any time is given by



$$x = v \cos (\varphi + \psi) \cdot t,$$

$$y = v \sin (\varphi + \psi) \cdot t - \tfrac{1}{2}gt^2.$$

Eliminating t we obtain, as the equation of the path,

$$y = x \tan (\varphi + \psi) - \frac{gx^2}{2v^2 \cos^2 (\varphi + \psi)}. \tag{1}$$

Let r be the range up the plane. Then the coördinates of the point where the projectile meets the inclined plane are

$$x = r \cos \varphi, \qquad y = r \sin \varphi. \tag{2}$$

Substituting these in (1), we find

$$r = \frac{2v^2}{g} \cdot \frac{\cos (\varphi + \psi) \sin \psi}{\cos^2 \varphi}. \tag{3}$$

From (1) we find

$$\frac{dy}{dx} = \tan (\varphi + \psi) - \frac{gt}{v \cos (\varphi + \psi)}. \tag{4}$$

This gives the direction at any time t. The projectile strikes the plane after a time

$$t = \frac{r \cos \varphi}{v \cos (\varphi + \psi)} = \frac{2v \sin \psi}{g \cos \varphi}.$$

Hence, when the projectile strikes the plane, we have

$$\frac{dy}{dx} = \tan (\varphi + \psi) - \frac{2 \sin \psi}{\cos \varphi \cos (\varphi + \psi)}$$

$$= \frac{\sin (\varphi + \psi) \cos \varphi - 2 \sin \psi}{\cos \varphi \cos (\varphi + \psi)}.$$

If the projectile strikes perpendicularly, $dy/dx = - \cot \varphi$. Hence,

$$\frac{\sin (\varphi + \psi) \cos \varphi - 2 \sin \psi}{\cos \varphi \cos (\varphi + \psi)} = - \frac{\cos \varphi}{\sin \varphi}.$$

From this equation, we find $\tan \psi = \frac{1}{2} \cot \varphi$. Substituting in (3), we find

$$r = \frac{2v^2 \sin \varphi}{g(1 + 3 \sin^2 \varphi)}.$$

Also solved by H. C. Feemster, H. S. Uhler, G. Paaswell, C. N. Schmall, O. S. Adams, Horace Olson, and L. A. Warren.

NUMBER THEORY.

242. Proposed by NORMAN ANNING, Chilliwack, B. C.

Find a function of n which is equal to A_k when $n \equiv k \pmod{p}$, $k = 1, 2, 3, \cdots, p$.

Solution by the Proposer.

Let θ be a primitive root of $x^p - 1 = 0$, then it is known that $\theta, \theta^2, \theta^3, \cdots, \theta^p$ are all the pth roots of unity and that $\theta^m + \theta^{2m} + \theta^{3m} + \cdots + \theta^{pm} = p$ or 0, according as m is or is not divisible by p. [See, for example, Burnside and Panton, *Theory of Equations*, Vol. I, pp. 95 and 96.] Consider now the expression:

$$\begin{aligned} f = \frac{1}{p} [&A_1\{\theta^{n-1} + \theta^{2(n-1)} + \cdots + \theta^{p(n-1)}\} \\ &+ A_2\{\theta^{n-2} + \theta^{2(n-2)} + \cdots + \theta^{p(n-2)}\} \\ &+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &+ A_p\{\theta^{n-p} + \theta^{2(n-p)} + \cdots + \theta^{p(n-p)}\}]. \end{aligned}$$

Since, for any integral values of n, the numbers, $(n-1)$, $(n-2)$, $\cdots$, $(n-p)$, are p consecutive integers, one and only one of them is divisible by p.

Suppose $(n-k)$ is divisible by p, *i. e.*, $n \equiv k \pmod p$. Then the coefficient of A_k is p and all the other coefficients are zero.

Hence, when $n \equiv k \pmod p$, $f = A_k$ as required.

244. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Determine the rational value of x that will render $x^2 + px + q$ a perfect square. What value of x will render $x^2 - 7x + 2$ a perfect square?

Solution by Harold T. Davis, Colorado Springs, Colorado.

Let $x^2 + px + q = y^2$. Then $x^2 + px + (p^2/4) + q - (p^2/4) - y^2 = 0$, or $(2x+p)^2 - (2y)^2 = p^2 - 4q$.

(1) Let $2x + p = z$ and $2y = w$. Then $z^2 - w^2 = p^2 - 4q$; or, if $4q > p^2$, $w^2 - z^2 = 4q - p^2$. Let a and b be complementary factors of $p^2 - 4q$. Then $z + w = a$ and $z - w = b$. Whence $z = (a+b)/2$ and $w = (a-b)/2$. Substituting these values in equations (1), we have

$$x = \frac{a+b-2p}{4} \text{ and } y = \frac{a-b}{4}.$$

Example. $x^2 - 7x + 2 = y^2$. Here $p^2 - 4q = 41$, the complementary factors of which are 41 and 1. Hence,

$$x = \frac{41+1+14}{4} = 14 \quad\text{and}\quad y = \frac{41-1}{4} = 10.$$

A complete discussion of the solution of the general equation of the second degree in two variables is given in Chrystal's *Algebra*, Part II, page 458.

Also solved by Horace Olson, Norman Anning, H. N. Carleton, O. S. Adams, J. A. Colson, J. L. Riley, N. Pandya and J. H. Weaver.

QUESTIONS AND DISCUSSIONS.

Send all communications to U. G. Mitchell, University of Kansas, Lawrence.

REPLIES.

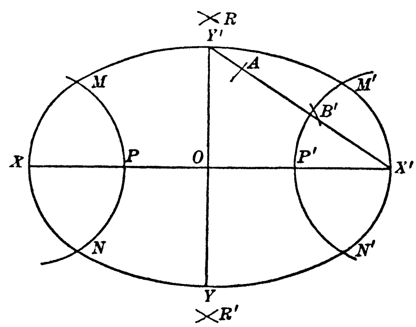
33. Under what conditions or to what extent is Mr. Iwerson's construction, given below, a useful or practical approximation to a true ellipse? What criterion can be given to measure definitely the degree of approximation?

Mr. Iwerson's approximate construction for an ellipse by ruler and compasses alone, having given the axes, was given in the November, 1916, issue of the Monthly, pp. 354, 355. The following corrections should be made: In the last two lines on p. 354, Ox should be OY, and Oy should be OX.

Note. In the February issue of the Monthly (pp. 90–92), we published a reply to this question by Professor Capron, of the U. S. Naval Academy. Before the February issue had come from the press, Professor Howland, of Wesleyan University, sent in the reply printed below. These two discussions are, accordingly, independent and from entirely different points of view. Professor Capron took as his primary measure of approximation the proportional errors in the radii of curvature at important points. Professor Howland has taken as his measure of approximation the ratio of the distance between the true and constructed curves (measured vertically or normally) to the semi-major axis. The two discussions seem to overlap in but one place. What Professor Capron has called the proportional error in the length of the minor axis and designated as E_1, corresponds to a maximum value of Professor Howland's relative divergence, d_2/a, which for some eccentricities occurs at $x = 0$. There is some slight difference in the formulas since in the one case the error is given relative to the semi-minor and in the other case relative to the semi-major axis.

The behavior of the arc MM', as brought out in the discussion below, is decidedly interesting. One would certainly not imagine *a priori* that the arc would cut the ellipse in four real distinct points.—U. G. M.

REPLY BY LEROY A. HOWLAND, Wesleyan University, Middletown, Conn.



In the figure, if the axes of the ellipse be taken as coördinate axes, the coördinates of the lettered points are readily found as follows:

$$X',\ (a, 0);\quad Y',\ (0, b);\quad B,\left(\frac{2ab^2}{a^2+b^2},\ \frac{a^2b-b^3}{a^2+b^2}\right);$$

$$A,\left(a-\frac{a^2}{\sqrt{a^2+b^2}},\ \frac{ab}{\sqrt{a^2+b^2}}\right);\quad M',\left(\frac{a+k}{2},\ \frac{(a-k)\sqrt{3}}{2}\right),$$

where

$$k=\frac{a^2-b^2}{\sqrt{a^2+b^2}};$$

$$P',\ (k, 0);\quad R',\ (0, -k\sqrt{3});\quad AB'=a-k;\quad MM'=a+k.$$

Since $\begin{vmatrix} \dfrac{a+k}{2}, & \dfrac{(a-k)\sqrt{3}}{2}, & 1 \\ k & 0 & 1 \\ 0 & -k\sqrt{3} & 1 \end{vmatrix} = 0$, M', P' and R' lie on a straight line and since R' and P' are the centers of curvature of the arcs MM' and $M'N'$ respectively, the reason becomes evident why these arcs meet (in M') at an angle of 0°.

The equation of the ellipse, the circle about P', and the circle about R' are, respectively,

(1) $$x^2/a^2+y^2/b^2=1,$$

(2) $$(x-k)^2+y^2=(a-k)^2,$$

(3) $$x^2+(y+k\sqrt{3})^2=(a+k)^2.$$

We are, therefore, interested in the three arcs

$$(1') \qquad y = \frac{b}{a}\sqrt{a^2 - x^2},$$

$$(2') \qquad y = \sqrt{(a-k)^2 - (x-k)^2}, \qquad \frac{a+k}{2} \leqq x \leqq a,$$

$$(3') \qquad y = \sqrt{(a+k)^2 - x^2} - k\sqrt{3}, \qquad -\frac{a+k}{2} \leqq x \leqq \frac{a+k}{2}.$$

The distances of the circular arcs from the ellipse are given by

$$d_1 = \sqrt{(a-k)^2 - (x-k)^2} - \frac{b}{a}\sqrt{a^2 - x^2},$$

$$d_2 = \sqrt{(a+k)^2 - x^2} - k\sqrt{3} - \frac{b}{a}\sqrt{a^2 - x^2}.$$

If we put $b/a = \lambda$, $x = ax'$, then

$$k = \frac{1 - \lambda^2}{\sqrt{1 + \lambda^2}}a,$$

and the ratios d_1/a and d_2/a, which we may call the relative divergences of the circular arcs, are functions of λ and x'.

If $a = b$ the construction is obviously exact, hence we shall assume $a > b$ or $\lambda < 1$.

Within its interval d_1 can easily be shown to be always greater than zero. To find its maximum, we have

$$\frac{d(d_1)}{dx} = -\frac{x-k}{\sqrt{(a-k)^2 - (x-k)^2}} + \frac{bx}{a\sqrt{a^2 - x^2}}.$$

The critical values are found to be $x = a$, which gives a minimum $d_1 \doteq 0$, and the roots of the cubic

$$(a^2 - b^2)x^3 + (a^2 - b^2)(a - 2k)x^2 + a^2k(k - 2a)x + a^3k^2 = 0.$$

This cubic has three real roots, one in each of the intervals

$$(-\infty, 0), \left(0, \frac{a+k}{2}\right), \quad \text{and} \quad \left(\frac{a+k}{2}, a\right).$$

There is, therefore, always one and only one critical value between $(a + k)/2$ and a. This gives a maximum value of d_1.

A discussion of d_2 can be made more complete, but is less simple, for the reason that in its interval it changes sign, in some cases as many as four times. To show this, we will find the intersections of

$$x^2/a^2 + y^2/b^2 = 1$$

and

$$x^2 + (y + k\sqrt{3})^2 = (a + k)^2.$$

Eliminating x^2 we have

$$(a^2 + b^2)y^2 - 2\sqrt{3}b^2\sqrt{a^2 + b^2}y + 2b^2(a\sqrt{a^2 + b^2} - a^2 + b^2) = 0,$$

whence

$$y = \frac{b^2\sqrt{3} \pm b(\sqrt{a^2 + b^2} - a)}{\sqrt{a^2 + b^2}}.$$

Both of these values of y can be shown to be greater than the ordinate of M' and hence all real intersections of these two curves occur along the arc under discussion. The corresponding values of x are given by the equations

$$x^2 = \frac{a^2(a - b\sqrt{3})[2\sqrt{a^2 + b^2} - a + b\sqrt{3}]}{a^2 + b^2},$$

$$x^2 = \frac{a^2(a + b\sqrt{3})[2\sqrt{a^2 + b^2} - a - b\sqrt{3}]}{a^2 + b^2}.$$

The expressions in brackets are always positive. Hence we have three distinct cases:

(1) $a < b\sqrt{3}$, two real distinct intersections;

(2) $a = b\sqrt{3}$, two real distinct and two coincident;

(3) $a > b\sqrt{3}$, four real distinct.

In case (1), d_2 becomes zero twice and changes sign twice. In case (2), d_2 becomes zero three times and changes sign twice. In case (3), d_2 becomes zero four times and changes sign four times. To find the maxima of d_2, we have

$$\frac{d(d_2)}{dx} = \frac{-x}{\sqrt{(a + k)^2 - x^2}} + \frac{bx}{a\sqrt{a^2 - x^2}}.$$

The critical values are

$$x_1 = -\sqrt{\frac{a^4 - b^2(a + k)^2}{a^2 - b^2}}, \qquad x_2 = 0, \qquad x_3 = \sqrt{\frac{a^4 - b^2(a + k)^2}{a^2 - b^2}},$$

$$\frac{d^2(d_2)}{dx^2} = -\frac{(a + k)^2}{[(a + k)^2 - x^2]^{3/2}} + \frac{ab}{(a^2 - x^2)^{3/2}},$$

$$\left.\frac{d^2(d_2)}{dx^2}\right|_{x=0} = -\frac{1}{a + k} + \frac{b}{a^2} = \frac{-a^2 + b(a + k)}{a^2(a + k)}.$$

This is evidently positive and gives a minimum so long as $a^2 < b(a + k)$, that is, so long as the critical points x_1 and x_3 are imaginary. Since d_2 is now negative, this minimum is a maximum of its absolute value. When $a^2 = b(a + k)$, the three critical points coincide at $x = 0$. It is found in this case that

$$\frac{d^2(d_2)}{dx^2} = \frac{d^3(d_2)}{dx^3} = 0,$$

but

$$\frac{d^4(d_2)}{dx^4} > 0,$$

and we still have a minimum d_2 or maximum $| d_2 |$. For $a^2 > b(a + k)$, $x = 0$ gives a maximum d_2, which has no interest for us until d_2 becomes positive for $x = 0$, *i. e.*, until $a > \sqrt{3}b$. This value is $a - b + (1 - \sqrt{3})k$ or, introducing $b/a = \lambda$,

$$\left|\frac{d_2}{a}\right|_{x=0} = 1 - \lambda + (1 - \sqrt{3})\frac{1 - \lambda^2}{\sqrt{1+\lambda^2}}.$$

This function of λ can be shown to have a negative derivative for $\lambda < 1/\sqrt{3}$, hence this maximum divergence increases as λ decreases and approaches as a limit

$$2 - \sqrt{3} = 0.268\cdots.$$

We also find that at x_1 and x_3, as soon as they become real, we have maxima for $| d_2 |$ equal to

$$k\sqrt{3} - \frac{\sqrt{a^2 - b^2}\sqrt{(a + k)^2 - a^2}}{a}.$$

or

$$\left|\frac{d_2}{a}\right|_{x=x_1} = \frac{1 - \lambda^2}{\sqrt{1 + \lambda^2}}[\sqrt{3} - \sqrt{2\sqrt{1 + \lambda^2} + 1 - \lambda^2}].$$

This function of λ is found to be an increasing function[1] for $0 < \lambda < 0.72\cdots$. Hence as λ decreases, $| d_2/a |$ decreases and approaches the limit 0.

The following typical cases indicate the degree of approximation of the construction:

λ	critical points	max. of $\left\|\frac{d_1}{a}\right\|$	max. of $\left\|\frac{d_2}{a}\right\|$
$\frac{3}{4}$	$x = 0.858a$	0.0222	——
	$x = 0$	——	0.0062
	$x = \pm\frac{\sqrt{-23}}{20}a$	——	——
$\frac{1}{\sqrt{3}}$	$x = 0.905a$	0.0217	——
	$x = 0$	——	——
	$x = \pm 0.506a$	——	0.0040
$\frac{5}{12}$	$x = 0.942a$	0.0156	——
	$x = 0$	——	0.0249
	$x = \pm 0.747a$	——	0.0015

[1] $a^2 > b(a + k)$, $1 > \lambda\left(1 + \frac{1 - \lambda^2}{\sqrt{1 + \lambda^2}}\right)$, $\sqrt{1 + \lambda^2} > \lambda(1 + \lambda)$, $\lambda^4 + 2\lambda^3 - 1 < 0$ or $\lambda < 0.72$.

The above discussion shows the vertical divergence between the true and the constructed ellipse. A better measure of the approximation, however, is the divergence along radii of the circular arcs.

If we transform our equation of the ellipse to axes through the point P' and then change to polar coördinates, we have:

$$\rho^2[a^2 - (a^2 - b^2)\cos^2\theta] + 2b^2k\rho\cos\theta + b^2(k^2 - a^2) = 0$$

or, introducing λ and letting $\cos\theta = t$,

$$\rho^2(1 + \lambda^2)[1 - (1 - \lambda^2)t^2] + 2\lambda^2(1 - \lambda^2)\sqrt{1 + \lambda^2}\, at\rho + \lambda^2(\lambda^4 - 3\lambda^2)a^2 = 0.$$

This gives for the ellipse

$$\frac{\rho}{a} = -\frac{\lambda^2}{\sqrt{1 + \lambda^2}}\left[\frac{(1 - \lambda^2)t - \sqrt{3 - \lambda^2 - 2(1 - \lambda^2)t^2}}{1 - (1 - \lambda^2)t^2}\right].$$

For the circle about P' we have

$$\rho' = a - k = a\left(1 - \frac{1 - \lambda^2}{\sqrt{1 + \lambda^2}}\right)$$

or

$$\frac{\rho'}{a} = 1 - \frac{1 - \lambda^2}{\sqrt{1 + \lambda^2}}.$$

$\dfrac{\rho' - \rho}{a} = n_1$ we may call the relative normal divergence. The critical values of θ are found from the equation

$$\sqrt{1 - t^2}\left\{[1 - (1 - \lambda^2)t^2]\left[1 - \lambda^2 - \frac{2(1 - \lambda^2)t}{\sqrt{3 - \lambda^2 - 2(1 - \lambda^2)t^2}}\right]\right.$$
$$\left. + 2(1 - \lambda^2)t[(1 - \lambda^2)t - \sqrt{3 - \lambda^2 - 2(1 - \lambda^2)t^2}]\right\} = 0.$$

$t = 1$ or $\theta = 0$ gives us the minimum $n_1 = 0$. Equating the other factor to zero and simplifying, we obtain

$$2(1 - \lambda^2)^2t^6 - (1 - \lambda^2)(5 - \lambda^2)t^4 + 2(2 - \lambda^2)t^2 - 1 = 0$$

or

$$[(1 - \lambda^2)t^2 - 1]^2[2t^2 - 1] = 0.$$

We have then the rather remarkable result that the value of θ giving a maximum n_1 is independent of λ and is always 45°. This maximum is

$$n_1 = 1 - \frac{1 + (\sqrt{2} - 1)\lambda^2}{\sqrt{1 + \lambda^2}},$$

and

$$\frac{d(\max n_1)}{d\lambda} = \frac{(\sqrt{2} - 1)\lambda}{(1 + \lambda^2)^{3/2}}[\sqrt{2} - 1 - \lambda^2].$$

As λ decreases, therefore, max n_1 increases until $\lambda^2 = \sqrt{2} - 1$, when it has the value $0.0148\cdots$. It then decreases, approaching 0 with λ.

In similar manner, the equation of the ellipse, referred to R' as pole and a horizontal through R' as initial line, is

$$\rho/a = \frac{1}{\sqrt{1+\lambda^2}}\left[\frac{\sqrt{3}(1-\lambda^2)s + \lambda\sqrt{2(1-\lambda^2)(2-\lambda^2)s^2 - (3-\lambda^2)(1-2\lambda^2)}}{(1-\lambda^2)s^2+\lambda^2}\right],$$

where $s = \sin\theta$. For the circle $\rho'/a = 1 + \dfrac{1-\lambda^2}{\sqrt{1+\lambda^2}}$.

$|(\rho' - \rho)/a| = n_2$ will be the relative normal divergence for these arcs. Critical values of θ are found from an equation which in simplified form becomes

$$\sqrt{1-s^2}\,[(1-\lambda^2)s^2+\lambda^2]^2\,[2(2-\lambda^2)s^2-3] = 0.$$

For $s = 1$, $\theta = 90°$, we have the vertical divergence discussed previously. The other real critical value is given by

$$s = +\sqrt{\frac{3}{2(2-\lambda^2)}}, \qquad \lambda^2 \lesseqgtr \tfrac{1}{2}.$$

The corresponding maximum is

$$|n_2| = \frac{\sqrt{2(2-\lambda^2)}}{\sqrt{1+\lambda^2}} - \frac{1-\lambda^2}{\sqrt{1+\lambda^2}} - 1.$$

The three typical cases, computed above, give the following results:

λ	max $\lvert n_1\rvert$ $s^2 = \frac{1}{2}$	max $\lvert n_2\rvert$ $s = 1$	max $\lvert n_2\rvert$ $s = +\sqrt{\frac{3}{2(2-\lambda^2)}}$
$\frac{3}{4}$	0.0137	0.0062	———
$\frac{1}{\sqrt{3}}$	0.0145	0.0000	0.0038
$\frac{5}{12}$	0.0106	0.0249	0.0014

DISCUSSIONS.

Relating to the Indeterminate Form 0/0.

By M. O. Tripp, Olivet College, Olivet, Michigan.

In the May, 1916, number of the Monthly (Vol. XXIII, p. 180) Professor J. W. Nicholson considers the equation

$$y = \frac{x^2 - a^2}{x - a}, \tag{1}$$

and draws the conclusion that for $x = a$, *y has a definite and also an indeterminate value.* The object of this note is to show that we are not warranted in drawing such a conclusion.

When we clear (1) of fractions by multiplying both sides by $x - a$ and consider $(x - a)/(x - a)$ as having the value unity, we are at liberty to do so only upon condition that $x \neq a$.

Geometrically it is clear that when $x = a$ we are not warranted in drawing the conclusion that $y = 2a$ and $y = 0/0$. For, let x take a series of values from $x = a - d$ to $x = a + d$, $(d > 0)$. When $x \neq a$, $y = x + a$. Hence the coordinates of a point on the curve (1) satisfy the equation $y = x + a$, when $x \neq a$; but when $x = a$ we are not warranted in drawing the conclusion that y is necessarily equal to $2a$.

If we trace the locus of (1) from $x = a - d$ to $x = a + d$, we follow the straight line $y = x + a$ until we come to the point where $x = a$, then expand upward and downward to infinity vertically. As we pass on through $x = a$ the locus again follows the curve $y = x + a$. When $x = a$ in (1) we get the result that y is always genuinely indeterminate and does not take a definite value.

Note. The above discussion is published, as was Mr. Nicholson's, because of the interest which the subject possesses for many instructors.

In general, we take for granted that a function exists only where it has been defined. Consequently it seems to us that no amount of argument can get anywhere concerning the function in question for $x = a$ until the function has been defined for $x = a$, and when the function has been defined for $x = a$ the cause for argument disappears.

Neither Mr. Nicholson nor Mr. Tripp defines the function in question for $x = a$. In equation (1) of either paper, $y = x + a$ for $x \neq a$ and is not defined for $x = a$. Mr. Nicholson says (1) consists of two loci, but forgets that he introduces one of them, $x - a = 0$, when he multiplies both members of (1) by $x - a$.—U. G. M.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Ind.

After April 20, 1917, all communications to the Secretary, Professor W. D. CAIRNS, should be addressed to 55 East Lorain Street, Oberlin, Ohio, whither he is returning after a sojourn of seven months at the University of Chicago.

Dr. V. M. SLIPHER, for a number of years chief assistant at the Lowell Observatory, Flagstaff, Arizona, has been promoted to the directorship of the observatory, succeeding the late Percival Lowell.

Dr. EDWARD KIRCHER, Benjamin Peirce instructor in mathematics at Harvard University for the past two years, has accepted an instructorship in mathematics at the University of Minnesota.

CHARLES J. WHITE, emeritus professor of mathematics at Harvard University, died on February 12, at the age of seventy-eight years.

CHARLES A. PITKIN, professor of mathematics and physics at Thayer Academy, South Braintree, Mass., since 1877, died on November 7, at the age of sixty-three years.

Professor FLORIAN CAJORI, president of the Mathematical Association of America, has a review of Macfarlane's "Lectures on Ten British Mathematicians of the Nineteenth Century" in *Science*, January 26.

According to press dispatches from Berlin, the Prussian Minister of Education has announced to the Budget Committee of the Reichstag that 10,950 public-school teachers have fallen during the present European war and that their places have been taken by women.

Professor L. E. DICKSON, of the University of Chicago, has retired from the editorial committee of the *Transactions of the American Mathematical Society*, his unexpired term being filled by the selection of Professor L. P. EISENHART. Professors G. A. BLISS and E. B. WILSON, associate editors of the *Transactions*, have also retired, and have been succeeded by Professors C. N. MOORE and F. R. SHARPE.

The prize awards by the Paris Academy of Science, 1916, included 2,000 francs to Professor N. E. NÖRLUND, of the University of Lund, for his work on linear difference equations. The Paris Academy also announces the following mathematical prizes for 1918: (1) The Poncelet prize of 2,000 francs, to the author of the work most useful to the progress of pure mathematics; (2) The Francoeur prize of 1,000 francs, for discoveries or works useful to the progress of pure or applied mathematics.

The Texas Mathematics Teachers' Bulletin, Vol. 2, No. 2, appeared on December 15, 1916. This periodical is edited by Adjunct Professor J. W. CALHOUN, and Associate Professor C. D. RICE, of the University of Texas. It is published as a *Bulletin* of the University of Texas, and is open to the teachers of mathematics in Texas for the expression of their views, the editors assuming no responsibility for statements of facts or opinions in articles not written by them. The contents of the present number will show the nature of the subjects discussed. The following subjects are treated: "The geometry original," by J. G. DUNLAP; "The mathematics of investment," by E. L. DODD; "Literal arithmetic," by C. D. RICE; "On postulational systems," A. A. BENNETT.

The British Mathematical Association held its annual meeting at London on January 5, at which the following papers were presented: "The school syllabus in geometry," by T. P. NUNN; "Some of the work of the teaching committee," by A. W. SIDDONS; "Technical education and its relations to literature and science," by A. N. WHITEHEAD; "An accuracy test set in some public schools," by A. W. SIDDONS; "The place of mathematics in education reconstruction," by P. ABBOTT.

Regular meetings of the Edinburgh Mathematical Society were held on December 8 and January 12 at which the following papers were read: "On a class of continued fractions" and "A method of solving algebraic equations," by L. R. FORD; "On certain determinants of Cayley and Sylvester," by E. T. WHITTAKER; "An addition to the slide rule," by E. M. HORSBURGH; "The apolar locus of two tetrads of points," by W. P. MILNE.

At the January meeting of the London Mathematical Society the following papers were presented: "Asymptotic formulæ in combinatory analysis," by G. H. HARDY and S. RAMANUJAN; "The singular solutions of ordinary differential equations of the first order," by M. J. M. HILL; "The nature of a moving electric charge and its lines of electric force," by H. BATEMAN; "The expansion of the variables of a hypergeometric equation in terms of the ratio of two solutions," by L. J. ROGERS; "A problem in the theory of diffraction," by H. J. PRIESTLEY.

The February number of the *Proceedings* of the National Academy of Sciences contains the following mathematical papers: "Natural and isogonal families of curves on a surface," by JOSEPH LIPKA; "Some problems of Diophantine approximation," by G. H. HARDY and J. E. LITTLEWOOD; "A note on the fitting of parabolas," by J. R. MINER.

With this issue of the MONTHLY is completed a synopsis of mathematical courses offered in the coming summer sessions of the various colleges and universities of this country. Eighteen programs have been received in response to the request in the February number of the MONTHLY and the circular letter issued in January. The following summer announcements have been received since the copy for the March number of the MONTHLY was made up:

UNIVERSITY OF CALIFORNIA. Summer session, June 25–August 4. By Professor T. M. PUTNAM: Functions of a complex variable; Seminar for graduate students.—By Professor E. R. HEDRICK (University of Missouri): Functions of a real variable; A survey of elementary mathematics.—By Professor B. M. WOODS: Mathematics of investment; Integral calculus.—By Dr. H. N. WRIGHT: Synthetic projective geometry; Differential geometry. The elementary courses in freshman mathematics will be given.

HARVARD UNIVERSITY. Summer session, July 2–August 11. By Professor M. BÔCHER: Plane analytic geometry, five hours.—By Professor D. JACKSON: Logarithms and trigonometry, five hours.

UNIVERSITY OF KENTUCKY. Summer session, June 11–July 25. By Professor P. P. BOYD: Integral calculus, six hours; Analytics, six hours; College algebra, six hours.—By Professor J. M. DAVIS: Differential calculus, six hours: Trigonometry, six hours. Courses are offered in high-school algebra and geometry.

THE UNIVERSITY OF MINNESOTA. Summer session, June 18–July 28. By Professor G. N. BAUER: Trigonometry, eight hours; Fundamental concepts of secondary mathematics, four hours.—By Professor W. H. KIRCHNER: Solid geometry, eight hours; Descriptive geometry, eight hours.—By Professor W. F. HOLMAN: Integral calculus, twelve hours.—By Professor W. D. REEVE: Teachers' course, eight hours; Algebra, eight hours.—By Professor H. L. SLOBIN: Theory of equations, eight hours; Differential calculus, eight hours.—By Mr. R. M. BARTON: Analytics, eight hours; Algebra, eight hours.

UNIVERSITY OF MISSOURI. Summer session, June 7–August 3. By Professor W. D. A. WESTFALL: Integral calculus, five hours; Calculus of variations, three hours; Synoptic course, three hours.—By Professor W. D. CAIRNS (Oberlin College): Analytic geometry, five hours; Elementary calculus, five hours; Advanced calculus, three hours.—By Professor L. INGOLD: College algebra, five hours; Trigonometry, five hours; Teaching of mathematics, three hours.

UNIVERSITY OF SOUTHERN CALIFORNIA. Summer session, July 2–August 11. By Professor PAUL ARNOLD: Teachers' course, five hours; Integral calculus, five hours.—By Professor H. C. WILLETT: Analytic geometry, five hours; Trigonometry, five hours.

THE UNIVERSITY OF TENNESSEE. Summer session, June 19–July 27. By Professor H. E. BUCHANAN: History of mathematics; Trigonometry; Teachers' course in algebra.—By Professor J. B. HAMILTON: College algebra; Solid geometry; Plane geometry; Elementary algebra.

UNIVERSITY OF WASHINGTON. Summer session, June 18–July 31. By Professor R. E. MORITZ: Elementary graphs, five hours; Mathematics of investment, five hours; Definite integrals, five hours.—By Professor A. F. CARPENTER: Trigonometry, five hours; Mathematics in the high schools, five hours.—By Professor E. T. BELL: Teachers' course in algebra, five hours; Theory of equations, five hours.

KIRKSVILLE (MO.) STATE NORMAL SCHOOL. Summer Session, May 30–Aug. 2. By Professor WM. H. ZEIGEL: Integral Calculus and Surveying.—By Professor BYRON COSBY: Teaching of Elementary Mathematics; Trigonometry.—By Professor G. H. JAMISON: College Algebra; Analytic Geometry.—By Professor CHAS. A. EPPERSON: Teaching of Secondary Mathematics; Theory of Equations.

Various members of the Association are announced as special instructors for the summer sessions in institutions other than their own. Among these are Professors W. D. CAIRNS, at the University of Missouri, E. R. HEDRICK, at the University of California, D. N. LEHMER and G. D. BIRKHOFF, at the University of Chicago, A. COHEN and B. F. FINKEL, at the University of Colorado.

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THE ORIGIN OF MATHEMATICAL INDUCTION.*

By W. H. BUSSEY, University of Minnesota.

INTRODUCTION.

A criticism often made of mathematics as a subject of study in our high schools and colleges is that it involves nothing of observation, experiment, and induction as these terms are understood in the natural sciences. Whether or not the old and well-developed branches of mathematics as taught in our schools have been made into such well-organized deductive disciplines that this criticism is just, it is true that the work of the original investigators who have developed mathematical science has involved a great deal of observation, experiment, and induction; induction being, according to the Century Dictionary, "the process of drawing a general conclusion from particular cases." Observation and experiment in mathematics do not involve costly and complicated apparatus as often in physics, astronomy, and the other sciences, pencil and paper being all that is ordinarily necessary, but they are just as truly observation and experiment.

In the natural sciences a law arrived at by observation and experiment has to be verified by subsequent experiment by the same or other observers, either directly by repetition of the same experiment or indirectly by testing some logical consequence of the law in question. But in mathematics it is often possible to give rigorous demonstrations of theorems arrived at by ordinary induction. One method of clinching an argument by ordinary induction is what has been called *mathematical induction.* A more significant name and one that is being used more and more is *complete induction.* It is not a method of discovery but a method of proving rigorously that which has already been discovered. It is one of the most fruitful methods in all mathematics. It has

* Read before the Minnesota Section of the Association at its second meeting, April 9, 1917.

applications in widely differing branches of mathematics, in algebra, trigonometry, calculus, theory of probability, theory of groups, etc. In American college algebras it is used in proving divisibility theorems like *$x^n - y^n$ is divisible by $x - y$ for all positive integral values of n*; in proving the binomial theorem for positive integral exponents; and in proving given formulas for the summation of series, like

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \tfrac{1}{4}n^2(n+1)^2.$$

A theorem provable by complete induction involves a statement about an integer, usually denoted by n, which is to be proved true for all values of the integer. The proof is in two parts. The first part proves that the theorem is true in a special case, that is for a special value of the integer n involved in the theorem. The second part of the proof is what has been called the argument from n to $n + 1$. It is the argument which justifies one in drawing a general conclusion from the special cases verified. For this reason it may properly be called the *induction argument.* The method is too well known to need any further explanation here. If one assumes that the reader understands the method it is not necessary every time to write out explicitly the argument by which the two parts of the proof taken together establish the theorem in question for all values of n. It is necessary only to exhibit both parts of the proof. Indeed it is quite customary for writers of mathematical books to give only the argument from n to $n + 1$ and to leave the rest to be supplied by the reader.

MAUROLYCUS'S USE OF COMPLETE INDUCTION.

Cantor in his *Vorlesungen über Geschichte der Mathematik*[1] says that Pascal was the originator of the method of complete induction.[2] But he has corrected this statement in a brief note in the *Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht.*[3] In this note he says that he has been informed by G. Vacca that Maurolycus[4] described and used the method in his arithmetic which was published in 1575. I quote from Cantor: "Ich wurde durch Herrn G. Vacca darauf aufmerksam gemacht dass schon Maurolycus in seiner Arithmetik von 1575 die Methode genau geschildert und von ihr Gebrauch gemacht hat. Aus Maurolycus aber entnahm sie erst Pascal. Darüber kann nicht der leiseste Zweifel obwalten da Pascal sich 1659 für den Satz

$$2\left[\frac{n(n+1)}{2}\right] - n = n^2$$

ausdrücklich auf Maurolycus beruft welcher gerade diesen Satz mittels vollständiger Induktion bewiesen hat."[5]

[1] Vol. II, p. 749.

[2] W. W. R. Ball's *History of Mathematics* has nothing to say about the origin of complete induction.

[3] Vol. XXXIII (1902), p. 536.

[4] *D. Francisci Maurolyci, Abbatis Messanensis, Mathematici Celeberrimi, Arithmeticorum Libri Duo.* Venice, 1575.

[5] The theorem referred to is equivalent to *twice the nth triangular number minus n equals n^2.* See page 202 of this paper.

Maurolycus in Book I of his arithmetic begins with the definitions of different kinds of numbers, namely, *even, odd, triangular, square, numeri parte altera longiores,* etc. By definition the nth triangular number is the sum of the integers from 1 to n inclusive and the nth *numerus parte altera longior* is $n(n-1)$. He arranges them in a table as follows:

Integers	*Even*	*Odd*	*Triangular*	*Square*	*N. P. A. L.*[1]
1	0	1	1	1	0
2	2	3	3	4	2
3	4	5	6	9	6
4	6	7	10	16	12
5	8	9	15	25	20
6	10	11	21	36	30
7	12	13	28	49	42
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
n	E	O	T	S	L

I have added at the bottom of each column a symbol for the nth number of that column which I find convenient in explaining the work of Maurolycus. Numbers in the same row Maurolycus calls collateral numbers. A number in one row is said by him to *precede* any number in the following row and to *follow* any number in the preceding row; *e. g.,* he calls 15 the triangular number following the even number 6.

I quote a number of Maurolycus's theorems for reference. In some cases I give the proofs as given by him. The numbering of the theorems is his.

Proposition IV. "The odd numbers are obtained from unity by successive additions of 2." (Maurolycus uses this in *Proposition VI* in the form $O_n + 2 = O_{n+1}$, *i. e.,* the nth odd number plus 2 equals the next odd number.)

Proposition VI. "Every integer plus the preceding integer equals the collateral odd number." [In symbols this is $n + (n-1) = O_n$.] Maurolycus's proof, freely translated, is this:

> "The integer 2 added to unity makes the integer 3 but when added to 3 it makes an amount greater by 2 and this (by virtue of *Proposition IV*) is the next odd integer, namely 5. Again since the integer 3 added to 2 makes 5, which is the collateral odd integer, when it is added to 4 the result will be greater by 2, that is (by virtue of *Proposition IV*), it will be the next odd integer which is 7. And in like manner to infinity as the proposition states."

This is not a very clear statement of a proof by mathematical induction but the idea is there. Maurolycus's ideas might be put more clearly as follows: The theorem is true by inspection in the case of the first two integers 1 and 2, *i. e.,* $2 + 1 = 3$ which is the odd integer collateral to 2. This is the first part of the induction proof. Maurolycus then takes up the special cases $3 + 2 = 5$ and $4 + 3 = 7$ and in doing so he shows by his repeated use of *Proposition IV* that

[1] *Numeri Parte Altera Longiores.*

the other part of the mathematical induction proof was in his mind. His *Proposition IV* furnishes the argument from n to $n+1$. In modern notation it would be put in this way:

If $n + (n - 1) = O_n$ (*i. e.*, if any integer plus the preceding one equals the collateral odd integer), the result of adding $1 + 1$ to the left side and 2 to the right side is $(n + 1) + n = O_n + 2$. But by Prop. IV, $O_n + 2 = O_{n+1}$. Therefore $(n + 1) + n = O_{n+1}$.

This argument from n to $n+1$ seems to have been in Maurolycus's mind. But if this were the only example of complete induction in his work it might not be a conclusive proof that he understood the method. *Proposition XV* is a much more convincing case. But before giving an account of it I wish to state several other of his theorems for reference and to discuss the proposition which Cantor says Pascal got from Maurolycus.

Proposition VIII. "Every triangular number doubled equals the following *numerus parte altera longior.*" (In symbols this is $2T_n = L_{n+1}$.)

Proposition X. "Every *numerus parte altera longior* plus its collateral integer equals the collateral square number." (In symbols, $L_n + n = S_n$.)

Proposition XI. "Every triangular number plus the preceding triangular number equals the collateral square number." (In symbols, $T_n + T_{n-1} = S_n$.)

This proposition, although stated somewhat differently by Cantor, is the one which Cantor says Pascal got from Maurolycus and which he says Maurolycus proved by complete induction. For since a triangular number is equal to the sum of the natural numbers in order, $T_{n-1} = T_n - n$; and it follows that $T_n + T_{n-1} = 2T_n - n$; or, since by the formula for the sum of an arithmetic progression[1] the nth triangular number is $[n(n + 1)]/2$, the equation $T_n + T_{n-1} = S_n$ is equivalent to

$$2\left[\frac{n(n+1)}{2}\right] - n = S_n = n^2,$$

which is the form that Cantor gives. But Cantor is wrong in saying that this theorem was proved by Maurolycus by complete induction. For Maurolycus's proof (in modern notation) is this:

By definition $T_n = T_{n-1} + n$. Therefore $T_n + T_{n-1}$, the left-hand member of the relation to be proved, is equal to $2T_{n-1} + n$ which equals $L_n + n$ (by *Proposition VIII*), and this equals S_n (by *Proposition X*).

Proposition XIII. "Every square number plus the following odd number equals the following square number." (In symbols, $S_n + O_{n+1} = S_{n+1}$.)

Proposition XV. "The sum of the first n odd integers is equal to the nth square number."[2] (In symbols, $O_1 + O_2 + O_3 + \cdots + O_n = S_n$.) Maurolycus's proof freely translated is this:

---

[1] In *Proposition VII*, which I have not quoted, Maurolycus uses the usual arithmetical progression device for proving $T_n = [n(n + 1)]/2$. He says in effect: $T_n = 1 + 2 + 3 + \cdots + n$ and also $T_n = n + (n - 1) + (n - 2) + \cdots + 2 + 1$, and therefore by addition $2T_n = n(n + 1)$.

[2] This theorem in the form $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ is often given in American college algebras as one of the first examples of complete induction to be proved by the student.

"By a previous proposition[1] the first square number (unity) added to the following odd number (3) makes the following square number (4); and this second square number (4) added to the 3d odd number (5) makes the 3d square number (9); and likewise the 3d square number (9) added to the 4th odd number (7) makes the 4th square number (16); and so successively to infinity the proposition is demonstrated by the repeated application of *Proposition XIII.*"

This is a clear case of a complete induction proof. *Proposition XIII* is used as a lemma. It furnishes the argument from n to $n + 1$. The first few special cases are mentioned in *Proposition XV* itself. In modern symbols the proof would be this:

1st. The theorem is true when $n = 1$. *2d.* Assume that it is true when $n = k$, *i. e.*, assume $O_1 + O_2 + \cdots + O_k = S_k$; add O_{k+1} to both sides of this equation and get $O_1 + O_2 + \cdots + O_{k+1} = S_k + O_{k+1}$ which equals S_{k+1}, by *Proposition XIII.*

Pascal's Use of Complete Induction.

Pascal repeatedly used the method of complete induction in connection with his arithmetical triangle[2] and its applications. Cantor's argument that Pascal borrowed the method from Maurolycus is valid in spite of the fact that he is in error in saying that Maurolycus proved *Proposition XI* of his arithmetic by complete induction. For Pascal does refer to Maurolycus for a proof of this proposition which shows that Pascal was familiar with the part of Maurolycus's arithmetic in which Maurolycus does use complete induction. It is in a letter from Pascal (writing under the pseudonym Amos Dettonville) to Carcavi[3] that Pascal refers to Maurolycus for the proof of the theorem that "*twice the nth triangular number minus n equals* n^2." Pascal says "*Cela est aisê par Maurolic.*" Pascal makes use of this theorem in connection with his work on centers of gravity.

I give the following as two interesting examples of Pascal's use of the method of complete induction. I use modern algebraic notation for the sake of brevity. Without modern notation it would be necessary to explain the construction of Pascal's arithmetical triangle and many of his theorems about it.

Theorem.[4] *The number of combinations of n things k at a time is to the number of combinations of n things $k + 1$ at a time as $k + 1$ is to $n - k$.* (In symbols, ${}_nC_k : {}_nC_{k+1} = k + 1 : (n - k)$.)

Proof.—First part. By inspection the theorem is true when $n = 2$. For then the only possible values of k and $k + 1$ are 1 and 2 respectively and ${}_2C_1 : {}_2C_2 = 2 : 1$.

Second part. Assume that the theorem is true when $n = q$. That is, assume[5]

$$(A) \qquad {}_qC_k : {}_qC_{k+1} = k + 1 : q - k$$

for all positive integral values of $k < q$. It is then to be proved that, on this assumption,

$$(B) \qquad {}_{q+1}C_j : {}_{q+1}C_{j+1} = j + 1 : q + 1 - j$$

for all positive integral values of $j < q + 1$.

[1] He refers to *Proposition XIII.*

[2] *Oeuvres Complètes de Blaise Pascal*, Paris, 1889, Vol. III, p. 243 ff.

[3] L. c., p. 376.

[4] This is Pascal's "Consequence XII," l. c., p. 248.

[5] This relation (A) is seen by inspection to be true when $k = q$, if we define ${}_sC_t = 0$ when $t > s$.

[(B) is obtained from (A) by replacing q in (A) by $q + 1$ and by using another letter for k to avoid confusion later.] The well-known relation[1]

$$(C) \qquad {}_NC_R = {}_{N-1}C_{R-1} + {}_{N-1}C_R$$

is needed to prove that (B) follows from (A).

By relation (C) the left-hand member of (B) is equal to

$$\frac{{}_qC_{j-1} + {}_qC_j}{{}_qC_j + {}_qC_{j+1}} = \frac{\dfrac{{}_qC_{j-1}}{{}_qC_j} + 1}{1 + \dfrac{{}_qC_{j+1}}{{}_qC_j}}.$$

On applying relation (A) to the minor fractions ${}_qC_{j-1}/{}_qC_j$ and ${}_qC_{j+1}/{}_qC_j$ this becomes

$$\frac{\dfrac{j}{q-j+1} + 1}{1 + \dfrac{q-j}{j+1}} = \frac{j+1}{q-j+1}$$

which was to be proved.

Another example from Pascal is this problem in the division of stakes in a gambling game. Two players A and B of equal skill, playing for a stake P, wish to leave the game table before finishing their game. Their scores and the number of points which constitute the game being given indirectly as follows: Player A lacks α points of winning and player B lacks β points. If $\alpha + \beta$ be denoted by n, Pascal says that the stakes should be divided between B and A in the ratio

$$({}_{n-1}C_0 + {}_{n-1}C_1 + {}_{n-1}C_2 + \cdots + {}_{n-1}C_{\alpha-1}) : ({}_{n-1}C_\alpha + {}_{n-1}C_{\alpha+1} + \cdots + {}_{n-1}C_{n-1}).$$

Since[2]

$${}_{n-1}C_0 + {}_{n-1}C_1 + {}_{n-1}C_2 + \cdots + {}_{n-1}C_{n-1} = 2^{n-1},$$

this is the same as saying that B's share of the whole stake is

$$\frac{P}{2^{n-1}}[{}_{n-1}C_0 + {}_{n-1}C_1 + {}_{n-1}C_2 + \cdots + {}_{n-1}C_{n-1}],$$

and A's share is

$$\frac{P}{2^{n-1}}[{}_{n-1}C_\alpha + {}_{n-1}C_{\alpha+1} + \cdots + {}_{n-1}C_{n-1}].$$

Proof.—First Part. The theorem is true in the special case in which $n = 2$. For in this case the scores of A and B are even and each lacks only one point of winning. They are of equal skill and so one is as likely to win the game as the other and the stake should be divided in the ratio 1 to 1. The theorem states that the stake should be divided in the ratio ${}_1C_0 : {}_1C_1$, which is $1 : 1$. The theorem is also true in the special case in which $n = 3$. In this case the score at the end of play must be such that one player lacks one point and the other lacks two points. Suppose that it is A who lacks the one point. Then B lacks two points. If the play were to continue for one more point and if A were to win that point he would win the game and be entitled to the whole stake P. But if he were to lose he would be entitled to $P/2$ by virtue of the special

[1] Fine, H. B., *College Algebra*, p. 404. This relation is true when $R = 0$ and when $N \geqq R$ if we define ${}_sC_0 = 1$ and ${}_sC_t = 0$ when $t > s$.

[2] It is customary to give ${}_NC_0$ the meaning ${}_NC_0 = 1$ by definition. The relation ${}_{n-1}C_0 + {}_{n-1}C_1 + \cdots + {}_{n-1}C_{n-1} = 2^{n-1}$ is more usually written ${}_{n-1}C_1 + {}_{n-1}C_2 + \cdots + {}_{n-1}C_{n-1} = 2^{n-1} - 1$. See Fine, H. B., *College Algebra*, p. 402.

case previously considered. The division of the stake should therefore be such that A will get the $P/2$ which he is entitled to in case he loses the next point and one half of the other $P/2$ which he has an even chance of winning. This is the same as saying that A should receive the arithmetic mean between P and $P/2$. The stake should therefore be divided between B and A in the ratio 1 to 3. The theorem gives the ratio ${}_2C_0 : ({}_2C_1 + {}_2C_2)$, which is $1 : 3$.

If A lacked 2 points and B one point the division between B and A would be in the ratio $3 : 1$. The theorem in this case gives $({}_2C_0 + {}_2C_1) : {}_2C_2$ which is $3 : 1$.

Second Part of the Proof. Assume the theorem true for n, *i. e.*, assume that when the points lacking for A and B have their sum equal to n the equitable division of the stake is as the theorem indicates. To prove that on this assumption the theorem is true when the sum of the points lacking for A and B is $n + 1$, suppose that A lacks k points and B lacks l points where $k + l = n + 1$. If the play were to continue and A were to win the next point, the sum of the points lacking after that would be exactly n points (*i. e.*, A would lack $k - 1$ points and B would lack l points) and by our assumption the rule given by the theorem may be applied. The result of applying the theorem in this case is

$$\frac{P}{2^{n-1}}[{}_{n-1}C_0 + {}_{n-1}C_1 + \cdots + {}_{n-1}C_{k-2}]$$

for B's share of the stake. (This comes from putting $k - 1$ for α in the statement of the theorem.) But if A were to lose then B would win and A would lack k points and B only $l - 1$ points. The sum of the points lacking would be exactly n and the rule given by the theorem may be applied as before, putting k for α. The result is

$$\frac{P}{2^{n-1}}[{}_{n-1}C_0 + {}_{n-1}C_1 + \cdots + {}_{n-1}C_{k-1}]$$

for B's share. As in the special case previously considered B's share of the stake should be the arithmetic mean between

$$\frac{P}{2^{n-1}}[{}_{n-1}C_0 + {}_{n-1}C_1 + \cdots + {}_{n-1}C_{k-2}] \quad \text{and} \quad \frac{P}{2^{n-1}}[{}_{n-1}C_0 + {}_{n-1}C_1 + \cdots + {}_{n-1}C_{k-1}]$$

which equals

$$\frac{P}{2^n}[2({}_{n-1}C_0 + {}_{n-1}C_1 + \cdots + {}_{n-1}C_{k-2}) + {}_{n-1}C_{k-1}].$$

This may be written in the form

$$\frac{P}{2^n}[{}_{n-1}C_0 + ({}_{n-1}C_0 + {}_{n-1}C_1) + ({}_{n-1}C_1 + {}_{n-1}C_2) + \cdots + ({}_{n-1}C_{k-3} + {}_{n-1}C_{k-2}) + ({}_{n-1}C_{k-2} + {}_{n-1}C_{k-1})].$$

But by virtue of the relation ${}_NC_R = {}_{N-1}C_{R-1} + {}_{N-1}C_R$, used in the proof of the preceding theorem of Pascal's, each binomial in this expression may be replaced by a single term so that B's share is

$$\frac{P}{2^n}[{}_nC_0 + {}_nC_1 + {}_nC_2 + \cdots + {}_nC_{k-2} + {}_nC_{k-1}]$$

which is just what the theorem gives for B's share.

This completes the induction argument for this theorem of Pascal's.

Other and More Recent Uses of Complete Induction.

The following examples will give some idea of the variety of uses which can be made of the method of complete induction.

1. (*a*) *$x^n - y^n$ is divisible by $x - y$.*
   (*b*) *$x^n - y^n$ is divisible by $x + y$ when n is even but not*[1] *when n is odd.*
   (*c*) *$x^n + y^n$ is divisible by $x + y$ when n is odd but not*[1] *when n is even.*

[1] The reference here (as elsewhere in the paper) is to *algebraic* divisibility. The statements obviously need not be true for the case of divisibility of the integers represented by the forms.

Negative divisibility theorems like the second parts of (*b*) and (*c*) are just as easily proved by complete induction as positive divisibility theorems such as (*a*).

2. $$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1) = \tfrac{1}{3}n(n+1)(n+2).$$

American college algebras contain many such examples in the summation of series.

3. *De Moivre's Theorem.* $(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$ *for positive integral values of* n.

4. *The binomial theorem for positive integral exponents.*

5. *If* $$F_n(x) \equiv (x+a_1)(x+a_2)(x+a_3)\cdots(x+a_n)$$
$$\equiv x^n + A_1x^{n-1} + A_2x^{n-2} + \cdots + A_{n-1}x + A_n,$$
then
$$A_1 = \Sigma a_1, \quad A_2 = \Sigma a_1a_2, \quad A_3 = \Sigma a_1a_2a_3, \quad \cdots, \quad A_n = a_1a_2\cdots a_n.$$

(See H. Weber and J. Wellstein, *Encyclopädie der Elementar-Mathematik,* volume I, p. 196.)

6. *Fermat's Theorem.* $a^n - a$ *is divisible by* n *if* a *is any integer and* n *is a prime integer.* (See Weber and Wellstein, volume I, p. 197.)

7. *The Polynomial Theorem.*
$$(x+y+z+\cdots)^n = \Sigma\frac{n!}{\alpha!\beta!\gamma!\cdots}\,x^\alpha y^\beta z^\gamma\cdots.$$

(See Weber and Wellstein, volume I, p. 198.)

8. *Any polynomial symmetric in* $x_1, x_2, \cdots, x_n$ *is equal to a polynomial in the elementary symmetric functions.* (See Weber and Wellstein, volume I, p. 232.)

9. *A necessary and sufficient condition that a polynomial in any number of variables vanish identically is that all its coefficients are zero.* (For a proof of this theorem for one variable and its extension by complete induction see Maxime Bôcher's *Introduction to Higher Algebra,* p. 5.)

10. *If* f_1 *and* f_2 *are polynomials in any number of variables of degrees* m_1 *and* m_2 *respectively, the product* f_1f_2 *will be of degree* m_1m_2. (See Bôcher's *Higher Algebra,* p. 6.)

11. *A necessary and sufficient condition that a polynomial* $f(x_1x_2\cdots x_n)$ *vanish identically is that it vanish throughout the neighborhood of a point* $(a_1a_2\cdots a_n)$. (See Bôcher's *Higher Algebra,* p. 10.)

12. *If* $f(x)$ *is a polynomial of the nth degree,*
$$f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \qquad (a_0 \neq 0),$$
there exists one and only one set of constants $\alpha_1\alpha_2\cdots\alpha_n$ *such that*
$$f(x) \equiv a_0(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n).$$

To prove this theorem by complete induction one needs the fundamental theorem of algebra that there is at least one value of x for which such a polynomial as $f(x)$ vanishes. (See Bôcher's *Higher Algebra,* p. 17.)

13. *If all the* $(r+1)$*-rowed principal minors of the symmetrical matrix*

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad (\text{where } a_{ij} = a_{ji})$$

are zero, and also all the $(r+2)$*-rowed principal minors, then the rank of the matrix is* r *or less.* (See Bôcher's *Higher Algebra*, p. 57.)

14. *The conjugate of the product of any number of matrices is the product of their conjugates taken in reverse order.* (See Bôcher's *Higher Algebra*, p. 65.)

15. *If* $y = u_1 u_2 \cdots u_n$ *and* y', u_1', u_2', *etc., denote first derivatives with respect to a variable* x, *then*

$$\frac{y'}{y} = \frac{u_1'}{u_1} + \frac{u_2'}{u_2} + \cdots + \frac{u_n'}{u_n}.$$

16. *Leibnitz's Theorem.*

$$\frac{d^n(uv)}{dx^n} = u_n v + n u_{n-1} v_1 + \frac{n(n-1)}{1\cdot 2} u_{n-2} v^2 + \cdots + n u_1 v_{n-1} + u v_n,$$

where u *and* v *are functions of* x *and the subscripts denote derivatives with respect to* x. *That is,*

$$u_1 = \frac{du}{dx}, \qquad u_2 = \frac{d^2u}{dx^2}, \qquad \text{etc.}$$

(For a proof of this theorem and some examples of its use see G. A. Osborne's *Differential and Integral Calculus*, pages 65–67.)

17. (*a*) *Limit* $(x_1 + x_2 \cdots + x_n) =$ *limit* $x_1 +$ *limit* $x_2 + \cdots +$ *limit* x_n.

(*b*) *Limit* $(x_1 x_2 \cdots x_n) =$ (*limit* x_1)(*limit* x_2) $\cdots$ (*limit* x_n).

(See W. F. Osgood's *Differential and Integral Calculus*, p. 16.)

18. *If*

$$y = \log x, \qquad \frac{d^n y}{dx^n} = \frac{(-1)^{n-1}(n-1)!}{x^n}.$$

Numerous examples like this are to be found in G. A. Osborne's *Differential Calculus*, pages 62–65.

QUADRATIC FACTORS OF POLYNOMIALS.

By A. F. FRUMVELLER, Marquette University.

All methods of finding quadratic factors of $f(x) = \sum_0^n (a_i x^i)$ will be empirical to a certain extent; if $x^2 + mx + n$ be the factor sought, any one of the divisors of a_0 is taken as n, and its associated number m remains to be found. The older method[1] (quoted, for example, in Wentworth, McLellan and Glashan's "Algebraic Analysis") consisted in arranging in a vertical column the expressions

$$\cdots, f(-3), f(-2), f(-1), f(0), f(1), f(2), f(3), \cdots$$

and placing after each of these numbers all its factors written with both plus and minus signs. If $r_i^{(s)}$ be one of these factors, lying in the horizontal row of $f(s)$, add s^2 to it. Then the numbers $r_i^{(s)} + s^2$ thus formed are kept on the line of $f(s)$, and constitute our actual working-table. If now an arithmetic sequence can be discovered running *vertically* through this table, the *difference* of this sequence is a provisional value of m, to be tried out by synthetic division. It is most laborious and unsatisfactory to attempt this scheme.

The method proposed by Professor Glenn, in this MONTHLY for October, 1916, locates m at the outset among the factors of a certain symmetric function of the coefficients, namely

$$P(x_i + x_j), \quad (i \neq j;\ i, j = 1, 2, 3, \cdots n);$$

the proper association of each m with its own n being then determined experimentally by synthetic division. He uses the upper limit L of the roots to cut down the number of the possible combinations, by excluding values of (m, n) that exceed $(2L, L^2)$. This is a great improvement; but there is still too much room for experiment, and the real problem of getting m as a $f(n)$ is not met. The ideal solution would be to assign a process or an equation which would directly associate with each assumed n its own m, or, at least, a minimum number of m's from which to choose. The methods presented herewith for that purpose seem to be new, and may be of interest on account of their rapid and easy application.

The Quartic. Let

$$f(x) \equiv x^4 + ax^3 + bx^2 + cx + d = (x^2 + mx + n)(x^2 + px + q).$$

Then (1) $a - m = p$, (2) $b - n = mp + q$, (3) $c = np + mq$, (4) $d = nq$. From (1) and (3), we get $m = (c - an)/(q - n)$, which from the nature of the problem must be integral. We now try every $n < L^2$, and check the m's found in (2), which, combining with (1), becomes

$$m^2 - am - (n + q - b) = 0.$$

Thus, for $\left\{\begin{matrix} & a & b & c & d \\ 1 & -6 & +3 & +22 & -6 \end{matrix}\right\}$, we find $(m, n) = (*, -1), (-4, 1),$

[1] See also Kronecker, Runge and Mandl in *Crelle's Journal*, Vols. 92, 99 and 113 respectively.

$\cdots$ and this last satisfies (2). Hence $x^2 - 4x + 1$ is a factor. Again, using $\left\{\begin{array}{ccccc} & a & b & c & d \\ 1 & +4 & -4 & -17 & +10 \end{array}\right\}$ (see Glenn, loc. cit.), we have $(m, n) = (*, 1)$, $(*, -1)$, $(*, 2)$, $(3, -2)$. This last pair satisfies (2), hence

$$f(x) = (x^2 + 3x - 2)(x^2 + x - 5).$$

For brevity and directness, this method leaves little to be desired.

The Quintic. Let

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = (x^2 + mx + n)(x^3 + px^2 + qx + r).$$

Then (1) $a - m = p$, (2) $b - n = mp + q$, (3) $c = np + mq + r$, (4) $d = nq + mr$, (5) $e = nr$. If we find q from (4) and (5), and also from (1) and (2), and compare these values, we get

$$m^2 + m\left(\frac{e}{n^2} - a\right) + \left(b - n - \frac{d}{n}\right) = 0,$$

where m is an integer. For each m found, compute p and q, and check in (3).

Using $\left\{\begin{array}{cccccc} & a & b & c & d & e \\ 1 & -1 & 6 & 9 & 1 & -4 \end{array}\right\}$, we find for $n = \pm 1, \pm 2$, m turns out fractional or imaginary; hence there is no quadratic factor.

Again, using $\left\{\begin{array}{cccccc} & a & b & c & d & e \\ 1 & -1 & -6 & 9 & 1 & -4 \end{array}\right\}$, for (m, n) we get $(*, 1)$, $(4, -1)$, $(-1, -1)$; and on computing p and q, the check is satisfied for $m = -1$, $n = -1$, $p = 0$, $q = -5$. Hence $f(x) = (x^2 - x - 1)(x^3 - 5x + 4)$.

The Sextic. The conditional equations are: (1) $a - m = p$, (2) $b - n = mp + q$, (3) $c = np + mq + r$, (4) $d = nq + mr + s$, (5) $e = nr + ms$, (6) $f = ns$. Hence

$$q = \frac{1}{n^3}\begin{vmatrix} d & m & 1 \\ e & n & m \\ f & 0 & n \end{vmatrix} = \begin{vmatrix} 1 & a - m \\ m & b - n \end{vmatrix},$$

and the m-eliminant is, therefore,

$$m^2\left(\frac{f}{n} - n^2\right) + m(an^2 - e) + (n^3 - n^2 b + dn - f) = 0.$$

Using $\left\{\begin{array}{ccccccc} & a & b & c & d & e & f \\ 1 & 2 & -5 & -7 & 8 & 3 & -2 \end{array}\right\}$, with (3) as a check, for $n = -1$ we have $m = 2$, or -1; for $n = -2$, $m = 1$. These numbers give the three quadratic factors.

It will be noticed that the m-eliminant is a quadratic for the *quintic* and *sextic*; a cubic for the *septic* and *octic*; for a polynomial of degree $2k$ or $2k - 1$, it is of degree $k - 1$. The use of determinants makes it easy to write out these equations. One check-equation will always be left over, by means of which the

$(k-1)$ m-values can be tested. Having the m-equation, we therefore insert $n < L^2$, and look for integral roots by synthetic division, as far as $m < 2L$, where L is the upper (lower) limit of the roots of $f(x)$. We are then sure that the m's thus found are correctly paired with their own proper n.

The following table gives the m-eliminants for degrees 4 to 9; the law of their structure is obvious, so that the table can be extended indefinitely.

4-ic: $\dfrac{1}{n^2}\begin{vmatrix} c & m \\ d & n \end{vmatrix} - |a - m| = 0.$ Check in the b-equation.

5-ic: $\dfrac{1}{n^2}\begin{vmatrix} d & m \\ e & n \end{vmatrix} - \begin{vmatrix} 1 & a-m \\ m & b-n \end{vmatrix} = 0.$ Check in the c-equation.

6-ic: $\dfrac{1}{n^3}\begin{vmatrix} d & m & 1 \\ e & n & m \\ f & 0 & n \end{vmatrix} - \begin{vmatrix} 1 & a-m \\ m & b-n \end{vmatrix} = 0.$ Check in the c-equation.

7-ic: $\dfrac{1}{n^3}\begin{vmatrix} e & m & 1 \\ f & n & m \\ g & 0 & n \end{vmatrix} - \begin{vmatrix} 1 & 0 & a-m \\ m & 1 & b-n \\ n & m & c \end{vmatrix} = 0.$ Check in the d-equation.

8-ic: $\dfrac{1}{n^4}\begin{vmatrix} e & m & 1 & 0 \\ f & n & m & 1 \\ g & 0 & n & m \\ h & 0 & 0 & n \end{vmatrix} - \begin{vmatrix} 1 & 0 & a-m \\ m & 1 & b-n \\ n & m & c \end{vmatrix} = 0.$ Check in the d-equation.

9-ic: $\dfrac{1}{n^4}\begin{vmatrix} f & m & 1 & 0 \\ g & n & m & 1 \\ h & 0 & n & m \\ i & 0 & 0 & n \end{vmatrix} - \begin{vmatrix} 1 & 0 & 0 & a-m \\ m & 1 & 0 & b-n \\ n & m & 1 & c \\ 0 & n & m & d \end{vmatrix} = 0.$ Check in the e-equation.

The General Polynomial. To find an explicit form of the m-equation for the general case, it will be expedient to use a more systematic notation, and to modify the preceding method somewhat, so as to avoid having two determinants in the same equation. Let

$$\begin{aligned} f(x) &\equiv x^k + a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots + a_0 \\ &= (x^2 + mx + n)(x^{k-2} + b_{k-3}x^{k-3} + \cdots + b_0). \end{aligned}$$

Performing the multiplication and equating coefficients, we get

$$\begin{aligned} a_{k-1} &= m + b_{k-3}, \\ a_{k-2} &= n + mb_{k-3} + b_{k-4}, \\ a_{k-3} &= nb_{k-3} + mb_{k-4} + b_{k-5}, \\ &\cdots\cdots\cdots\cdots \\ a_1 &= nb_1 + mb_0, \\ a_0 &= nb_0. \end{aligned}$$

This is a consistent set of k linear equations between the $(k-1)$ quantities $[1, b_{k-3}, b_{k-4}, \cdots, b_1, b_0]$. Eliminating these, we have at once as our m-equation

$$\text{(I)} \qquad 0 = \begin{vmatrix} a_{k-1} & m & 1 & 0 & 0 & \cdots & 0 & 0 \\ a_{k-2} & n & m & 1 & 0 & \cdots & 0 & 0 \\ a_{k-3} & 0 & n & m & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1 & 0 & 0 & 0 & 0 & \cdots & n & m \\ a_0 & 0 & 0 & 0 & 0 & \cdots & 0 & n \end{vmatrix}.$$

This is of the $(k-1)$th degree in m, whereas the equations formerly obtained were only of degree $(k-3)/2$ or $(k-4)/2$, so that the number of m's belonging to a given n is doubled; but the present equation is more readily handled for the purpose we have in view, namely, to display the inner structure of the m-eliminant.

Let us use the symbols

$$\Delta_2, \Delta_3, \cdots \text{ for } \begin{vmatrix} m & 1 \\ n & m \end{vmatrix}, \begin{vmatrix} m & 1 & 0 \\ n & m & 1 \\ 0 & n & m \end{vmatrix}, \cdots.$$

Then, on continuously developing (I) in terms of its last row, we get

$$\text{(II)} \qquad \begin{aligned} 0 &= a_0\Delta_{k-1} - na_1\Delta_{k-2} + \cdots + (-1)^{k-1}n^{k-1}a_{k-1}\Delta_0 \\ &= \sum_{\lambda=0}^{\lambda=k-1} [(-1)^\lambda n^\lambda a_\lambda \Delta_{k-\lambda-1}]; \quad (\Delta_1 = m;\ \Delta_0 = 1). \end{aligned}$$

It remains to evaluate Δ_r. Let us write down a few of the first determinants of this form, to observe the law, noting that $\Delta_r = m\Delta_{r-1} - n\Delta_{r-2}$.

$$\begin{aligned} \Delta_1 &= m, & \Delta_5 &= m^5 - 4m^3n + 3mn^2, \\ \Delta_2 &= m^2 - 1n, & \Delta_6 &= m^6 - 5m^4n + 6m^2n^2 - 1n^3, \\ \Delta_3 &= m^3 - 2mn, & \Delta_7 &= m^7 - 6m^5n + 10m^3n^2 - 4mn^3, \\ \Delta_4 &= m^4 - 3m^2n + 1n^2, & \Delta_8 &= m^8 - 7m^6n + 15m^4n^2 - 10m^2n^3 + 1n^4. \end{aligned}$$

If one desires an explicit general formula for Δ_r it may be found as follows: The above coefficients seem to be arithmetic sequences of order 0, 1, 2, 3, $\cdots$ and a simple induction shows that this is really so. If we let $T_q^{(j)}$ stand for the *qth term of an arithmetic sequence of order j,* we have

$$\begin{cases} \Delta_{2h} = m^{2h} - T_{2h-1}^{(1)}m^{2h-2}n + \cdots + (-1)^h T_1^{(h)} m^0 n^h, \\ \Delta_{2h+1} = m^{2h+1} - T_{2h}^{(1)}m^{2h-1}n + \cdots + (-1)^h T_2^{(h)} mn^h; \end{cases}$$

or more briefly,

$$\text{(III)} \qquad \Delta_r = \sum_{j=0}^{j=h} [(-1)^j T_{r-2j+1}^{(j)} m^{r-2j} n^j], \quad (r = 2h, \text{ or } 2h+1).$$

The T's can be found from the following table:

0th order:	1		1		1		1		1		1		1		1		1		...
1st " :		1		2		3		4		5		6		7		8			...
2d " :			1		3		6		10		15		21		28				...
3d " :				1		4		10		20		35		56					...
4th " :					1		5		15		35		70		126				...
5th " :						.		.		.		.		.		.			...

The formula for the qth term in the line j is[1]

$$T_q^{(j)} = d_1 + C_1^{q-1}d_2 + C_2^{q-1}d_3 + \cdots + C_j^{q-1}d_j.$$

In our case, the d's themselves are binomial coefficients of order j, so that

$$T_q^{(j)} = \sum_{i=0}^{i=j} (C_i^{q-1}C_i^j),$$

where $C_0^s = C_0^0 = 1$, and $C_k^s = 0$, $(k > s)$. Inserting this in (III), we obtain

$$\Delta_r = \sum_{j=0}^{j=h}\left((-1)^j \sum_{i=0}^{i=j}[C_i^{r-2j}C_i^j]m^{r-2j}n^j\right), \quad (r = 2h, \text{ or } 2h+1).$$

Finally we make use of the proposition that $T_q^{(j)} = H_j^q = C_j^{q+j-1}$, so that

$$C_s^{n-s} = \sum_{i=0}^{i=s}[C_i^{n-2s}C_i^s],$$

and get as our definite result

$$\text{(IV)} \qquad \Delta_r = \sum_{j=0}^{j=h}((-1)^j C_j^{r-j}m^{r-2j}n^j).$$

Equations (II) and (IV) completely solve the problem of associating the n of a quadratic factor of $f(x)$ with its own proper m's, and display the nature of this association.

Incidentally, we get in (IV) the expansion of an interesting determinant. If we replace 1 by r, and put $\Delta_q \equiv D_q(n, m, r)$, we have

$$D_q(n, m, r) = \sum_{j=0}^{j=h}[(-1)^j C_j^{q-j}m^{q-2j}n^j r^j].$$

Among the special forms, we note $D_q(n, m, -1)$, which is a continuant, and $D_q(1, \pm 1, \pm 1)$, which gives us curious summation formulas.

[1] Fine, *College Algebra*, p. 365.

ON THE CONSTRUCTION OF CERTAIN CURVES GIVEN IN POLAR COÖRDINATES.

By R. E. MORITZ, University of Washington.

1. *Definitions.* Consider the parametric equations in cartesian coördinates,

$$y = a \cos pt + k, \tag{1}$$

$$x = qt. \tag{2}$$

(1) represents a simple harmonic motion, t the time, y the distance of the vibrating point from a fixed point in the line along which the vibration takes place, a the amplitude and $2\pi/p$ the period of the vibration, k the distance of the mean point of vibration from the fixed point. (2) represents a uniform linear (translatory) motion at right angles to the direction of the line along which the simple harmonic motion takes place, q the uniform velocity of the point, x its distance from a fixed point at any given time t. (1) and (2) taken simultaneously represent the motion resulting from the composition of these two motions. This resultant motion we shall call a *translatory-harmonic* motion, the locus of this motion the *linearly-harmonic* curve, whose equation

$$y = a \cos \frac{p}{q} x + k \tag{3}$$

is obtained by eliminating the parameter t from the component equations (1) and (2).

Definition 1. A translatory-harmonic motion ($y = a \cos pt + k$, $x = qt$) *is the motion of a point which has simple harmonic motion* ($y = a \cos pt + k$) *along a line, while at the same time the line moves with a constant velocity* q *at right angles to itself. The locus of the resultant motion is the linearly-harmonic curve* $y = a \cos (p/q)x + k$.

Let us now consider the parametric equations in polar coördinates,

$$\rho = a \cos pt + k, \tag{4}$$

$$\theta = qt. \tag{5}$$

Like (1), (4) represents a simple harmonic motion, ρ being the distance of the vibrating point from a fixed point in the line along which the vibration takes place, while a, p, t and k have the same meaning as in (1). (5) represents a uniform angular (rotatory) motion, q the uniform angular velocity, θ the vectorial angle of the rotating point at any given time t. (4) and (5) taken simultaneously represent the motion resulting from the composition of these two motions. This resultant motion we shall call a *rotatory-harmonic* motion, the locus of this motion is the *cyclic-harmonic* curve

$$\rho = a \cos \frac{p}{q} \theta + k, \tag{6}$$

obtained by eliminating the parameter t from the component equations (4) and (5).

Definition 2. A rotatory-harmonic motion ($\rho = a \cos pt + k$, $\theta = qt$) *is the motion of a point which has simple harmonic motion* ($\rho = a \cos pt + k$) *along a line, while at the same time this line rotates with a constant angular velocity* q *about one of its fixed points. The locus of the resultant motion is the cyclic-harmonic curve* $\rho = a \cos (p/q)\theta + k$.

2. *Construction of Cyclic-harmonic Curves by Points.* The foregoing definition furnishes a convenient method of constructing by points any cyclic-harmonic curve whose equation is given.

Let the equation of the curve be $\rho = a \cos (p/q)\theta + k$.

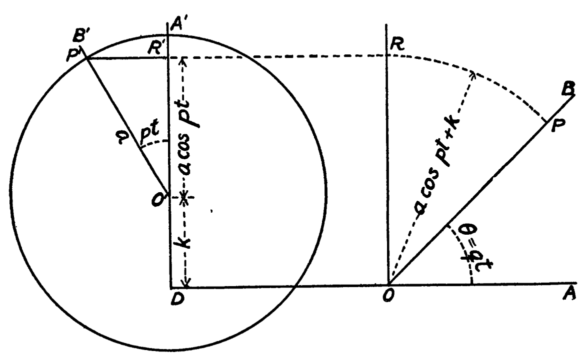


FIG. 1.

Let O (Fig. 1) represent the origin of coördinates, OA the initial line. At a point D on AO produced, chosen at any convenient distance from O, construct a perpendicular DA' and on it take DO' equal to k. Select any convenient unit of angular measure and construct the AOB ($= \theta$) and $A'O'B'$ ($= \theta'$) equal to qt and pt units respectively, t being any arbitrarily assumed integer. On $O'B'$ take $O'P'$ equal to a and let R' be the projection of P' on DA'. With O as a center and DR' as a radius describe an arc. Then P, the intersection point of this arc with OB in case DR' is positive, or with BO produced in case DR' is negative, that is in case R' falls below D, is a point on the required curve; for in either case

$$\rho = OP = OR = DR' = DO' + O'R' = k + a \cos pt, \quad \theta = qt,$$

from which on eliminating t, $\rho = a \cos (p/q)\theta + k$.

By choosing the unit of angular measure sufficiently small, and taking in turn $t = 0, 1, 2, 3$, etc., we may thus construct as many points of the curve as desired, at intervals small at will. Figs. 2, 3, 4, 5, show the method applied to the construction of the cyclic-harmonics $\rho = a \cos \frac{3}{2}\theta + k$, for the values $k = 3a$, $k = a$, $k = a/2$, $k = 0$, respectively.

3. *Classification of Cyclic-harmonic Curves.* Let the ratio p/q determine the species of the curve $\rho = a \cos (p/q)\theta + k$. An inspection of the preceding figures

discloses certain properties which are independent of the particular value of the ratio p/q employed and which are therefore common to all the species. Fig. 2 has an open center and it follows from the mode of construction, as is otherwise obvious from the form of the equation, that the curve is confined between two circles whose radii are $k - a$ and $k + a$ respectively. Fig. 3 consists of leaves which meet in cusps at the center, the axial diameter of each leaf is $k + a$. Fig. 4 consists of two sets of leaves, the axial diameters of the larger set being $k + a$, those of the smaller set $k - a$. Fig. 5 consists of a single set of equal leaves whose axial diameters equal a.

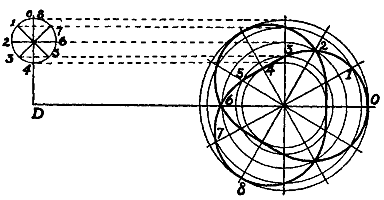


FIG. 2. Curtate Cyclic-harmonic,
$\frac{p}{q} = \frac{3}{2}, \quad \frac{k}{a} = 3.$

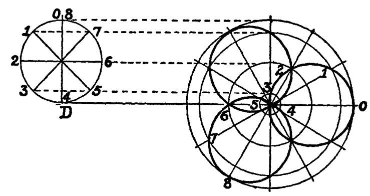


FIG. 3. Cuspitate Cyclic-harmonic
$\frac{p}{q} = \frac{3}{2}, \quad \frac{k}{a} = 1.$

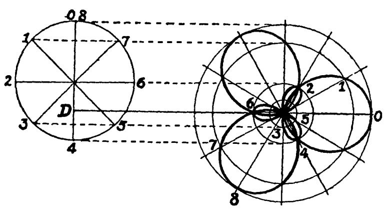


FIG. 4. Prolate Cyclic-harmonic,
$\frac{p}{q} = \frac{3}{2}, \quad \frac{k}{a} = \frac{1}{2}.$

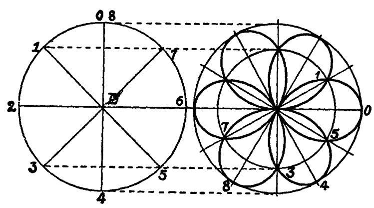


FIG. 5. Foliate Cyclic-harmonic,
$\frac{p}{q} = \frac{3}{2}, \quad \frac{k}{a} = 0.$

These properties serve as a convenient basis for the further classification of the cyclic-harmonic curves of any species. We shall call a cyclic-harmonic curtate if $k > a$, cuspitate if $k = a$, prolate if $0 < k < a$, equi-foliate (or foliate) if $k = 0$.

The cyclic-harmonic curves embrace as special cases a considerable number of familiar curves. Pascal's conchoids constitute one species, $p/q = 1$. The cardioid and Münger's double egg curves are cuspitate cyclic-harmonics, $p/q = 1$ and 2 respectively. The common limaçon and Freeth's nephroid are prolate cyclic-harmonics, $p/q = 1$ and $\frac{1}{2}$ respectively. All roses (Rosenkurven, rosaces) are equi-foliate cyclic-harmonics. The linearly harmonic (simple harmonic) curves will be shown to be degenerate forms of curtate cyclic-harmonic curves.

4. *The Cyclo-harmonograph.* The foregoing definition of rotatory harmonic motion suggests a simple mechanism for constructing any cyclic-harmonic curve kinematically.

A wheel W_1, center C, carries a crank-pin R which, as the wheel rotates, slides in a slotted cross-bar ST of a cross-head HK perpendicular to ST. This cross-head is constrained to move in the direction of HK by means of two fixed guides F and G. As the wheel W_1 moves with constant angular velocity about its center C, any point P in HK will have simple harmonic motion in the direction of HK.

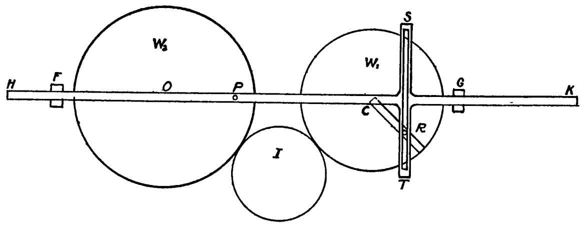


FIG. 6.

Let now the mechanism thus far described be made to revolve about a fixed wheel W_2, center O, by means of an idle-wheel I which connects the circumferences of the wheels W_1 and W_2. Any point P in HK will then receive rotatory harmonic motion and a pencil or pen-point placed at P will trace out a cyclic-harmonic curve in the plane of the paper.

To deduce the equation of the curve traced out by P, let the dotted lines in Fig. 7 represent the initial position of the mechanism, which is so chosen that the crank-pin R_0 is in the line of centers OC_0 of the wheels W_2 and W_1 and such that C_0 is between O and R_0.

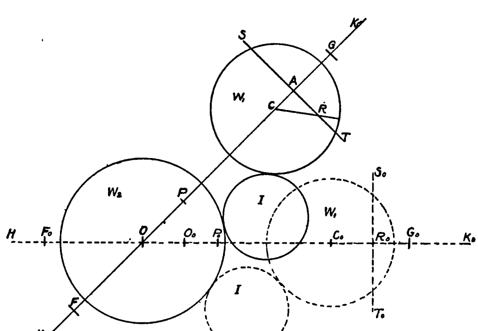


FIG. 7.

Take $R_0O_0 = C_0O$, the distance between the centers of the two wheels W_1 and W_2, and let k denote the distance of any point P_0 on H_0K_0 from O_0.

Let C represent the center of the wheel W_1 in any other position, $P\ R\ H\ K\ F\ G\ S\ T$ the corresponding positions of $P_0\ R_0\ H_0\ K_0\ F_0\ G_0\ S_0\ T_0$ respectively, and A the intersection of ST with HK.

Take O for the origin of coördinates, OK_0 for the initial line, and denote the coördinates of the point P by ρ, θ. Furthermore let $C_0R_0 = CR = a$, and

angle $RCK = \phi$. Then

$$OP = \rho = OC + CA - PA, \quad OC = OC_0 = O_0R_0,$$

$$CA = CR \cos\phi = a \cos\phi, \quad PA = P_0R_0 = O_0R_0 - k.$$

Hence

$$OP = \rho = O_0R_0 + a\cos\phi - (O_0R_0 - k), \text{ or } \rho = a\cos\phi + k.$$

Now let p and q represent the radii of the wheels W_2 and W_1 respectively, then obviously $q\phi = p\theta$, $\phi = (p/q)\theta$, and we have

$$\rho = a \cos\frac{p}{q}\theta + k \tag{7}$$

as the general equation of the locus of the point P.

The form of equation (7) shows that the locus of P is independent of the distance between the centers of the wheels W_1 and W_2, but depends only on the dimensions of the two wheels, the distance of the crank-pin R from the center C and the arbitrary distance O_0R_0. The single wheels W_1 and W_2 may therefore be replaced by two trains of wheels of various diameters, the idle-wheel I being used to connect at will any wheel of either train with any wheel of the other. A sliding carriage on HK and a crank-pin adjustable to various distances CR makes it possible to assign to k and c arbitrary values within the physical limits of the mechanism. Finally, it is immaterial whether the wheel W_2 is kept fixed while W_1 revolves about it, or whether the centers of both wheels remain fixed and both wheels be allowed to revolve, the paper on which the curve is traced being attached to the face of the wheel W_2. For practical reasons the latter is the more advantageous arrangement.

5. *Linearly-harmonic Curves as Degenerate Forms of Cyclic-harmonics.* Fig. 8 shows a portion of a curtate cyclic-harmonic curve. Plainly, as the arc which

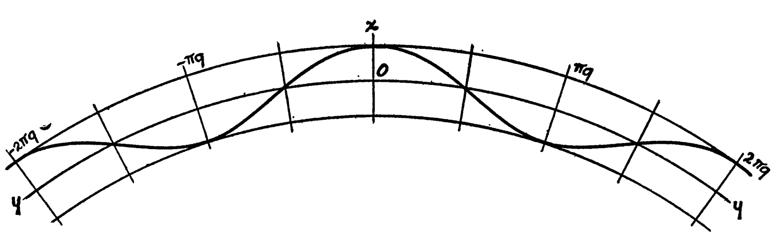


FIG. 8.

forms the directrix of this curve degenerates into a straight line the curve itself will degenerate into a linearly harmonic curve.

In Fig. 7 let x, y denote the rectangular coördinates of any point ρ, θ of the cyclic-harmonic curve $\rho = a\cos(p/q)\theta + k$, referred to C_0 as a new origin with C_0K_0 as the positive direction of the new x-axis. Put $k = l = OC_0$, the distance between the centers of the wheels W_2 and W_1. We then have

$$x + l = \rho \cos \theta, \qquad y = \rho \sin \theta, \qquad \rho = \sqrt{(x + l)^2 + y^2},$$

$$\frac{p}{q}\theta = \cos^{-1}\frac{\rho - l}{a} = \frac{p}{q}\tan^{-1}\frac{y}{x + l}. \tag{8}$$

Now let p increase indefinitely while q remains constant, then since $l = p + q + a$ constant, l also will increase indefinitely, and it is easy to see that

$$\lim_{l \doteq \infty} \frac{\rho - l}{a} = \frac{x}{a}, \qquad \lim_{l \doteq \infty} \left[\frac{p}{q}\tan^{-1}\frac{y}{x + l}\right] = \frac{y}{q},$$

hence equation (8) becomes in the limit

$$\cos^{-1}\frac{x}{a} = \frac{y}{q},$$

or

$$x = a \cos\frac{y}{q},$$

the equation of a linearly-harmonic curve, amplitude a, wave-length $2\pi q$.

6. *The Division of the Circle into Any Number of Parts.* The cyclo-harmonograph effects a complete solution of the classic problem of cyclotomy by determining the vertices of a regular polygon of any odd number of sides. It is easily shown that the cyclic harmonic curve $\rho = a \cos (p/q)\theta + k$ has a set of $p(q - 1)$ nodes corresponding to the angles

$$\theta_{\lambda, \mu} = \left(\lambda + \frac{q}{p}\mu\right)\pi, \qquad \lambda = 1, 2, 3, \cdots, q - 1; \qquad \mu = 0, 1, 2, \cdots, p - 1.$$

These nodes lie in sets of $q - 1$ on the p straight lines $\theta_\lambda = (\lambda/p)\pi$, in which $\lambda = 0, 1, 2, \cdots, p - 1$, and in sets of p on the $q - 1$ circles $\rho_\mu = a \cos (\mu/q)\pi + k$, in which $\mu = 1, 2, 3, \cdots, q - 1$. It is clear therefore that any set of these nodes situated on the same circle determines the vertices of a regular polygon with n sides. In particular the cyclic-harmonic $\rho = a \cos (p/2)\pi + k$ has the p nodes $\theta_\lambda = (\lambda/p)\pi$, in which $\rho_\lambda = k$, $\lambda = 1, 3, 5, \cdots, 2p - 1$.

7. *The Number of Species.* The number of species of cyclic-harmonics which it is possible to describe with a given cyclo-harmonograph depends of course on the number of wheels in the train of gears employed in the mechanism. Suppose that there are n wheels in each train and that the diameters of these wheels are proportional to the numbers $1, 2, 3, \cdots, n$. Let the diameters of the two wheels which the idle-wheel connects be as p is to q. Assume p constant and greater than q, with this assumption there are $\phi(p)$ admissible values of the ratio p/q, $\phi(p)$ being the totient function of p, that is, the number of numbers which are less than p and have with it no common divisor other than unity. Now let p take all values from 1 to n and we obtain $\Sigma_1^n \phi(n)$ species of curves under the restriction $n \geqq p > q$. Evidently there is an equal number of species under the restriction $n \geqq q > p$. Besides these there is the case $p/q = 1$. The total

number of species within the range of a cyclo-harmonograph having n wheels in each train of gears is therefore $1 + 2\,\Sigma_1^n\,\phi(n)$.

8. *Explanation of Plates.* The plates which follow contain four each of six species of cyclo-harmonic curves corresponding to the values $p/q = 2$, $1/2$, $9/2$, $2/9$, $8/5$, and $10/9$ respectively. Of each set of four, the first is curtate, the second cuspitate, the third prolate, and the fourth equi-foliate. The pole is taken at the center of each figure, the polar axis extending to the right. In the cases $k = 0$, k has actually been chosen slightly different from zero in order to exhibit more clearly the cusps at the origin.

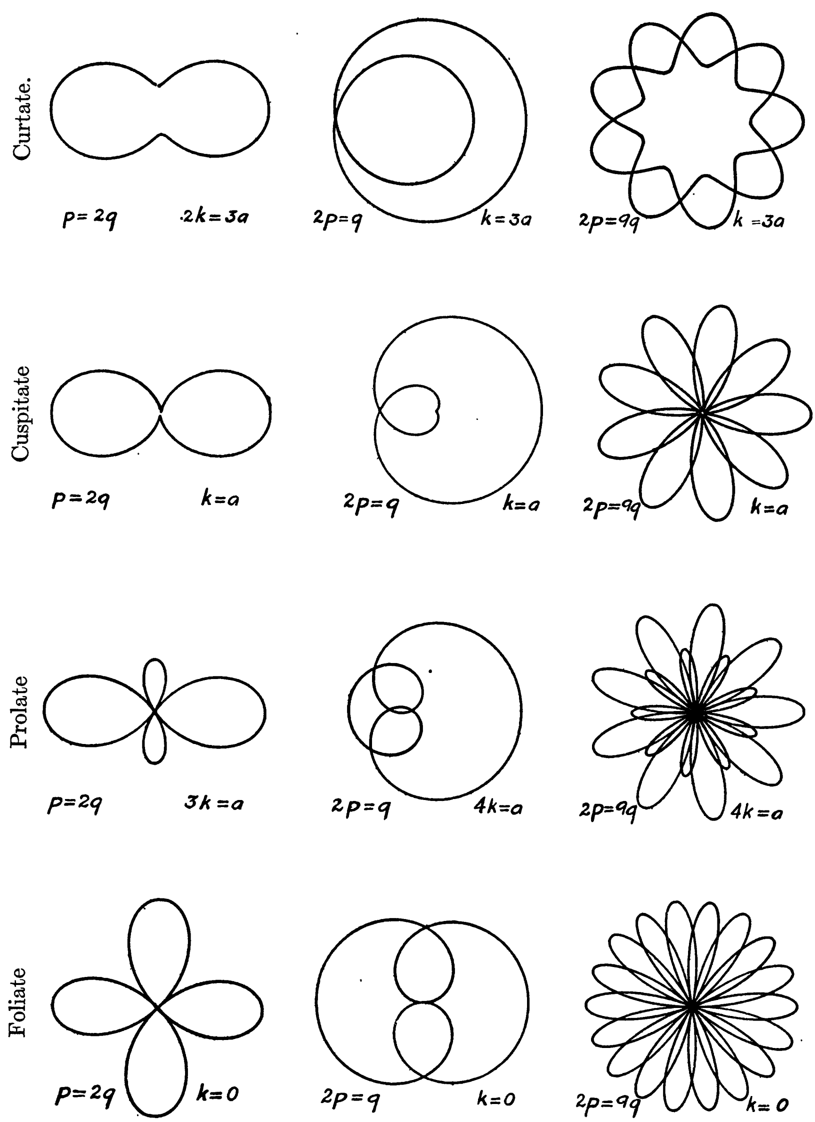


PLATE I.

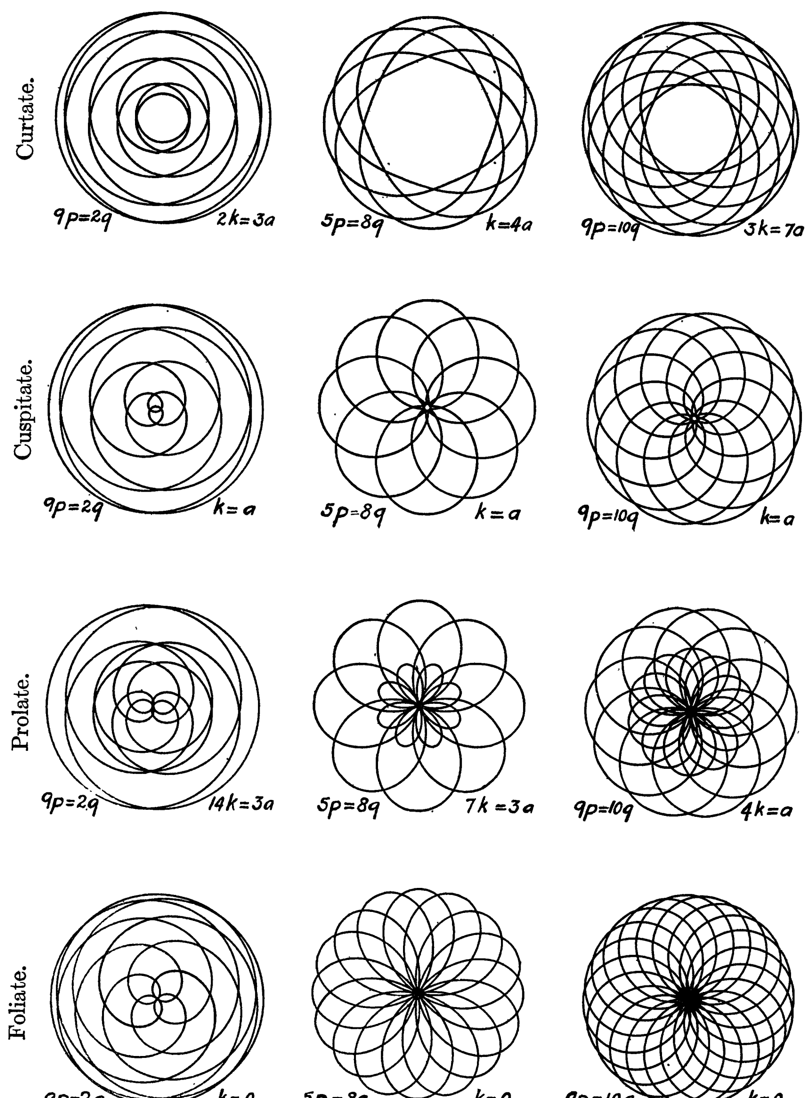

Curtate.
9p=2q
2k=3a
5p=8q
k=4a
9p=10q
3k=7a
Cuspitate.
9p=2q
k=a
5p=8q
k=a
9p=10q
k=a
Prolate.
9p=2q
14k=3a
5p=8q
7k=3a
9p=10q
4k=a
Foliate.
9p=2q
k=0
5p=8q
k=0
9p=10q
k=0

Plate II.

MATHEMATICAL PROBLEMS IN RELATION TO THE HISTORY OF ECONOMICS AND COMMERCE.*

By DAVID EUGENE SMITH, Columbia University.

If students of the history of economics and commerce wish to find a new and interesting field for exploration, and one which is certain to yield results that are worth the labor of cultivation, they will do well to consider the history of problems in arithmetic and algebra as set forth in the manuscripts and early printed books that have come down to us. No doubt some of this field has already been explored, but it is quite certain that only a small portion has thus far come under cultivation. The manuscripts on arithmetic from the thirteenth century to the beginning of printing, the large number of books printed before problems began to represent past as well as contemporary conditions, and the more original text-books of later periods contain a considerable amount of material on the history of commerce and economics that no one seems yet to have studied with any thoroughness.

Considered more broadly, a very good history of civilization could be written from the wide range of problems of mathematics. The emerging of humanity from the stage of mysticism, the development of the science of astronomy, the comprehension of the laws of mechanics and of physics in general, the transition from the agricultural to the industrial stage of a nation, the development of military science, and the rise of commerce are a few of the chapters that might be based upon the problems in mathematics which are easily accessible.

Returning, however, to commercial and economic history, a few examples will be cited to give some notion of the material at the disposal of the student. The problems in the manuscripts and early printed books on arithmetic in the fifteenth century tell us that Venice was then the center of the silk trade, although Bologna, Genoa, and Florence were prominent. Florence was the chief Italian city engaged in the dyeing of cloth. "Nostra magnifica Città di Venetia," as Tartaglia so affectionately and appropriately called her, carried on her chief trade with Lyons, London, Antwerp, Paris, Bruges, Barcelona, Montpellier, and the Hansa towns, besides the cities of Italy. Chiarini (Florence, 1481) indicates the following as the most important cities with which Florence had extensive trade, his spelling being here preserved: Alessandria degypto, Marsilia, Mompolieri, Lisbona, Parigi, Bruggia, Barzalona, Londra, Gostatinopoli, and Dommasco, with the countries of Tunizi, Cypri, and Candia. Tartaglia gives Barcelona, Paris, and Bruges as the leading cities connected with Genoa in trade a half century later.

We also know, from Chiarini's commercial arithmetic (1481), the most important commodities of Florentine trade in the decade before America was discovered. These were rame (brass), stoppa (tow), zoplhi (sulphur), smeriglio

* Extracts from a paper read at an informal session held on the evening between the meetings of the American Mathematical Society and the Colloquium at Cambridge, Mass., September 5, 1917.

(emery), lana (wool), ghalla (gall), trementina (turpentine), sapone (soap), risi (rice), zucchari (sugar), cannella (cinnamon), piombo (lead), lini (flax), pece (pitch), acciai (thread), canapa (hemp), incenso (incense), indachi (indigo), mace (mace), cubeba (cubebs), borage (borax), and the ever-present saffron, the "king of plants," then everywhere used as a *sine qua non* in daily life, and now almost forgotten.

The problems also tell us the cost of the luxuries and the necessities of life. Spanish linen was worth, for example, 94 to 120 ducats per hundred-weight, while Italian linen ran as high as 355 ducats and Saloniki linen as high as 380 ducats. French linen was much cheaper than the latter, selling for 140 ducats. The arithmetics tell us that the linen was baled and sent from Venice to towns like Brescia on muleback.

The problems "delle pigione" tell us that the houses of the bourgeoisie rented in Siena, in 1540, at about 25 to 30 lire per year, while a century later they rented in Florence for from 120 to 300 lire. We also have the prices of sugar, ginger, pepper, and other commodities, showing that these three, for example, were only within the reach of the wealthy.

Hotel life in a grand establishment is also revealed in various problems, of which this one, printed in 1561, is a fair type: "Item / Wenn in einem Gasthause weren 8 Kamern / in jglicher Kamer stünden 12 Bette / in jglichem Bette legen 3 Geste / vnd ein jglicher Gast gebe dem Hausgefinde 6 ϑ trinckgelt / Wie viel thuts in einer Summa?" The conditions are not at all exaggerated, as many travelers in remote parts of the world to-day can testify.

The early printed arithmetics also show interesting changes in commercial customs and a general rise in the standards of business integrity. For example, the chapter on Die Regel Fusci, very common in the sixteenth century, and relating in part to the adulteration of foods and drugs, would hardly be acceptable to-day either in school or in trade.

The arithmetics also tell very completely the story of the transition from the era of barter to that of the sale of goods for a monetary equivalent. The chapters on "Il baratto," "Stichrechnung," "Troquer," and "Manghelinghe" are among the most interesting ones to be found in these books. They tell us of the influence of the great fairs, they give us lists of these commercial centers, they reveal the relative values of the various commodities in general use, they tell us of the custom of barter in futures (the forerunner of our dealing in futures to-day), and incidentally they explain why a truckman is so called in our time.

An extensive and interesting history of exchange could be written from the problems of arithmetic, including the "cambio menuto, ouer commune," the "cambio reale," the interesting "cambio secco" ("change sec," or "trockener Wechsel") which "non ha humore, ne foglie ne frutto," as Sarava wrote in 1561, and the "cambio fittitio." To these various forms we can trace our standard systems of to-day, and in the study of the "cambio secco" we can understand the law of Henry VII which says that "Eny bargayne . . . by the name of drye exchange . . . be utterly voide."

The transition from partnership in its various forms to the corporations of to-day may well be studied in the problems of the commercial arithmetic, and there may also be followed the genesis, partial decay, and present status of equation of payments. Profit and loss, to-day the most vital topic of business arithmetic, has a long and varied history, and the economics of the problem may be studied in the older books, free from all the modern features of overhead, cost of doing business, and profit on the selling price.

Not only to the economist and the student of commerce is the field a rich one, but it is well worth the study of anyone who may be possessed of doubt as to the relation of mathematics to the daily life of the race. Not only can the history of the problem easily be made the history of commerce and economics, but the history of mathematics can easily be made the history of civilization.

ORGANIZATION OF THE MARYLAND–VIRGINIA–DISTRICT OF COLUMBIA SECTION OF THE ASSOCIATION.

As a result of preliminary correspondence, a group of Maryland mathematicians held a meeting in New York at the time of the December meeting of the Association and presented a petition to the council for authority to organize a section of the Association in Maryland, Virginia, and the District of Columbia. Professor Abraham Cohen was designated as temporary secretary.

The authority being granted by the council, arrangements were completed for a meeting to organize the Section at Johns Hopkins University on March 3, 1917. Among the 38 persons present at this meeting were the following members of the Association:

O. S. Adams, U. S. Coast and Geodetic Survey, Washington, D. C.
Clara L. Bacon, Goucher College, Baltimore, Md.
Harry Bateman, Johns Hopkins University, Baltimore, Md.
Lillian O. Brown, Hood College, Frederick, Md.
J. A. Bullard, U. S. Naval Academy, Annapolis, Md.
A. B. Coble, Johns Hopkins University, Baltimore, Md.
A. Cohen, Johns Hopkins University, Baltimore, Md.
H. A. Converse, Baltimore Polytechnic Inst., Baltimore, Md.
J. B. Eppes, U. S. Naval Academy, Annapolis, Md.
Angelo Hall, U. S. Naval Academy, Annapolis, Md.
W. M. Hamilton, U. S. Nautical Almanac Office, Washington, D. C.
R. A. Harris, U. S. Coast and Geodetic Survey, Washington, D. C.
A. W. Hobbs, Baltimore, Md.
L. S. Hulburt, Johns Hopkins University, Baltimore, Md.
W. D. Lambert, U. S. Coast and Geodetic Survey, Washington, D. C.
A. E. Landry, Catholic University of America, Washington, D. C.
Frank Morley, Johns Hopkins University, Baltimore, Md.
S. F. Norris, Baltimore City College, Baltimore, Md.

R. E. Root, U. S. Naval Academy, Annapolis, Md.
W. F. Shenton, Johns Hopkins University, Baltimore, Md.
E. R. Smith, The Park School, Baltimore, Md.
J. J. Tanzola, U. S. Naval Academy, Annapolis, Md.
H. R. Tolley, U. S. Dept. of Agriculture, Washington, D. C.

After the adoption of a constitution for the Section, the following officers were elected for the coming year: President, Professor ABRAHAM COHEN, of Johns Hopkins University; Secretary-Treasurer, Professor RALPH E. ROOT, U. S. Naval Academy; Member of the Executive Committee, Mr. WALTER D. LAMBERT, U. S. Coast and Geodetic Survey.

The program consisted of two papers, each followed by a discussion. The first paper, by Professor R. E. Root, was on "The aims and possibilities of this local section," and the discussion was led by Professor A. E. Landry, Mr. W. D. Lambert and Professor Frank Morley. Mr. E. R. Smith and Professor A. B. Coble also took part in the discussion. The second paper, by Professor L. S. Hulburt, was on "A college or university course for teachers of secondary mathematics," and the discussion was led by Professor Clara L. Bacon and Mr. E. R. Smith.

After the program there was a social hour at the Hopkins Club, where the members of the section were the guests of the University at a luncheon.

The constitution of the section provides for holding at least two meetings each year, one in the spring and another in the fall, and the executive committee may call additional meetings when desirable.

RALPH E. ROOT,
Secretary.

SECOND ANNUAL MEETING OF THE OHIO SECTION.

The second annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, on April 6, 1917, in connection with the meetings of the Ohio College Association, The Ohio Academy of Science, The Ohio Society of College Teachers of Education, and The Association of Ohio Teachers of Mathematics and Science. Chairman Focke occupied the chair, being relieved by Professor R. B. Allen for an interval.

The following thirty-four persons were registered, all but the last six being members of the Association:

R. B. Allen, Kenyon College; F. Anderegg, Oberlin College; W. E. Anderson, Wittenberg College; G. N. Armstrong, Ohio Wesleyan University; C. L. Arnold, Ohio State University; C. B. Austin, Ohio Wesleyan University; Grace M. Bareis, Ohio State University; Mrs. W. E. Beckwith, Western Reserve University; R. D. Bohannan, Ohio State University; Louis Brand, University of Cincinnati; A. G. Caris, Defiance College; O. L. Dustheimer, Baldwin-Wallace College; Theo. M. Focke, Case School of Applied Science; Harriet E. Glazier, Western College for Women; M. E. Graber, Heidelberg University; Harris

Hancock, University of Cincinnati; E. J. Hirschler, Bluffton College; H. W. Kuhn, Ohio State University; C. C. Morris, Ohio State University; Anna H. Palmie, Western Reserve University; S. E. Rasor, Ohio State University; Hortense Rickard, Ohio State University; Mary E. Sinclair, Oberlin College; S. A. Singer, Capital University; K. D. Swartzel, Ohio State University; C. J. West, Ohio State University; Forbes B. Wiley, Denison University; D. T. Wilson, Case School of Applied Science.

Non-members: P. Biefeld, Denison University; Jacob Bowers, Columbus Trades School; L. D. Parker, Cedarville College; Anna B. Peckham, Denison University; T. Elmer Trott, Mt. Union College; R. B. Wildermuth, Capital University.

The following program was carried out as arranged:

Friday Afternoon.

1. Chairman's Address. "A geometrical presentation of Taylor's series." Professor T. M. Focke, Case School of Applied Science.
2. "The application of mathematics to the biological and social sciences." Professor C. J. West, Ohio State University.
3. "The actual and apparent path of comet b 1916 (Wolf), and the actual path of Mellish's comet (1917a)." Professor Paul Biefeld, Denison University (Introduced by the Chairman).
4. "The application of Fourier's series to an isoperimetric problem." Professor E. J. Hirschler, Bluffton College.
5. "Formal seminvariants of binary forms." Professor C. C. Morris, Ohio State University.
6. "Senior year mathematics for engineering students." Professor Louis Brand, University of Cincinnati.

Friday Evening.

An informal meeting in the nature of a round table was held at the Ohio Union. The following subjects were proposed for discussion:

1. Should there be any absolute requirements in mathematics for the A.B. degree beyond one year of high school algebra and one of plane geometry?
2. How to discover and encourage real mathematical ability.
3. Continuation of the discussion of the afternoon papers.

Each of the first three papers in the afternoon program was illustrated by a number of lantern slides. A photograph of those present at the close of the afternoon meeting was taken. The members of the section dined together at the Ohio Union on Friday evening. Many remained and participated in the meetings of the Association of Ohio Teachers of Mathematics and Science on Saturday.

At the business meeting of the Section, the following officers were elected for the ensuing year:

Chairman, Professor Forbes B. Wiley, Denison University.

Secretary-Treasurer, Professor G. N. Armstrong, Ohio Wesleyan University.

Third member of the Executive Committee, Professor C. L. Arnold, Ohio State University.

Professor Morris emphasized the opportunity and the responsibility of the Ohio Section in connection with the meetings of the American Mathematical Society and the Mathematical Association of America to be held in Cleveland, September 4th to 7th, 1917, and of the Central Association of Science and Mathematics Teachers at Columbus, November 30th to December 1st, 1917.

Abstracts of Papers and Discussions.

1. In his paper on "A geometrical presentation of Taylor's series," Professor Focke suggests that the order of contact of curves be studied before taking up Taylor's theorem. It is then proposed to lead up to the Maclaurin expansion by finding the equation of a curve of the form

$$y = A + Bx + Cx^2 + Dx^3 + \cdots + Rx^n,$$

which shall have a contact of the nth order with $y = f(x)$ at the point where $x = 0$. Then, since the values of A, B, C, etc., turn out to be independent of n, the order of contact of the two curves can be made as high as we choose, provided $f(x)$ and whichever of its derivatives are needed exist when $x = 0$. The necessity of determining the interval of convergency when the number of terms increases without limit was shown by diagrams.

2. Professor West noted that in statistical work which is primarily a matter of accurate counting there is little need for more than the simplest processes of arithmetic. The attempt to draw conclusions true for a whole population, when only a limited number of individuals can be observed, gives rise to a multiplicity of involved problems which cannot be resolved without the aid of highly developed methods. Have real wages increased during the last twenty years; that is, will the laborer's pay purchase more now than at that time? Has the yield of a certain strain of corn been increased? These are typical statistical problems of the more recent kind. In this paper the development of statistical methods is traced and the mathematical problems which arise are outlined. The opportunities which the theory of mathematical statistics offers as a field for mathematical research is especially emphasized.

3. Professor Biefeld presented the report on comets not as a contribution in itself, but rather to suggest an interesting problem for the teacher in practical astronomy or celestial mechanics to be solved by students in his classes in connection with the study of orbits. Namely; the elements of a newly discovered comet being given, to construct:

1. A diagram of the actual path of the comet about the sun to scale. For this it is necessary to compute the true anomaly and radii-vectores for a series of dates.

2. To compute the ephemeris giving the true right ascensions and declinations

and the geocentric distances of the comet, and from the former to plot the apparent path among the fixed stars. It affords an interesting exercise, then, to watch the movement of an object in the sky, find its positions, and compare with computations.

As there are from 2 to 10 telescopic comets every year, new and interesting variations of the problem are brought out. The solution of the problem was illustrated by slides in relation to the above comets.

4. Professor Hirschler outlined a proof for the theorem that a circle encloses a larger area than any other simply closed regular curve of the same length. The curve is represented in parametric form with the length of arc measured from a fixed point as parameter. With the help of a number of recent theorems on Fourier's series its area is calculated in terms of the coefficients of the Fourier expansions for $x = \varphi(s)$ and $y = \psi(s)$. This area is compared with the area of a circle of length l found by integrating the equation $x'^2 + y'^2 = 1$ in terms of the same coefficients.

5. In his paper on "Formal seminvariants of binary forms," Professor Morris showed how the theory of numbers had been applied to the theory of algebraic invariants and seminvariants. He exhibited several fundamental systems for the cubic for different moduli and spoke of the progress that has been made on the quartic.

6. Three types of courses for senior engineering students were considered by Professor Brand:

1. Courses dealing with some phase of engineering with a well developed mathematical theory.
2. Courses presenting a branch of applied mathematics serviceable in engineering practice.
3. Courses in pure mathematics chosen to introduce important concepts not dealt with in elementary courses. As representative of the first type, a course given at the University of Cincinnati on the analysis of statically indeterminate structures was described in some detail. After urging the claims of vector analysis under the second class, the last was exemplified by a course on the number system of algebra, the material being outlined and its advantages discussed.

The evening meeting, held in the large committee room of the Ohio Union, was attended by about twenty-five persons. The discussion of the first topic was so animated and so interesting that the others were not reached and adjournment was forced by necessity of closing the building. Professors Caris, Allen, Bohannan, Morris, Wiley, West, Graber, Glazier, and Anderegg participated in the discussion, both sides of the question being represented. The fact that authority and tradition no longer shield the subjects of the educational curriculum from questioning and attack was emphasized. The claims and counterclaims as to the value of the study of mathematics found adequate presentation, as did also

those of the more recently introduced subjects of study. While strong faith in the validity of the claims of mathematics was apparent, yet there was evident a deep conviction that the teacher could justify himself and his subject only by intelligent, sympathetic, and earnest teaching.

G. N. ARMSTRONG,
Secretary.

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

Functions of a Complex Variable. By E. J. TOWNSEND, Professor and Head of the Department of Mathematics in the University of Illinois. Henry Holt and Company, New York, 1915. vii + 384 pages.

The present volume is one of the American Mathematical Series of which several have already appeared and of which Professor Townsend is the editor-in-chief. There are but few books in the English language that treat primarily of functions of a complex variable, and when we seek those of an elementary nature which are suitable for an introductory course, to be used by students who have had no further mathematical training than what is generally given in a first course in calculus, the number is very limited. There are the well-known texts, "*Introduction to Analytic Functions*" by Harkness and Morley and "*Theory of Functions of a Complex Variable*" by Burkhardt, translated by Rasor. A book as carefully and well written as the present volume, and which deals with such a fundamental and important field in mathematics, will undoubtedly be well received by both teachers and students.

In Chapter I there is a brief discussion of rational and irrational numbers; then follow the introduction of the complex number and the graphical representation of the same. The fundamental operations of addition, subtraction, multiplication, division, raising to powers, and the extraction of roots are explained for complex numbers, both analytically and graphically. The limitations of the graphic method for extraction of roots are illustrated by means of an example.

The beginning of Chapter II is devoted to the definitions and classifications of functions. The notions and properties of limits of sequences of real numbers are briefly stated, and the ideas are then extended to the realm of complex numbers. In article 13 several fundamental theorems relating to ordinary continuity of $f(z)$ with respect to z, uniform continuity in a region S, and uniform convergence along an arc are stated and proved.

In Chapter III differentiation and integration are taken up. Line integrals and their more general properties are discussed. The path of integration is defined as a curve having the property that it is monotone by segments of a finite number. This lends clearness to the discussions that follow, and obviates difficulties that would arise if a more general curve were chosen. Green's theorem for functions of two real variables, Cauchy's theorem, Cauchy's integral

formula, and the Cauchy-Riemann differential equations are then introduced. The chapter ends with a discussion of Laplace's differential equation and some of its applications to problems in mathematical physics.

Chapter IV is one of the longest in the book and deals with the mapping of given configurations from the Z-plane upon the W-plane and conversely. Detailed discussions are given for the simpler functions such as z^n, e^z, $\log z$, $\sin z$, $\cos z$, $\sinh z$, $\cosh z$. Functions of the above form when the argument is real are familiar to the student, but now much new information is added when the variable is complex.

In Chapter V the linear fractional transformations are taken up, beginning with the simpler forms and leading up to the general case. It is shown that the general linear fractional transformation has the properties belonging to a group. The transformations of the plane into itself due to the linear transformations are regarded as a problem in kinematics and the resulting motions are classified as parabolic, hyperbolic, elliptic, and loxodromic.

Chapter VI deals with infinite series. The more important properties of such series with complex terms are considered. The laws of operations with series are explained and illustrated with problems worked out in the text. Double series, uniform convergence, differentiation and integration of series, and power series are taken up in succession.

Chapter VII takes up the discussion of single-valued functions. It begins with a rather thorough treatment of analytic continuation. By the results thus obtained the author formulates more exactly his definition of an analytic function. Following this we have the treatment of singular points and zero points, Laurent's expansion, residues, the fundamental theorem of algebra, rational functions, transcendental functions, Mittag-Leffler's theorem, infinite products and a rather brief discussion of periodic functions.

Chapter VIII, the last in the book, treats of multiple-valued functions. Here the author points out clearly the distinction between multiple-valued analytic functions and multiple-valued expressions representing more than one single-valued analytic function. It is next shown that branch-points are characteristic of the former class. The advantage of the Riemann surface in representing multiple-valued functions is then brought out, and the surface is applied to some typical cases. After discussing some general properties of Riemann surfaces, interesting physical applications to the potential are given. Finally, in Art. 69, the general case of the algebraic function is discussed.

At the end of each chapter there is a list of well-graded exercises to be worked out by the student. It is only by the working out of a large number of such problems in connection with the theory that a student can hope to get a clear idea of the subject. Very excellent geometrical figures are given throughout the text. The typography and general appearance of the book are good. There are but few errors in printing, and those that exist should be of no serious trouble even to a beginner.

HANS H. DALAKER.

UNIVERSITY OF MINNESOTA,

College Algebra with Applications. By E. J. WILCZYNSKI. Edited by H. E. SLAUGHT. Allyn and Bacon, New York, 1916. xx + 507 pages.

"The material included in this book," says the author in his suggestions to the instructor, "probably contains everything ever given under the title *College Algebra* in any American college." The problem of making a selection that will suit the needs of all teachers of algebra is one that might well daunt the most ingenious maker of textbooks. Professor Wilczynski has wisely refrained from undertaking a solution of this problem and has instead given a great storehouse of materials from which the teacher can select to suit his own needs. To the reviewer this seems a very reasonable thing to do. Nothing is so exasperating as the constant worry of some earnest souls about the exact value of this or that tiny theorem in a course. Shall he lay stress on this problem? Shall he insist on a thorough mastery of this principle in the tenth week of the course? Is this the best sequence in presenting the subject? After many years of arranging courses and of trying to find out what material is most available for students the reviewer has given the problem up, as having too many indeterminates, and not enough exact equations. The best he has been able to do is to try to standardize his methods a little, and even in this he has found that one plan of attack will succeed very well with one class and fail with another. It is a problem quite analogous to that of the physician who has first to determine as best he can what is the matter with his patient, and then has the further problem as to the exact drug to use to get the right reaction from the particular patient involved.

The book contains sixteen chapters, beginning with one on the number system that will do much to clear up the difficulties which students (and teachers too) find with the subject of irrationals and complex numbers. Following this come chapters on linear functions, quadratic functions, functions of higher degrees, fractional functions, irrational functions and power functions. Throughout appear many applications to such subjects as the measurement of length, time, mass, the theory of the vernier, slide rule, logarithmic paper, velocity, acceleration, mass, density, and much else of great value to the teacher who is looking for "vitalizing material" for the subject. The author has undertaken to discuss these applications "as carefully as if the book were intended as a treatise on chemistry or physics." There is a chapter on determinants of the first, second and third orders, and another on determinants of higher orders separated from the first by a chapter on permutations and one on probabilities.

Chapter XIII, on simultaneous quadratics, is very full and clear and will be found one of the most helpful in the book. The last three chapters are devoted to the subject of limits, series, both finite and infinite, with a careful statement of the usual tests for convergence. An appendix contains a short table of logarithms and a mortality table.

The reviewer would be slow to predict an immense demand for Professor Wilczynski's book. He feels almost certain that it will not receive half the use which it deserves, the reason being that the average teacher does not want to select. He likes to have an exact statement of just what he is to teach, and the

student likes to have an exact statement of just what he is to learn. This is unfortunate, and not altogether the fault of the teacher. The teacher who omits a chapter from a book is apt to be looked upon with suspicion as a *franc-tireur*, or it may be darkly hinted that he does not understand the chapter well enough to teach it. If it should turn out that there is a large demand for this book the reviewer will take it as a sign that the teacher of mathematics is learning to be less dependent on his textbook. In any event the book will be found in the library of every progressive teacher of algebra.

D. N. LEHMER.

UNIVERSITY OF CALIFORNIA.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

ALGEBRA.

481. Proposed by ROGER E. MOORE, University of Wisconsin.

Show that if the coefficients in the binomial expansion of $(a - b)^n$ (n being a positive integer) be multiplied each to each by the corresponding terms of an arithmetical progression of $(n + 1)$ terms, then the algebraic sum of the $(n + 1)$ products will be zero.

482. Proposed by C. F. GUMMER, Kingston, Ontario.

Find the necessary and sufficient condition that the infinite sequences of positive quantities $(a_1, a_2, \cdots)$ and $(b_1, b_2, \cdots)$ may be such that the series $a_1x_1 + a_2x_2 + \cdots$ and $b_1x_1 + b_2x_2 + \cdots$ either both converge or both diverge when the x's are any positive quantities.

GEOMETRY.

510. Corrected statement (given incorrectly in the March issue).

Show how to find the equation of a line perpendicular to a side of the triangle of reference and passing through a given point, in a system of homogeneous coördinates, using the condition that two lines be parallel in this system but not the condition that two lines be perpendicular. Illustrate the method by using it to find the trilinear coördinates of the points of contact of an escribed circle of the triangle.

514. Proposed by VINCENTE MILLS, Manila, P. I.

Given an equilateral triangle, the length of the sides being unknown, and a point within, the distances from which to the vertices are given, required the length of a side of the triangle and the angles subtended at the given point by the sides of the triangle.

515. Proposed by C. F. GUMMER, Kingston, Ontario.

Show how to cut up a square carpet and make it into three equal square carpets. Estimate the total length of seam in comparison with a side of the original carpet.

CALCULUS.

429. Proposed by N. P. PANDYA, Sojitra, India.

Trace the curve given by the solution of

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = \frac{dy}{dx} + \frac{1 - 4x^2}{(1 - x^2)^{3/2}}.$$

430. Proposed by G. PAASWELL, New York City.

Revolve a circle about a chord (not a diameter). Select a system of rectilinear coördinates with this chord as one axis and the origin as the intersection of the chord and the circumference. Term this axis the z axis and pass a plane through the x (or y) axis. Find the area of this surface intercepted by this plane and the xz (or yz) plane.

MECHANICS.

346. Proposed by WILLIAM HOOVER, Columbus, Ohio.

Half the length of one of the equal parts of a uniform heavy string resting in equilibrium over a smooth horizontal indefinitely thin peg is cut off; determine the instantaneous change in the pressure on the peg.

347. Proposed by E. B. ESCOTT, Kansas City, Missouri.

A cord $ABCD$ is suspended from points A and D, which are 20 feet apart in horizontal distance. D is 4 feet lower than A. At B and C are suspended weights of 100 and 200 lbs. $AB = 8$ feet, $BC = 10$ feet, and $CD = 12$ feet. Find angles α, β, γ made by AB, BC, and CD, respectively, with the horizontal. Also find tensions T_1, T_2, and T_3 in AB, BC, and CD.

NUMBER THEORY.

265. Proposed by J. W. NICHOLSON, Louisiana State University.

If the roots of $x^4 - ax^2 + bx + c = 0$ are rational, prove that $4(a + yz) - 3(y + z)$ is a perfect square, y and z being any two roots of the equation.

245. Proposed by NORMAN ANNING, Chilliwack, B. C.

Show that $x^2 + y^2 = (a_1a_2\cdots a_n)^n$ has $4(n + 1)^m$ solutions in integers, in 2^{m+2} of which x and y are relatively prime, the a's being primes of the form $4k + 1$ and n a positive integer.

Note.—The proposer of this problem has changed it to read as above instead of the statement as previously published in the May, 1916, issue.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

469. Proposed by T. H. GRONWALL, New York City.

Show that the equation

$$f(x) \equiv 2ax^4 + (1 - b)x^3 + b(1 - b)x - 2ab = 0,$$

where $0 < b < 1$, $a > 0$ and $a^2 > b$, has only one positive root and that it lies between the roots of

$$g(x) \equiv x^2 - 2ax + b = 0.$$

SOLUTIONS BY D. R. CURTISS, Northwestern University.

I. By Descartes' Rule of Signs, $f(x)$ has only one positive root. Let x_1 be the smaller of the two roots of $g(x) = 0$, and x_2 be the larger. We shall have solved the problem if we establish the inequalities

$$f(x_1) < 0, \qquad f(x_2) > 0. \tag{1}$$

Let us now give b a constant value and let a vary from $\sqrt{b}$ to ∞. If X is either x_1 or x_2 we shall have $g(X) = X^2 - 2aX + b = 0$, and by differentiating with respect to a we obtain

$$\frac{dX}{da} = \frac{X}{X - a}. \tag{2}$$

We now have

$$\frac{df(X)}{da} = f'(X)\frac{dX}{da} + 2(X^4 - b) = \frac{f'(X)\cdot X + 2(X - a)(X^4 - b)}{X - a}$$

or, in expanded form,

$$\frac{df(X)}{da} = \frac{2X^5 + 6aX^4 + 3(1 - b)X^3 + b(1 - b)X + 2ab - 2bX}{X - a}. \tag{3}$$

The roots of $g(x) = 0$ satisfy the relations $0 < x_1 < a < x_2 < 2a$, from which we can show that the numerator on the right side of (3) is positive for both values of X. In fact, all the terms are positive except the last; but if $X = x_1 < a$, we have $2ab - 2bX > 0$, so that the numerator is positive in this case, while if $X = x_2$ we have, since $b < a^2$, and $a < x_2 < 2a$,

$$2X^5 + b(1 - b)X - 2bX + 2ab = (2X^5 - b^2X) + (2ab - bX),$$
$$> (2a^4 - a^4)X + b(2a - 2a) > 0.$$

Since $x_1 - a$ is negative and $x_2 - a$ is positive, we thus have

$$\frac{df(x_1)}{da} < 0, \qquad \frac{df(x_2)}{da} > 0.$$

It follows that $f(x_1)$ has its greatest value, and $f(x_2)$ its least value, when $a = \sqrt{b}$, in which case $x_1 = x_2 = a$, and $f(x_1) = f(x_2) = 0$. The relations (1) follow at once.

II. We will show that the positive root of $f(x) = 0$ (there can be only one, by Descartes' Rule) lies between $x = \sqrt{b}$ and $x = a$. The interval $[\sqrt{b}, a]$ is, however, wholly comprised between the two roots of $g(x) = 0$, since $g(0)$ and $g(\infty)$ are positive, while

$$g(\sqrt{b}) = 2b - 2a\sqrt{b} = 2\sqrt{b}(\sqrt{b} - a)$$

is negative, and $g(a) = b - a^2$ is also negative.

We now prove that $f(\sqrt{b}) < 0$ and $f(a) > 0$. In the first place,

$$f(\sqrt{b}) = 2ab^2 + 2b\sqrt{b}(1 - b) - 2ab.$$

Since $\sqrt{b} < a$, the substitution of a for $\sqrt{b}$ in the second term above gives

$$f\sqrt{b} < 2ab^2 + 2ab(1 - b) - 2ab = 0.$$

On the other hand, we have

$$f(a) = 2a^5 + a^3(1 - b) + ab(1 - b) - 2ab = 2a^5 + a^3 - ab(a^2 + 1 + b).$$

If we replace each b above by a^2 we obtain the inequality

$$f(a) > 2a^5 + a^3 - a^3(a^2 + 1 + a^2) = 0.$$

Also solved by E. J. Moulton, J. E. Rowe, H. H. Conwell, J. A. Bullard, J. L. Riley, Horace Olson, J. W. Baldwin, and the Proposer.

470. Proposed by ERNEST W. BROWN, Yale University.

There are n numbers each lying between $-\frac{1}{2}$ and $+\frac{1}{2}$, such that any value of each between these limits is equally probable. What is the probability that their sum will lie between $s - \frac{1}{2}$ and $s + \frac{1}{2}$, where s is an integral multiple of $\frac{1}{2}$.

Solution By C. F. Gummer, Kingston, Ontario.

Consider the more general problem where the n numbers are chosen at random from the intervals $(a_1, b_1), \cdots (a_n, b_n)$, and the sum is to lie in the interval (a, b).

Let $c_1 = b_1 - a_1, \cdots, c_n = b_n - a_n$.

Let $f_n(x) \cdot dx$ be the probability that the sum of the first r numbers will lie between x and $x + dx$.

Then $f_1(x) = 1/c_1$ when $a_1 < x < b_1$, and zero in other cases, and

$$f_r(x) = \int_{x-b_r}^{x-a_r} \frac{f_{r-1}(\xi) \cdot d\xi}{c_r}.$$

To evaluate $f_n(x)$, let us use the notations

$$\phi(x) = 1 \text{ for } x > 0, \qquad \phi(x) = -1 \text{ for } x < 0, \qquad \phi_r(x) = x^r\phi(x).$$

The function $\phi(x)$ is discontinuous but integrable. Since $\int_0^\alpha \phi(x)dx = \phi_1(\alpha)$, therefore, $\int_\alpha^\beta \phi(x)dx = \phi_1(\beta) - \phi_1(\alpha)$. In like manner,

$$\int_\alpha^\beta \phi_r(x)dx = \frac{\phi_{r+1}(\beta) - \phi_{r+1}(\alpha)}{r+1}.$$

Now

$$f_1(x) = \frac{1}{2c_1}\{\phi(x - a_1) - \phi(x - b_1)\}.$$

Hence,

$$f_2(x) = \frac{1}{c_2}\int_{x-b_2}^{x-a_2} f_1(\xi)d\xi$$

$$= \frac{1}{2c_1c_2}\{\phi_1(x - a_1 - a_2) - \phi_1(x - a_1 - b_2) - \phi_1(x - a_2 - b_1) + \phi_1(x - b_1 - b_2)\}.$$

We thus obtain by induction

$$f_n(x) = \frac{1}{2c_1 \cdots c_n \underline{|n-1}}\Sigma(-1)^k\phi_{n-1}(x - \alpha_1 - \beta_2 - \cdots - \kappa_n),$$

where, in the course of the summation, each of the letters α, β, $\cdots$, k is taken to represent either a or b independently of the others, and k is the number of them standing for b in any term.

The probability that the sum lies between a and b is then

(1) $$\int_a^b f_n(x)dx = \frac{1}{2c_1 \cdots c_n \underline{|n}}\Sigma(-1)^{k'-1}\phi_n(\omega - \alpha_1 - \cdots - \kappa_n),$$

where ω stands for a or b, and if it is b it is counted in k'.

We may modify the formula (1) as follows: If a is greater than $b_1 + b_2 + \cdots + b_n$ ($b > a$, $b_r > a_r$), the probability is zero. But the ϕ_n's now reduce to nth powers. Hence,

$$\frac{1}{2c_1c_2 \cdots c_n \underline{|n}}\Sigma(-1)^{k'-1}(\omega - \alpha_1 - \cdots - \kappa_n)^n = 0$$

for all values of a and b beyond certain limits, and is therefore identically zero. On adding this zero quantity to the right-hand side of (1) we cause all the ϕ_n's which are not mere nth powers to cancel, and double the others.

Hence, the probability is

$$\frac{1}{\underline{|n}\, c_1c_2 \cdots c_n}\Sigma(-1)^{k'-1}(\omega - \alpha_1 - \cdots - \kappa_n)^n,$$

the summation covering all the cases where $\omega - \alpha_1 - \cdots - k_n$ is positive.

It is easy to deduce for the special case propounded that the probability is

$$\frac{1}{\underline{|n}}\left\{\left(s + \frac{n+1}{2}\right)^n - {}_{n+1}C_1\left(s + \frac{n-1}{2}\right)^n + \cdots + (-1)^r{}_{n+1}C_r\left(s + \frac{n+1}{2} - r\right)^n\right\},$$

where r is the next integer below $s + (n+1)/2$, and s may be any real number.

GEOMETRY.

Solutions of 497 were received from J. B. Reynolds, Elijah Swift, R. M. Mathews, and H. R Howard after selections for publication were made.

498. Proposed by FRANK R. MORRIS, Glendale, California.

To trisect an angle ABC, on BA and BC take D and E equidistant from B. Using DE as a diameter draw the semicircle $DFGE$. With the same radius and D and E as centers draw arcs locating the points F and G on this semicircle. Connect F and G with B. Prove that this method trisects a right angle and a straight angle and that it does not trisect an oblique angle.

SOLUTION BY ROGER A. JOHNSON, Western Reserve University.

Let $x = \angle EBG = \angle FBD$, $y = \angle GBF$, and $\alpha = \angle EBD$. Then $\angle BDF = 150° - \alpha/2$, and $\angle BGF = 90° - y/2$. Applying the law of sines,

$$\frac{\sin x}{\sin\left(150 - \frac{\alpha}{2}\right)} = \frac{FD}{BF}, \qquad \frac{\sin y}{\sin\left(90° - \frac{y}{2}\right)} = \frac{GF}{BF};$$

whence

$$\frac{\sin x}{\sin y} = \frac{\sin\left(150° - \frac{\alpha}{2}\right)}{\sin\left(90° - \frac{y}{2}\right)}. \tag{1}$$

Now in order that $x = y$, we must have either

$$150° - \frac{\alpha}{2} = 90° - \frac{y}{2}, \tag{a}$$

or

$$150° - \frac{\alpha}{2} = 180° - \left(90° - \frac{y}{2}\right). \tag{b}$$

Letting $y = \alpha/3$, we obtain (a) $\alpha = 180°$, (b) $\alpha = 90°$, as the only values of α for which this construction effects the trisection. Conversely, it is very easily seen directly that either a right angle or a straight angle is actually trisected.

To determine roughly the magnitude of the error for angles in general, we may safely replace $\sin(90° - y/2)$ by $\sin(90° - \alpha/6)$; we have then

$$\frac{\sin x}{\sin y} = \frac{\sin\left(30° + \frac{\alpha}{2}\right)}{\sin\left(90° - \frac{\alpha}{6}\right)}. \tag{2}$$

For small angles, the construction is far from accurate; in fact,

$$\lim_{\alpha=0} \frac{\sin x}{\sin y} = \frac{\sin 30°}{\sin 90°} = \frac{1}{2},$$

so that if α is small, it is divided approximately in *fourths*. We find roughly, when $\alpha = 30°$, $60°$, $90°$, $120°$, $150°$, and $180°$, $(\sin x)/(\sin y) = .710$, $.879$, 1.000, 1.064, 1.066, and 1.000, respectively. Again, by setting $x = \alpha/2 - y/2$ in (1) and solving for $y/2$, we find

$$\cot\frac{y}{2} = 2\cot\frac{\alpha}{2} + \sqrt{3};$$

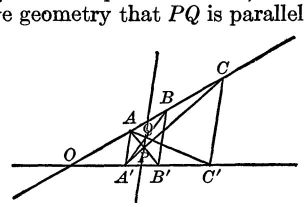
whence we may compute the value of y corresponding to any value of α. For example, if $\alpha = 10°$, $y = 4° 40'$ and $x = 2° 40'$; if $\alpha = 60°$, $y = 21° 46'$ and $x = 19° 7'$; if $\alpha = 135°$, $y = 42° 41'$ and $x = 46° 10'$.

Also solved by HORACE OLSON.

500. Proposed by R. T. MCGREGOR, Bangor, California.

OABC, *OA'B'C'* are two straight lines such that *AA'*, *BB'*, *CC'* are parallel. *AB'*, *A'C* meet in *P*; *A'B* and *AC'* meet in *Q*. Show by synthetic projective geometry that *PQ* is parallel to *AA'*. Milne's *Projective Geometry*, Chap. I, Ex. 20.

SOLUTION BY CLARIBEL KENDALL, University of Colorado.



Consider A, B, C as the first, third and fifth sides, respectively of a hexagon; and A', B', C' as the fourth, second, and sixth sides, respectively, of the same hexagon. (12), (45); (34), (61) meet in P and Q, respectively; hence, (23), (56) must meet in a point collinear with P and Q.

This follows as a special case of Pascal's Theorem for a hexagon inscribed in a conic.* The conic is in this case degenerate, two straight lines. But (23), (56) are parallel and hence must have an infinitely distant point in common with PQ. Therefore PQ is parallel to AA'.

Also solved by MARJORIE L. BROWN, O. S. ADAMS (two methods), F. E. WOOD, NATHAN ALTSHILLER, HANNAH SUFFIN, and H. H. CONWELL.

CALCULUS.

416. Proposed by CHARLES N. SCHMALL, New York City.

If A be a point on a cycloid and C the corresponding position of the center of the generating circle, show that AC envelops another cycloid half the size of the first.

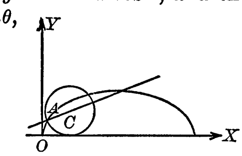
SOLUTION BY A. M. HARDING, University of Arkansas.

The coördinates of any point on the cycloid are $x = a\theta - a\sin\theta$, $y = a - a\cos\theta$, and the coördinates of the center of the generating circle are given by $x = a\theta$, $y = a$. The equation of AC is

$$\frac{x - a\theta}{\sin\theta} = \frac{y - a}{\cos\theta},$$

or

$$y - a = \cot\theta(x - a\theta).$$



The equation of a cycloid half the size of the given cycloid and having a cusp at O is

$$x = \frac{a\varphi}{2} - \frac{a}{2}\sin\varphi, \qquad y = \frac{a}{2} - \frac{a}{2}\cos\varphi.$$

We propose to show that AC is always tangent to this cycloid.

The equation of any tangent to this cycloid is

$$y - \frac{a}{2}(1 - \cos\varphi) = \cot\frac{\varphi}{2}\left[x - \frac{a}{2}(\varphi - \sin\varphi)\right].$$

Let $\varphi = 2\theta$. This equation then becomes

$$y - a\sin^2\theta = \cot\theta[x - a\theta + a\sin\theta\cos\theta]$$

or

$$y - a = \cot\theta(x - a\theta),$$

which is the same as the equation of AC.

Also solved by C. N. SCHMALL, ELIJAH SWIFT, G. W. HARTWELL, HORACE OLSON, O. S. ADAMS, M. R. GAFFET, R. H. HOWARD, J. B. REYNOLDS, and SHIMPEI NISHIMURA.

417. Proposed by H. S. UHLER, Yale University.

To the degree of approximation indicated show that $(\sqrt{-1})^{\sqrt{-1}} = 0.207879576351$.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

It is not difficult to show, as required in Todhunter's *Plane Trigonometry*, Ed. 1913, pp. 320–21, Examples 266, 275, "that $(a + bi)^{\alpha+\beta i}$ will be wholly real or imaginary if

$$(\beta/2)\log(a^2 + b^2) + \alpha\tan^{-1}(b/a)$$

is (I) zero, or an even multiple of $\pi/2$; or (II) an odd multiple $\pi/2$". In the problem, $a = \alpha = 0$, and $b = \beta = 1$, the conditions corresponding to (I), 0 being called an even number. $(\sqrt{-1})^{\sqrt{-1}} = e^{-\frac{1}{2}\pi}$, and the numerical value can be tested by using enough decimals in the values of e and π.

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* Cremona, *Elements of Projective Geometry*, translated by Leudensdorf, Articles 88 and 153, 3d edition.

The problem is of some historical interest, being due to John Bernoulli. The distinguished American astronomer, G. W. Hill, proposed it in *The Analyst*, Vol. II, January, 1875, p. 31, Ex. 56, with no reference to its history. It was solved by Walter Siverly, and Henry Heaton in the succeeding number of *The Analyst*, by the following method:

By Euler's Theorem, $\cos\theta + i\sin\theta = e^{i\theta}$, where θ may be $(4n+1)(\pi/2)$. Assume $n = 0$. Then $i = e^{(\pi/2)i}$ and $(i)^i = e^{-\frac{1}{2}\pi} = 0.2078795763507$.

Note.—By an oversight, this problem was placed under Calculus.

Solutions were also received from J. B. REYNOLDS, PAUL CAPRON, O. S. ADAMS, and E. B. ESCOTT.

Mr. Escott, using Steinhauser's 20-place tables, gets .20787957635076190854687 while Professor Reynolds, using Hutton's 20-place tables, gets .2078795763491790781.—EDITORS.

MECHANICS.

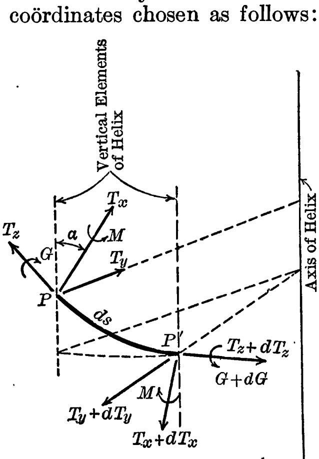
300. Proposed by V. M. SPUNAR, Chicago, Ill.

A helical spring is composed of twenty turns of steel wire 0.258″ in diameter, the diameter of the coil being 3″. If the spring is compressed by a force of 50 lbs., what is the maximum stress in the coil, its axial compression and its resilence?

SOLUTION BY G. PAASWELL, N. Y. City.

This solution follows the method given by Love in his treatise on Elasticity.

At any point, P, of the helix, take a system of rectilinear coördinates chosen as follows: the tangent, as the z axis; the radial element as the y axis; and the normal to the two above, as the x axis. The motion of the curve may be analyzed into a curvature k about the x axis described by the z axis and a twist, t, about the z axis described by the x axis. These are, respectively, if the pitch of the helix is a, $\cos^2 a/r$, $(\sin a\cos a)/r$, where r is the radius of the coil. This curvature, k, causes a bending moment, M, and the twist, t, causes a torsional moment, G.



If s is the total length of the helix; n, the number of turns and h the height of the coil; then $h = s\sin a$ and $s\cos a = 2\pi nr$.

Assume the spring subjected to an elongating force R and consider a small free portion of length ds (see figure). The twist from P to P' is tds and the curvature, kds. From ordinary statics,

$$\Sigma_x = \Sigma_y = \Sigma_z = 0,$$

whence

$$T_x - (T_x + dT_x) + tds(T_y + dT_y) = 0, \qquad (1)$$

$$T_y - (T_y + dT_y) - tds(T_x + dT_x) + kds(T_z + dT_z) = 0, \qquad (2)$$

$$T_z - (T_z + dT_z) - kds(T_y + dT_y) = 0, \qquad (3)$$

$$\Sigma_{\text{mom. } x} = \Sigma_{\text{mom. } y} = \Sigma_{\text{mom. } z} = 0,$$

$$(T_y + dT_y)ds = 0; \text{ whence } T_y = 0 \text{ and } (T_x + dT_x)ds + Gkds - Mtds = 0.$$

From the above, we get at once $T_x - Mt + Gk = 0$ and $tT_x - kT_z = 0$, or $T_x \tan a = T_z$.

The applied loading deforms the helix into a new one with pitch a' and radius r'. The new curvature and twist is, then, t', k'. Assuming that the original curve was under no stress, the bending and torsional moments are given, respectively, by the difference in the curvatures and twists, multiplied by the flexural and torsional rigidities. Then, $M = EI(k' - k)$ and $G = E_sI_s(t' - t)$, where E is Young's Modulus; E_s is the shearing modulus, I is the moment of inertia of the wire about a diameter; and I_s is the polar moment of inertia of the wire. If Poisson's ratio is taken as 1/4, then $E_s = \frac{2}{5}E$. Representing EI by A, then $M = A(k' - k)$ and $G = \frac{4}{5}A(t' - t)$. (6) can now be written $T_x = A\{t'(k' - k) - \frac{4}{5}k'(t' - t)\}$. Since the resultant stress must equal R, $R = T_x \sec a'$. Likewise, the resultant moment is

$$W = M\cos a' + G\sin a'.$$

There is, finally,

$$R = A\left\{\frac{\sin a'}{r'}\left(\frac{\cos^2 a'}{r'} - \frac{\cos^2 a}{r}\right) - \frac{4}{5}\frac{\cos a'}{r'}\left(\frac{\sin a' \cos a'}{r'} - \frac{\sin a \cos a}{r}\right)\right\}$$

and

$$W = A\left\{\cos a'\left(\frac{\cos^2 a'}{r'} - \frac{\cos^2 a}{r}\right) + \frac{4}{5}\sin a'\left(\frac{\sin a' \cos a'}{r'} - \frac{\sin a \cos a}{r}\right)\right\}.$$

The distortion of the helix will change the total angle φ, turned by the helix into $\varphi + d\varphi$; where φ is $2\pi n$; and the height of the helix from h to $h + dh$. Since $\varphi = (s \cos a)/r$ then $\varphi + d\varphi = (s \cos a')/r$ and $h + dh = s \sin a'$. Taking $d\varphi$ and dh as infinitesimal we have $s \cos a' = s \cos a - \tan a dh$. Since

$$\frac{1}{r} = \frac{\varphi}{\sqrt{s^2 - h^2}},$$

then

$$\frac{1}{r^2} = \frac{\varphi + d\varphi}{\sqrt{s^2 - (h + dh)^2}}.$$

After proper reductions and substitutions

$$R = \frac{A}{s^3}\left(\frac{4s^2 + h^2}{5(s^2 - h^2)}\varphi^2 dh - \frac{\varphi h}{5}d\varphi\right) \quad \text{and} \quad W = \frac{A}{S^3}\left(-\frac{\varphi h}{5}dh + \frac{5s^2 - h^2}{5}d\varphi\right).$$

There are two possible cases: the terminal is held fast against twist and hence $d\varphi = 0$; and the terminals are free to turn and the resultant moment W is zero. The former is the usual case in practice and will be the only one discussed here. Either condition, with R given, renders the problem a determinate one. Introducing the former condition, then

$$R = \frac{A}{5s^2}\frac{4s^2 + h^2}{s^2 - h^2}\varphi^2 dh \quad \text{and} \quad W = -\frac{A\varphi h}{5S^3}dh.$$

If p is the radius of the wire and S is the unit shear, we have, since the section is a circle,

$$S = pW/I_s.$$

The resilience must equal the work done, whence Res. $= \frac{1}{2} Rdh$.

With the conditions as given in the problem, dh is first to be found, and noting that $s^2 - h^2 = \varphi r^2$, there is

$$dh = \frac{5s^3 r^2 R}{A(4s^2 + h^2)}.$$

Further, since h^2 is usually infinitesimal in comparison with $4s^2$, the formula may be simply written

$$dh = \frac{5sr^2 R}{4A}.$$

This is the expression used in the numerical substitution below.

Note, that within reasonable limits of the pitch, s is practically constant so that the shortening of the spring is independent of the pitch.

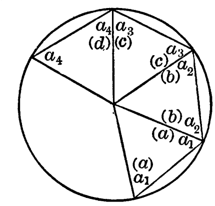
With the data given in the problem, $r = 1.5''$; $p = 0.129''$; $R = 50$ lbs. and E is assumed as 30,000,000 lbs. so that $A = 6{,}570$, $n = 20$. The horizontal projection of the length of the spring is $2\pi nr$, which is equal to 188.5″, so, that for small pitches, s may be taken about 190. The shortening is then 4.1″. Substituting this in the expression for W and then introducing this value of W for finding S, the maximum stress, *i. e.*, shear, is 855 lbs. per sq. in. The resilience is found to be 102.5 in. lbs.

329. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

A smooth circular table is surrounded by a smooth circular rim. Show that the ball, whose coefficient of restitution is e projected along the table from a point in the rim making an angle $\tan^{-1}e^{3/2}$ with the radius through the point, will return to the point of projection after three rebounds.

I. Solution by G. Paaswell, N. Y. City.

Before attacking this special problem let us establish the general case of k points of contact, *i. e.*, of $k - 1$ rebounds. Denote the reciprocal of e by r and $\tan a$ by s (referring to the figure). From the laws of elastic impact, $\tan b = r \tan a$; then $\tan c = r^2 \tan a$; $\tan d = r^3 \tan a$ and so forth. From the figure, $2a + 2b + 2c + \cdots = (k - 2)\pi$; whence



$$\tan^{-1} s + \tan^{-1} rs + \tan^{-1} r^2 s + \cdots = \frac{\pi}{2}(k - 2)$$

It may be shown by developing a few cases and then by induction that

$$\tan^{-1} a_1 + \tan^{-1} a_2 + \tan^{-1} a_3 + \tan^{-1} a_4 + \cdots + \tan^{-1} a_k$$

$$= \tan^{-1}\left(\frac{{}_1\Sigma_k - {}_3\Sigma_k + {}_5\Sigma_k \cdots \pm {}_k\Sigma_k}{1 - {}_2\Sigma_k + {}_4\Sigma_k \cdots \pm {}_{k-1}\Sigma_k}\right) = \frac{\pi}{2}(k - 2),$$

when k is odd and

$$= \tan^{-1}\left(\frac{{}_1\Sigma_k - {}_3\Sigma_k + {}_5\Sigma_k \cdots \pm {}_{k-1}\Sigma_k}{1 - {}_2\Sigma_k + {}_4\Sigma_k \cdots \pm {}_k\Sigma_k}\right) = \frac{\pi}{2}(k - 2),$$

when k is even.

${}_p\Sigma_k$ represents the sum of the k arguments taken p at a time. Taking the tangent of both members $\left(\text{the second member is } (k - 2)\frac{\pi}{2}\right)$ and noting that when k is odd, $\tan(k - 2)\frac{\pi}{2}$ is infinite and hence the denominator must vanish, and similarly, when k is even, the numerator of the corresponding fraction must vanish, we have

$${}_1\Sigma_k - {}_3\Sigma_k + {}_5\Sigma_k \pm \cdots\cdot {}_{k-1}\Sigma_k = 0 \ (k \text{ is even}),$$

$$1 - {}_2\Sigma_k + {}_4\Sigma_k \pm \cdots\cdot {}_{k-1}\Sigma_k = 0 \ (k \text{ is odd}).$$

When the sequence of a's forms a geometric progression it may be shown without difficulty that

$${}_p\Sigma_k = r^{p(p-1)/2} s^p \frac{1 - r^{-A_p}}{1 - r}; \qquad A_p = \frac{k!}{(k - p)!p!};$$

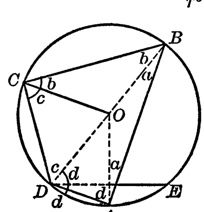
and the condition equations become

$$s\frac{1 - r^{A_1}}{1 - r} - r^3 s^3 \frac{1 - r^{A_3}}{1 - r} + r^{10} s^5 \frac{1 - r^{A_5}}{1 - r} \cdots \pm r^{(k-1)(k-2)/2} s^{k-1} \frac{1 - r^k}{1 - r} = 0, \ k, \text{ even};$$

$$1 - rs^2 \frac{1 - r^{A_2}}{1 - r} + r^6 s^4 \frac{1 - r^{A_4}}{1 - r} \cdots \pm r^{(k-1)(k-2)/2} s^{k-1} \frac{1 - r^k}{1 - r} = 0, \ k, \text{ odd}.$$

These are the general equations for any number of rebounds. Thus, $k = 2$ (one rebound) $s(1 + r) = 0$, or $s = 0$, $a = 0$; $k = 3$ (two rebounds), $1 - rs^2 \dfrac{1 - r^3}{1 - r} = 0$, $s^2 = \dfrac{1}{r(1 + r + r^2)}$

$a = \tan^{-1}\sqrt{\dfrac{e^3}{1 + e + e^2}}$; $k = 4$ (3 rebounds, the given case), $s\dfrac{1 - r^4}{1 - r} - r^3 s^3 \dfrac{1 - r^4}{1 - r} = 0$; $1 - s^2 r^3 = 0$, $s = \dfrac{1}{r^{3/2}}$; whence $a = \tan^{-1} e^{3/2}$.



II. Solution by Charles A. Hutchinson, Wittenberg College, Springfield, Ohio.

Let A be the point of projection, and B, C, D, the points of first, second and third rebounds, respectively.

Assume that after the third rebound, the ball does not pass along DA, but along DE.

Then $\tan a = \sqrt{e^3}$, $\tan b = 1/e \cdot \tan a = \sqrt{e}$, $\tan c = 1/e \cdot \tan b = 1/\sqrt{e}$, and $\tan f = 1/e \cdot \tan c = 1/\sqrt{e^3} = \cot a$. Hence, $f = 90° - a$.

Since $\tan c = 1/\sqrt{e} = \cot b$, therefore, $c = 90° - b$. Hence, $a + b + c + f = 180$. But $a + b + c + d = 180°$. Hence, $d = f$.

Hence, DA and DE coincide and the ball does return to A.

Also solved by HORACE OLSON, A. M. HARDING, and HAROLD T. DAVIS.

NUMBER THEORY.

230. Proposed by E. B. ESCOTT, Ann Arbor, Michigan.

Find three numbers such that their sum, the sum of their squares, and the sum of their cubes shall be a cube.

SOLUTION BY E. T. BELL, Seattle, Washington.

Presumably the proposer wished three *integers*, (x, y, z) satisfying the conditions. No restrictions being imposed, we may assume one of the cubes to be zero, and choose $(x + y + z) = 0$, which simplifies the problem very considerably, reducing it in effect to the solution in integers (m, v) of

(1) $$m^2 = 36v^3 + 1.$$

A solution (x, y, z) is:

$$\begin{aligned} x &= 146, & x^2 &= 21\ 316, & x^3 &= 3\ 112\ 136,\\ y &= -1\ 314, & y^2 &= 1\ 726\ 596, & y^3 &= -2\ 268\ 747\ 144,\\ z &= 1\ 168, & z^2 &= 1\ 364\ 224, & z^3 &= 1\ 593\ 413\ 632,\\ w &= -876, & & & w^3 &= -672\ 221\ 376, \end{aligned}$$

whence

$$x + y + z = 0^3,$$
$$x^2 + y^2 + z^2 = x^3,$$
$$x^3 + y^3 + z^3 = w^3 = (-6x)^3,$$

which may be most readily checked from the resolutions into prime factors:

$$x = 2 \times 73; \quad y = -2 \times 3^2 \times 73; \quad z = 2^4 \times 73; \quad w = -2^2 \times 3 \times 73;$$

clearly, (xt^3, yt^3, zt^3), where t is an arbitrary integer, is also an integral solution.

More generally, (m, v) being any solution of (1), and k arbitrary,

(2) $$\begin{aligned} x &= 18(m^2 + 3)(m + 1)^2(m - 1)^2k^3,\\ y &= -9(m^2 + 3)(m + 1)^3(m - 1)^2k^3,\\ z &= 9(m^2 + 3)(m + 1)^2(m - 1)^3k^3, \end{aligned}$$

give

(3) $$\begin{aligned} x + y + z &= 0^3,\\ x^2 + y^2 + z^2 &= \{2^3 3^4 k^2 v^4 (m^2 + 3)\}^3 = \{2^5 3^4 v^4 k^2 (3^2 v^2 + 1)\}^3,\\ x^3 + y^3 + z^3 &= \{-2^5 3^7 (m^2 + 3) v^7 k^3\}^3 = \{-2^7 3^7 v^7 k^3 (3^2 v^2 + 1)\}^3, \end{aligned}$$

which, for $k = 2^{-4}3^{-2}$, and the solution $(m, v) = (17, 2)$ of (1) gives the solution first stated.

Note.—The analysis for solution 2, which would take about two pages of the MONTHLY, can be written out if thought of sufficient interest, but it involves nothing new. Incidentally, the above solution carries with it that of many other curious indeterminate systems, when (3) is combined with Newton's formulæ for the sums of like powers of the roots of an algebraic equation; thus, we are shown how to find integers (a, b, c, p, r) satisfying

$$a^5 + b^5 + c^5 = 30p^3,$$
$$a^7 + b^7 + c^7 = 28r^3.$$

There is an infinite chain of such results, which are all more or less in the nature of accidents.

246. Proposed by ALBERT A. BENNETT, Princeton University.

Prove that

$$\frac{1}{\sqrt{b}}\left[\left(\frac{a+\sqrt{b}}{2}\right)^n-\left(\frac{a-\sqrt{b}}{2}\right)^n\right]$$

is an integer for every positive integral value of n, whenever a is an odd integer, positive or negative, and $b \equiv 1 \pmod{4}$.

SOLUTION BY PAUL CAPRON, U. S. Naval Academy.

Taking out the factor

$$\frac{a+\sqrt{b}}{2}-\frac{a-\sqrt{b}}{2}=\sqrt{b},$$

we have

$$\frac{1}{\sqrt{b}}\left[\left(\frac{a+\sqrt{b}}{2}\right)^n-\left(\frac{a-\sqrt{b}}{2}\right)^n\right]=\sum_{s=0}^{n-1}\left(\frac{a+\sqrt{b}}{2}\right)^{n-s-1}\left(\frac{a-\sqrt{b}}{2}\right)^s.$$

Let

$$\frac{a+\sqrt{b}}{2}=x,\qquad \frac{a-\sqrt{b}}{2}=y.$$

Then this expression is

$$\text{(I)}\qquad (x^{n-1}+y^{n-1})+xy(x^{n-3}+y^{n-3})+x^2y^2(x^{n-5}+y^{n-5})+\cdots+L;$$

where $L=x^{(n-1)/2}y^{(n-1)/2}$ if n is odd; and $L\equiv x^{(n-2)/2}y^{(n-2)/2}(x+y)$ if n is even.

Let $a=2k+1$, $b=4l+1$, k and l being integers. Then

$$xy=\frac{a^2-b}{4}=k^2+k-l,$$

an integer, so that any positive integral power of xy is an integer.

Also $x+y=a$ and $x^2+y^2=(x+y)^2-2xy=a^2-2(k^2+k-l)$ are integers. Further,

$$x^p+y^p=(x+y)^p-\Sigma\frac{p!}{s!(p-s)!}x^sy^s(x^{p-2s}+y^{p-2s});$$

so that if x^p+y^p is integral for a positive integral value of p, it is integral for a value of p greater by 2; but it is integral for $p=1$ and for $p=2$, and so is integral for any positive integral value of p.

Thus each term of (I) is integral, and I is an integer.

Also solved by ELBERT H. CLARKE.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence.

DISCUSSIONS.

I. RELATING TO INFINITESIMAL METHODS IN GEOMETRY.

BY E. B. WILSON, Massachusetts Institute of Technology.

The old method of using infinitesimals to find by geometric considerations various properties of figures was given a prominent place by W. E. Byerly in his *Differential Calculus*, but has been almost wholly discarded by recent writers. It is a question worthy of discussion whether the present neglect of the method is entirely wise. An example or two may be given.

1. *Consider the problem of the maximum triangle inscribed in a convex curve, or the minimum triangle circumscribed about the curve.*

If ABC is the maximum triangle inscribed in the curve, a displacement of any vertex, say C, along the curve, can only change the area of the triangle by an infinitesimal of higher order than the first, and hence the altitude from C to AB must change only by an infinitesimal of higher order. This means that the tangent to the curve at C must be parallel to the side AB. We have, therefore, the proposition that the maximum triangle is that for which the tangents to the curve at the vertices are parallel to the opposite sides of the triangle. Incidentally, the geometry of the figure shows that the triangle formed by the three tangents has its three sides bisected by the points of tangency.

If the minimum triangle circumscribed about the curve is desired, let PQR be the triangle. The displacement of the point of tangency of one side must change the area of the triangle by an infinitesimal of higher order. A consideration of the small triangles added to and subtracted from PQR by the change of one side to a neighboring point of tangency, shows at once that the triangle PQR has its sides bisected by their points of tangency. Incidentally the geometry of the construction shows that the inscribed triangle formed by joining the three points of tangency has its sides parallel to the sides of PQR.

It is thus seen that the solutions for the maximum inscribed triangle and the minimum cricumscribed triangle are identical in so far as the determination of three points on the curve is concerned. Is there any *easy* way of reaching these results by the exclusively analytical methods now in vogue?

As applied to the circle, it is a matter of the simplest plane geometry to show that the above construction calls for equilateral triangles. Is there any easier way than this for finding the maximum and minimum triangles for the special case of the circle?

Any ellipse may be regarded as the orthogonal projection of a circle. In such a projection areas are multiplied by a constant, the cosine of the angle between the planes. The eccentric angle in the ellipse is identical with the central angle in the circle. Hence we see that the maximum triangle inscribed in the ellipse is any triangle whose vertices are equally spaced along the ellipse when measured by the eccentric angle (or by the sectorial area; *i. e.*, the central radii to the vertices of the triangle divide the ellipse into three sectors of equal area). The minimum triangle about the ellipse needs no special attention, because of the general theorem mentioned above.

2. *Consider the problem of finding the point P in the plane of three points ABC such that the sum of the distances $PA + PB + PC$ shall be a minimum.*

If this problem is solved by the straightforward analytical method, the analysis appears complicated, owing to the three radicals used to express the distances.

The geometric solution is easy. Let P be the point. Displace P along a line perpendicular to one of the lines PA, PB, PC, say PC. Then PC changes by an infinitesimal of the second order, and $PA + PB$ must change by an

infinitesimal of higher order than the first; that is, $PA + PB$ is constant up to and including the first order. Hence (as shown by the laws of reflection for light, or from the properties of the ellipse), PA and PB must make equal angles with PC. By symmetry, the angles subtended at P by the three sides of the triangle ABC must all be 120°.

It is now easy to construct the point P. Draw upon two sides of the triangle arcs of 120°. Their intersection determines P, and the arc of 120° on the third side of the triangle passes through P. (The construction breaks down when one angle of the triangle is greater than 120°, and in this case the vertex of that angle is the solution of the problem.)

I should be glad to see other analytic solutions of the problem which are as simple. See the solution by Dunham Jackson in the January 1917 MONTHLY, which appeared after this note was sent to the editor.

II. RELATING TO A PROBLEM IN MINIMA DISCUSSED BY PROFESSOR DUNHAM JACKSON IN THE JANUARY, 1917, NUMBER (P. 46) OF THIS MONTHLY.

BY ROGER A. JOHNSON, Western Reserve University, Cleveland, Ohio.

The problem is to determine a point the sum of whose distances from the vertices of a given triangle shall be a minimum.

Professor Jackson has apparently overlooked the fact that this problem admits an elementary geometric solution due to Steiner (*Collected Works*, Vol. II, p. 729). The following is a development of this solution; for more detailed treatment and historical notes, see Emmerich, *Die Brocard'sche Gebilde*, § 42.

Lemma I. The sum of the distances to the sides of an equilateral triangle, from a point inside the triangle, is constant and equal to the altitude. Moreover, if from any point the perpendiculars to the sides of the equilateral triangle make angles of 120° (and not 60°) with one another, the point is inside the triangle.

Lemma II. Let ABC be a triangle, each angle of which is less than 120°. There is a point P, and only one, such that $\angle BPC = \angle CPA = \angle APB = 120°$.

Lemma III. Through A, B, C, draw MN, NL, LM, perpendicular to PA, PB, PC, respectively. Then LMN is an equilateral triangle and P is inside it.

THEOREM. *If Q is any point of the plane, other than P, then*

$$QA + QB + QC > PA + PB + PC.$$

For $QA + QB + QC$ is greater than the sum of the perpendiculars from Q to MN, NL, LM; and the sum of these perpendiculars is greater than or equal to $PA + PB + PC$, according as Q lies outside or inside triangle LMN.

The point P is one of two points called *isogonic centers*, which have numerous interesting properties (cf. Emmerich, *l. c.*). If any angle of the triangle is greater than 120°, it falls outside the triangle and therefore fails to yield a minimum. As Professor Jackson indicates, it is not hard in this case to show that the vertex of the obtuse angle is the desired point. To do this by the same method we have

just used, we assume that angle B is greater than 120°. Through A and C, respectively, let XAY and XBZ be drawn perpendicular to BA and BC respectively; through B draw a line YZ so that $\triangle XYZ$ is isosceles, and $\angle YXZ$ is less than 60°; then $ZY < XY$.

Let PF, PG, PH, be perpendicular to XY, YZ, ZX, respectively.

Then

$$XY \cdot PF + XZ \cdot PH + YZ \cdot PG = XY \cdot BA + XZ \cdot BC.$$

Let $m = XY = XZ$. Then, since $YZ \cdot PG < XY \cdot PG$,

$$m(PF + PG + PH) > m(BA + BC),$$

and

$$PF + PG + PH > BA + BC.$$

Also, as before, $PA + PB + PC \geqq PF + PG + PH$, which establishes the result.

III. Relating to the Exponential Function.

By Otto Dunkel, Washington University.

An interesting treatment of the elementary transcendental functions was given by A. Hurwitz in the *Mathematische Annalen*, Vol. 70, 1911, entitled "Über die Einführung der elementaren transzendenten Funktionen in der algebraischen Analysis." The method used by Hurwitz was somewhat similar to the one used by Professor Huntington in his article "An Elementary Theory of the Exponential and Logarithmic Functions" in the September, 1916, number of the Monthly, pp. 241–246, except that Hurwitz discussed first the function $\log x$ and then derived sequences for the definition of e^x of the type used by Huntington. About the time of the appearance of Hurwitz's article the writer developed a treatment of the exponential function similar to the one given by Huntington but somewhat simpler in the fact that the inequality $(1 + d)^m > 1 + md$ was not required. This makes the proof more elementary, as the proof of this inequality in Huntington's article is made to depend partly upon the binomial theorem and partly upon an additional proof for a remaining case. This advantage is gained by using sequences in which each exponent is double the preceding, for then it is a very easy matter to prove the increasing and decreasing character of the two sequences, and having done this the sequences themselves supply the place of the above mentioned inequality in the subsequent reasonings. A brief sketch of how this can be effected is given below.

Given the sequences

$$(A)\quad 1 + x,\ \left(1+\frac{x}{2}\right)^2,\ \left(1+\frac{x}{4}\right)^4,\ \left(1+\frac{x}{8}\right)^8,\ \cdots,\ \left(1+\frac{x}{m}\right)^m,\ \left(1+\frac{x}{2m}\right)^{2m},\ \cdots$$

$$(B)\quad (1-x)^{-1},\ \left(1-\frac{x}{2}\right)^{-2},\ \left(1-\frac{x}{4}\right)^{-4},\ \left(1-\frac{x}{8}\right)^{-8},\ \cdots,\ \left(1-\frac{x}{m}\right)^{-m},\ \left(1-\frac{x}{2m}\right)^{-2m},\ \cdots,$$

where x is any real number, and considering only the terms for which $m > |x|$, we shall prove the following: (a) Sequence (A) increases; (b) Sequence (B) decreases; (c) Each A is less than the corresponding B and the difference approaches zero as m increases.

We have for any real value of x

$$\left(1+\frac{x}{2m}\right)^2 = 1+\frac{x}{m}+\frac{x^2}{4m^2} > 1+\frac{x}{m},$$

and, since $1 + x/m$ is positive, we have also

$$\left(1+\frac{x}{2m}\right)^{2m} > \left(1+\frac{x}{m}\right)^m.$$

This proves (a), and (b) follows at once from the above inequality by replacing x by $-x$ and reversing the resulting inequality.

To prove (c) we have

$$\left(1-\frac{x}{m}\right)^{-m} - \left(1+\frac{x}{m}\right)^m = \left(1-\frac{x}{m}\right)^{-m}\left[1-\left(1-\frac{x^2}{m^2}\right)^m\right]$$

$$< \left(1-\frac{x}{m}\right)^{-m}\frac{x^2}{m} \quad \text{when } m > x^2.$$

It is easily seen that each of the two factors on the right in the equality is positive and hence the first part of (c) is true. The inequality is seen to be true by replacing x in (A) by $-x^2/m$ and using (a). Since the factor $(1 - x/m)^{-m}$ decreases and x^2/m approaches zero as m increases the second part of (c) is true.

If we call the common limit of the sequences (A) and (B) $\exp x$, we have the important inequalities

$$1+h < \left(1+\frac{h}{m}\right)^m < \exp h < \left(1-\frac{h}{m}\right)^{-m} < (1-h)^{-1}, \quad |h| < 1, \quad m = 2^t,$$

from which follow the remaining properties of the function. For example, since

$$\frac{\left(1+\frac{x}{m}\right)^m\left(1+\frac{y}{m}\right)^m}{\left(1+\frac{x+y}{m}\right)^m} = \left[1+\frac{xy}{m(m+x+y)}\right]^m,$$

if we take m so large that $xy/(m + x + y) = h$ is less than unity in absolute value, the above inequalities show that

$$\frac{\exp x \exp y}{\exp(x+y)} = 1.$$

The derivative of $\exp x$ follows at once from the same inequalities. There is therefore an advantage in beginning the sequences with $1 + x$ and $(1 - x)^{-1}$

respectively. The sequences given here contain, of course, those in Huntington's paper.

IV. Relating to Infinite Products for sin z and cos z.

By M. B. Porter, University of Texas.

As the elementary method for obtaining these products primarily applies only to real variables and involves the evaluation of a somewhat troublesome limit and as such proofs as those of Borel's *Fonctions Entières,* p. 82, require a recondite theorem of Hadamard concerning functions of finite order, it seemed to the writer that the following considerations might interest the readers of the Monthly.

We start with the polynomial

$$f_n(z) = z \prod_{m=-n}^{m=n} \left(1 - \frac{z}{mc}\right), \quad c \text{ real},$$

which evidently approaches the analytic limit

$$F(z) = z \prod_{1}^{\infty} \left(1 - \frac{z^2}{n^2c^2}\right)$$

uniformly as n becomes infinite.

Evidently $f_n'(z)$ has all of its roots real by Rolle's theorem and since it is an even function these roots are equal in pairs though of opposite sign, and separate the roots of $f_n(z) = 0$; if we denote the ith root by c_i we have

$$(i - 1)c < c_i < ic. \tag{1}$$

Now consider

$$\frac{f_n'(z)}{f_n(z)} = \sum_{-n}^{n} \frac{1}{z - mc} \equiv \sum_{m=k-s}^{m=k+s+1} \frac{1}{z - mc} + R_n(z). \tag{2}$$

As n becomes infinite this is evidently convergent and uniformly so if we cut out the neighborhoods of the point $z = mc$; and since

$$\left| \sum_{-n-\nu}^{n+\nu} \frac{1}{z - mc} - \sum_{-n}^{n} \frac{1}{z - mc} \right| < \epsilon,$$

for n large enough, it is evident that the zeros of $f_n'(z) = 0$ are as close as we please to $(k + \frac{1}{2})c$ inside of a circle as large as we please if n is taken large enough; for $(k + \frac{1}{2})c$ is obviously a root of the function defined by the last summation in (2). Now

$$f_n'(z) \equiv \prod_{i=-n}^{i=n} \left(1 - \frac{z}{c_i}\right) = \prod_{i=-m}^{i=m} \left(1 - \frac{z}{c_i}\right) \prod_{i=m+1}^{i=n} \left(1 - \frac{z^2}{c_i^2}\right) (m < n).$$

Comparing the last product on the right with

$$\prod_{m+1}^{n}\left(1-\frac{z^2}{m^2c^2}\right),$$

we see that it is as near unity as we please while the first product is as close to

$$\prod_{-m}^{m}\left(1-\frac{z}{(m+\frac{1}{2})c}\right)$$

as we please if n is large enough.

Thus we see that

$$\operatorname*{Lim}_{n=\infty} f_n'(z) = F'(z) = \operatorname*{Lim}_{n=\infty}\prod_{-n}^{n}\left(1-\frac{z}{(m+\frac{1}{2})c}\right).$$

By the same reasoning we show that

$$F''(z) = -\frac{8}{c^2}\sum_{1}^{\infty}\frac{1}{(2n+1)^2}F(z), \tag{3}$$

and integrating the differential equation (3) we have

$$F(z) = \sin\frac{\sqrt{8\Sigma}}{c}z = kz\prod_{1}^{\infty}\left(1-\frac{z^2}{n^2c^2}\right).$$

The constant k is seen to be $\sqrt{8\Sigma}/c$ and $\sqrt{8\Sigma}$ can be evaluated by noting that since sine x has the period 2π and the product the period $2c$ we have

$$\sqrt{8\Sigma} = \pi.$$

The product for $\cos \pi z/c$ can be gotten in the same way, since

$$F'(z) = \cos\frac{\sqrt{8\Sigma}}{c}z$$

is the solution of (3) required.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Indiana.

Assistant Professor C. B. UPTON has been promoted to an associate professorship of mathematics at Teachers College, Columbia University.

Mr. W. H. WILSON, University of Illinois, has accepted an instructorship in mathematics at the Massachusetts Institute of Technology.

Professor A. M. HARDING has been promoted to a full professorship of mathematics at the University of Arkansas.

Dr. G. R. CLEMENTS, of the University of Wisconsin, has been appointed instructor in mathematics at the U. S. Naval Academy, Annapolis.

Professor E. H. MOORE, University of Chicago, has been appointed chairman of the Mathematical Committee of the National Research Council.

Word has been received announcing the death of Mr. J. A. COLSON, at Searsport, Me., February 13, 1917. Mr. COLSON was a charter member of the Association.

Dr. J. W. NICHOLSON, for forty years professor of mathematics in the University of Louisiana, and author of school and college text-books, died on March 22. Dr. NICHOLSON was a charter member of the Association.

Professor WILLIAM BEEBE, of the mathematical faculty of Yale University, died on March 11, aged sixty-six. He graduated from Yale in 1873, and had been teaching in that institution since 1876.

The fourth meeting of the Kansas Section was held at the University of Kansas on Saturday, March 17, 1917. An important program was carried out, a full report of which will appear in the June issue of the MONTHLY.

The Benjamin Peirce instructorships in mathematics at Harvard University have been filled by the reappointment of Dr. W. L. HART and the appointment of Dr. T. A. PIERCE. Mr. B. H. BROWN and Mr. J. L. WALSH have been appointed instructors in mathematics; and Dr. L. R. FORD, of the University of Edinburgh, has been appointed instructor in actuarial mathematics.

At the meeting of the Royal Society, February 22, one mathematical paper was presented, entitled "The ordinary convergence of restricted Fourier series," by Professor W. H. YOUNG.

Three papers of interest to teachers of mathematics appear in the March issue of *School Science and Mathematics* as follows: "Recent tendency in the teaching of elementary applied mathematics," by J. R. YOUNG; "Algebraic derivation of the law of cosines," by A. BABBITT; "A method of demonstrating and teaching the trigonometric functions," by S. C. MITCHELL.

On December 27–29, 1916, the Indian Mathematical Society held its first conference at Madras. An account of this meeting, the secretary's report and the presidential address are contained in the first number of Vol. IX of the *Journal of the Indian Mathematical Society*.

Professor A. GÉRARDIN, writing from Paris under date of February 24, 1917, reports the death of his two collaborators, Professors Barisien and Rudis, and the serious illness of Professor M. H. Brocard.

The MONTHLY still has on hand some copies of the reprints of Professor Huntington's paper on the "Logical skeleton of elementary dynamics," which may be had from the Secretary of the Association, Professor W. D. Cairns, 55 East Lorain St., Oberlin, Ohio, at ten cents each, postage prepaid.

The 191st regular meeting of the American Mathematical Society was held in New York on April 28, 1917. There were fourteen papers on the printed program, including one by Professor MAURICE FRÉCHET, of the University of Poitiers, and one by Professor LUIGI BIANCHI, of the University of Pisa.

Professor H. N. RUSSELL, Princeton University, has been appointed chairman of Section A (Mathematics and Astronomy) for the Pittsburgh meeting of the American Association for the Advancement of Science, to be held December 28, 1917, to January 2, 1918. Professor G. A. MILLER, University of Illinois, was elected from the Association at Large to represent Section A upon the Council for the same meeting.

Professor W. GILLESPIE, of the department of mathematics, at Princeton University, has gone with a party of students to the war zone for work under the direction of the Y. M. C. A. for the benefit of the English soldiers.

Professor R. C. ARCHIBALD, Brown University, writes that he would be glad to learn from any of the readers of the MONTHLY if a copy of a Russian translation of Steiner's little book, *Die geometrischen Konstructionen ausgeführt mittelst der geraden Linie und eines festen Kreises,* can be found in America. The book was translated into Russian by P. EROCHIN and R. HOLZBERG, and was edited by D. SINTZOR.

UNIVERSITY OF NEBRASKA. Summer session, June 11 to August 3. By Professor W. C. BRENKE: Calculus, five hours; Teachers' course, three hours; College algebra and plane trigonometry, five hours each.

UNIVERSITY OF ILLINOIS. The following mathematical courses are announced for the summer session of 1917: By Professor G. A. MILLER: Introduction to group theory; Theory of equations and determinants.—By Professor ARNOLD EMCH: Projective geometry; Constructive geometry.—By Dr. G. E. WAHLIN: Advanced calculus; Integral calculus.—Dr. E. W. CHITTENDEN: Differential calculus.—By Mr. H. D. FRARY: Analytical geometry.—By Mr. R. F. BORDEN: Plane trigonometry.—By Mr. C. H. RICHARDSON: College algebra.

UNIVERSITY OF WISCONSIN. Summer Session, June 25 to August 3.—By Professor C. S. SLICHTER: Mechanics, five hours; Hydrostatics, three hours; Algebra, five hours.—By Professor L. W. DOWLING: Theory of functions, three hours; Higher geometry, five hours; Analytic geometry, five hours.—By Assistant Professor ARNOLD DRESDEN: Calculus of variations, three hours; Definite

integrals, five hours; Calculus, five hours.—By Dr. T. M. SIMPSON: Differential equations, five hours; Commercial Algebra, five hours; Trigonometry, five hours. —By Assistant Professor W. W. HART: Determinants, three hours; Solid geometry, five hours; Teachers' course, five hours.

At the regular meeting of the London Mathematical Society held on February 1, the following papers were read: "The significance of a certain algebraic fraction in the theory of distributions" and "The number of ways of pairing off the members of two identical sets of different quantities," by Major MCMAHON; and "Curves of constant torsion," by W. H. SALMON. At the meeting held on March 1, papers were presented by A. E. JOLLIFFE on "Some properties of the quadrangle formed by the points of contact of the tangents to a nodal cubic," and by E. H. NEVILLE, on "Indicatrices of curvature."

The January, 1917, number of *The Rice Institute Pamphlet* has appeared containing the lectures delivered at the inauguration of Rice Institute by Professor EMIL BOREL, director of scientific studies at the Ecole Normale and professor of mathematics at the University of Paris, and by Senator VITO VOLTERRA, professor of mathematical physics and celestial mechanics at the University of Rome. Professor BOREL delivered lectures upon (1) "Aggregates of zero measure," and (2) "Monogenic uniform non-analytic functions"; Professor VOLTERRA's lectures were: (1) "The generalization of analytic functions" and (2) "On the theory of waves and Green's method."

It is remarkable that the French mathematicians can continue to issue a number of their periodicals. The May and June, 1916, numbers of the *Annales Scientifiques de L'École Normale Supérieure* have recently reached this country, containing lengthy memoirs. The January and February numbers of the *Bulletin des Sciences Mathématiques* are also at hand. Besides the reports and reviews these numbers of the *Bulletin* contain the following short papers: "Upon a theorem of Bertrand," by P. APPELL; and "A direct demonstration of the last theorem of Poincaré," by T. DANTZIG, of Indiana University.

The recently formed Mathematical Club at the University of North Carolina reports an enthusiastic membership numbering thirty. The plan of the programs is to meet once a month, having two papers presented, one upon a mathematical subject, and one upon some allied subject. The following subjects have been presented: "Mathematical requirements for electrical engineering students," "The differential coefficient," "The logic of mathematics," "Inscription of a 17-gon in a circle," "The teaching of mathematics in the high school" and "A note on linear equations."

In *Nature*, March 8, Professor JOSEPH LARMOR gives a short account of the life of the renowned French geometer, Professor GASTON DARBOUX, who died in Paris in February. DARBOUX was born at Nîmes, August 13, 1842; he re-

ceived his early training at the lycée of his native town, being engaged at the school from 6 A. M. to 8 P. M. Later he studied in the lycée at Montpelier, and in 1861 passed the examinations at the head of his class for admission to the Ecole Polytechnique and the École Normale. Intending to engage in teaching, he chose the latter institution and began his studies with great inclination to the study of geometry, mastering the classic works of Monge, Gauss, Poncelet, Dupin, Lamé and Jacobi. In 1864 his memoir on orthogonal surfaces appeared in the *Comptes Rendus*, and two years later this same subject served as his dissertation for the doctorate in mathematical sciences at the Sorbonne. DARBOUX then began his career as a teacher of mathematics, first at the Collège de France, then at the Lycée Louis de Grande, and later as an assistant to Liouville at the Sorbonne. In 1880 he began his life work as professor of higher geometry at the Sorbonne, a position especially created for the renowned geometer Chasles, in 1846. During fourteen years following his graduation from the Sorbonne, DARBOUX published many original memoirs on partial differential equations, and theory of surfaces. In later years this early work was elaborated and published, 1887–1896, in the four classical volumes on infinitesimal geometry known as General Theory of Surfaces. DARBOUX was highly honored by being made a member of the Paris Academy of Sciences, 1884, and since 1900 its perpetual secretary. He was elected a foreign member of the Royal Society, 1900, and in 1916 received its high tribute by being awarded the Sylvester Medal.

The eighth regular meeting of the American Mathematical Society at Chicago was held on Friday and Saturday, April 6–7, 1917. In addition to sixteen papers on the program, there was held on Friday afternoon a symposium on "The Lebesgue Integral," conducted by Professor G. A. BLISS, of the University of Chicago, and Professor T. H. HILDEBRANDT, of the University of Michigan. Detailed synopses of these papers were printed with full references to the literature, and distributed in advance of the meeting. This symposium proved to be a most interesting and edifying part of the program. The dinner on Friday evening was, as usual, accompanied with informal talks on topics of interest to the mathematical fraternity, including particularly the new relation of the *Annals of Mathematics* to the Mathematical Association of America, and the contribution which the mathematicians of the United States might make to the government in the present war crisis. As the outcome of the discussion on the latter point, it was voted that the officers of the Society and of the Association present be requested to make recommendations to their respective Councils in the matter of organizing the resources of the mathematical constituency of the United States in the service of the government. Accordingly, the officers above mentioned joined in recommending that the Councils of the two organizations empower their officers to carry forward this program, in particular, to make a census of the members, if the government advises this, in order to list the various activities in which members are able, experienced, and willing to aid the government. The Council of the Association has since authorized such action as the officers deem wise by a practically unanimous vote.

At the Massachusetts Institute of Technology, a committee of the faculty has been appointed to consider ways of improving methods of instruction. Dr. C. R. MANN, who for the past two years has been preparing a report on engineering education under the auspices of the Carnegie Foundation for the Advancement of Teaching, has been called to the Institute to be chairman of the committee. In the *Educational Review*, January 17, Dr. MANN has published "A study of engineering education," reprinted from a Bulletin of the Society for the Promotion of Engineering Education. This is a very interesting report, tracing briefly the progress of education during colonial times, showing the increase of the demand for technical education, and the founding of the Rensselaer Polytechnic Institute in 1824, the first engineering school in the United States. The report then refers to the great industrial development of the period from 1825 to 1860, the Morrill Act of 1862 and the founding of the state colleges of mechanic arts in the years 1862–1875.

The report also shows some interesting studies on the capabilities of students in technical schools. Under the head of "What freshmen know and can do," it is shown that 90 per cent. of those tested could solve the simplest linear equation in algebra, while only one third of the freshmen could substitute and correctly reduce a simple, fractional expression containing x, a, b, when $x = (a + b) \div 2$. But in "What the schools do to freshmen," it appears that of 2,000 students who entered technical schools in 1911, only 732 graduated in 1915; the mortality in particular studies seems rather high, 52 per cent. passing in physics and a like number in mechanics, 45 per cent. in calculus, 43 per cent. in modern languages and English, and 34 per cent. in chemistry.

The large number of failures in the cases studied would amply justify the maintenance of such a committee as Dr. MANN is to head at the Massachusetts Institute of Technology to consider ways of improving methods of instruction. No doubt other institutions of higher education would profit in the effectiveness of their service to the people should similar investigations be instituted. Too often do instructors assume the attitude of examiner rather than that of real service to the students. The student has not learned how to work, and hence needs help. When 50 per cent., or 40 per cent., or 25 per cent., of a class in mathematics fail, the instructor should first be quite sure that the fault is wholly with the students.

Much interest is being manifested among colleges and universities in the maintenance of mathematical clubs for the especial benefit of undergraduate students. A large number of printed programs of these clubs has been received by the MONTHLY, and many inquiries have been made relative to material for such programs. The programs received by the editors are fashioned along similar lines, and include a range of subjects which should create much interest in the common knowledge of mathematics.

We append a collection of subjects suitable for club programs. (*A*) *Geom-*

etry: (1) Three historic problems; (2) "Flatland"; (3) Non-Euclidean geometry; (4) The fourth dimension; (5) The straight line; (6) Curve tracing; (7) The probability curve; (8) Imaginary loci; (9) Optical images; (10) The nine point circle; (11) Circles connected with a triangle; (12) Expressions for the area of a triangle; (13) Proofs of the Pythagorean theorem; (14) The planimeter; (15) Ancient geometry; (16) Modern geometry. (*B*) *Arithmetic and Algebra:* (17) Mathematical games; (18) The slide rule; (19) Magic squares; (20) Development of arithmetic and algebra; (21) Calculation of π and e; (22) Adding and multiplying machines; (23) Mathematical symbolism; (24) Numerical properties of color and sound; (25) Logarithms; (26) Mathematics of insurance and statistics; (27) Fermat's last theorem; (28) Theory of errors; (29) Mathematics of the calendar; (30) Properties of the numbers 9 and 37; (31) Number systems; (32) Remarkable numbers. (*C*) *Historical and Pedagogical:* (33) Mathematics of the Hindus; (34) Egyptian and Greek mathematics; (35) Mathematics of the Renaissance; (36) Japanese mathematics; (37) The development of trigonometry; (38) The German and French secondary schools; (39) Mathematics in the German universities; (40) Mathematics in American universities; (41) Vocational mathematics; (42) The development of the calculus; (43) Lives of great mathematicians (many subjects available); (44) Controversy between Leibnitz and Newton; (45) Definitions in mathematics; (46) "A Budget of Paradoxes," De Morgan; (47) Mathematical periodicals; (48) Zeno's arguments; (49) Opportunities for the study of graduate mathematics; (50) Why study mathematics?

Another series of topics is in preparation for publication and this is to contain references to the literature which may be helpful to clubs making use of the topics.

Notes on the Association.

The membership of the Association grows with interesting speed. Since the Charter Membership list was completed on April 1, 1916, there have been added in September, 1916, 13 individual members and 8 institutional members; in December, 1916, 15 individual members and 16 institutional members; in February, 1917, 13 individual members and one institutional member; in April, 1917, 22 individual members and one institutional member. Thus the total additions to the Charter list are 63 individual and 26 institutional members. The names, hitherto unpublished, which were voted on by the Council in April, are as follows:

Kansas State Agricultural College to institutional membership.

Samuel Beatty, Lecturer, University of Toronto, Toronto, Can.

Sister Brigetta, Instructor in mathematics, College of St. Scholastica, Duluth, Minn.

H. N. Carleton, West Newbury, Mass.

Chancellor A. B. Chace, Brown University, Providence, R. I.

H. T. Davis, Instructor in mathematics, High School, Colorado Springs, Colo.

Dr. William S. Dennett, New York, N. Y.
Tomlinson Fort, Professor of mathematics, University of Alabama, University, Ala.
Mary E. Helwig, Instructor in mathematics, High School, Kansas City, Kansas.
Glenn James, Instructor in mathematics, Purdue University, West LaFayette, Ind.
H. P. Kean, Professor of mathematics, McKendree College, Lebanon, Ill.
C. H. Lehmann, Student, Cooper Union, New York, N. Y.
J. S. Mikesh, Director, Junior College, Hibbing, Minn.
O. J. Ramler, Instructor in mathematics, Catholic University of America, Washington, D. C.
W. H. Rittenhouse, M.E., Philadelphia, Pa.
S. T. Sanders, Assistant Professor of mathematics, Louisiana State University, Baton Rouge, La.
W. G. Simon, Graduate student, University of Chicago, Chicago, Ill.
C. E. Smith, Professor of mathematics and science, Northland College, Ashland, Wis.
G. T. Street, Instructor in mathematics, Denison University, Granville, Ohio.
H. R. Tolley, Scientific assistant, Office of Farm Management, U. S. Department of Agriculture, Washington, D. C.
T. E. Trott, Professor of mathematics, Mt. Union College, Alliance, Ohio.
J. K. Whittemore, Instructor in mathematics, Sheffield Scientific School, Yale University, New Haven, Conn.
E. H. Worthington, Instructor in mathematics, University of Pennsylvania, Philadelphia, Pa.

The Committee on Libraries, of which Professor W. B. Ford is chairman, has made good progress and will be able to report very soon, possibly in time for the June issue, certainly for the September issue.

The Committee on Fortschritte and Revue Semestrielle, of which Professor E. V. Huntington is chairman, has been able to make very little progress. The only information the committee has been able to gather comes from a London bookdealer, who stated on April 12, 1917, that he had been advised that numbers 1 and 2 of volume 24 of the Revue Semestrielle had appeared. Number 1 of volume 25 has not been issued so far as is now known.

The Program Committee and the Committee on Arrangements for the Summer meeting in Cleveland on September 7 and 8 will make preliminary reports for publication in the June MONTHLY, which will be mailed to members during the last week in June. The June issue will be the last to appear previous to the meeting. The final announcements will be mailed about the middle of August under second-class postage to the addresses of members on the MONTHLY mailing list, unless the Secretary is advised of changes of addresses prior to that time.

The Kentucky Section of the Mathematical Association of America has been organized. The teachers of higher mathematics in Kentucky have maintained a local state organization for nine years; this organization now becomes a part of the larger Association, having been an early applicant for admission as a section. Professor E. L. REES, of the University of Kentucky, is secretary of the section. The spring meeting of the section was held on May 11, 1917, at Berea, Ky.

The Rocky Mountain Section of the Association has been organized and will include the states of Colorado and Wyoming. The first meeting was held on April 7, 1917, and will be reported in the June issue. There are now nine sections of the Association, namely, in Kansas, Ohio, Missouri, Iowa, Indiana, Kentucky, Minnesota, Maryland, Virginia, District of Columbia, and Colorado-Wyoming. Many of the sections hold meetings twice a year. Two spring meetings of sections are reported in this issue, and four more will be reported in the June issue.

The editors of the *Annals of Mathematics* report a most gratifying response to the circular sent out by the Association in March announcing the enlargement of the *Annals* for 1917–1918, and asking for subscriptions on the new basis of cooperation between the *Annals* and the Association. Over three hundred and thirty new subscriptions had been received when this was written, aside from sixty or seventy requests for samples copies of the issue containing Professor Dickson's article on the Fermat Theorem. The new volume of the *Annals* on the enlarged basis begins in October, 1917, and subscriptions from members of the Association at the one-half rate ($1.50) may be sent at any time to the Editors at Princeton, New Jersey. Payment may be deferred, if desired, until October first. This opportunity is open to new members of the Association whenever they may join.

The Secretary and Managing Editor have just made up a shipment of several boxes of books and journals to the library of the Association, which is at present to be housed in a private alcove of the library of Oberlin College, where it will be under the supervision of the Secretary and will be catalogued and cared for by the library staff of Oberlin College. The journals comprise the exchange list of the MONTHLY, which is already well extended and is slowly growing, together with certain sets of journals extending back varying periods which have been contributed by the Managing Editor. These include the *Bulletin of the American Mathematical Society*, *School Science and Mathematics*, *The Mathematics Teacher*, and *Science*. It is hoped that these and other partial sets may be completed. A list of copies or volumes wanted will be published as soon as an inventory can be made of those in hand. The complete set of the MONTHLY contributed jointly by Professors E. H. MOORE and H. E. SLAUGHT has now been bound and deposited in the library.

The number of books is not yet extensive, but it is hoped that all publishers

in this country who print college and university mathematical books will be interested to contribute copies to the library. In order to insure proper credit to all such donors the Secretary has prepared a suitable printed label to be attached to the inside front cover of all books contributed. These will be supplied to publishers, to authors, or to any members or friends of the Association who may wish to become donors. All such contributions may be sent to The Mathematical Association of America in care of the Oberlin College Library, Oberlin, Ohio.

In particular, any persons who may have back numbers or volumes of the MONTHLY, especially back of 1910, are requested to send an inventory of the same to the Secretary. He is continually asked to supply missing copies for those who are making up sets, and he will pay cash for these whenever they can be resold.

SPECIAL ANNOUNCEMENT.

The MATHEMATICAL ASSOCIATION of AMERICA invites all members who have constructive ideas upon the topic: "The treatment of the applications in college courses in mathematics," to submit brief papers on that subject before June 15. All papers will be passed upon by the program committee and a suitable selection will be made for presentation at the summer meeting. This is a subject that should arouse much interest. Address all communications to the Chairman of the Committee, Professor C. S. Slichter, University of Wisconsin, Madison, Wis.

The following from The New Era Printing Company explains the delay of the April issue:

"The delay was caused by the slow freights. We had exhausted the supply of cover paper, and while an order had been placed in ample time and we received the bill of lading on the 9th of April, the paper did not reach us until the last of the month. We regret the delay which was unavoidable."

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Carolyn Rodgers, *John Wanamaker Commercial Institute, New York City.* I find it a most comprehensive book and have asked that it be put into our school as our text. The authors are to be congratulated on the quality and arrangement of the subject-matter, and the publishers deserve much credit for the excellent style and type of the book.

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ALGEBRAICAL DEVELOPMENTS AMONG THE EGYPTIANS AND BABYLONIANS.

By LOUIS C. KARPINSKI, University of Michigan.

INTRODUCTION.

The purpose of this discussion of algebraical developments among the Egyptians and Babylonians is to show that much of the material of our elementary algebra was long ago anticipated, to some extent, in the Orient. Similar anticipations of algebraical reasoning are indicated in the material, such as we have, which shows the progress of mathematics in ancient India and China and Greece.

In Egypt and India and probably in Babylon, too, the numerical phases of mathematical problems received the greater attention; in Greece the geometrical phases were primarily studied. However, by these diverse routes many of the same problems were studied. The first degree equation in one unknown quantity appeared in analytical garb in Egypt, but the same problem is solved geometrically in Greece; simultaneous quadratics (leading to pure quadratics) appear in old Egyptian papyri, solved numerically, while similar problems are given geometrical solution by the ancient Greeks. Numerical application of the Pythagorean proposition appears before the time of Pythagoras in Egypt and probably in India, while the triumph of a logical, geometrical proof is reserved for the Hellenic people as a part of their most noteworthy contribution to the development of human intelligence. The study of arithmetical and geometrical progressions took a somewhat analogous course, appearing on the Ganges, on the Nile, on the Tigris and Euphrates, and, in most complete development, by the Aegean Sea. The conclusion which is suggested by the concordant development of mathematics in ancient times among diverse civilizations, as well as by the dominant position

of the subject in ancient systems of education, is that the human mind has an intrinsic interest in the subject, not conditioned by time or place, but dependent simply upon the development of the reasoning faculty.

Our brief survey will seek to reveal, as far as possible, the fact that this development of algebraical reasoning is a vital and integral part of the history of civilization; the mathematics can not be entirely dissociated from the other sciences that were studied by these ancient peoples, and the references which are made to other scientific achievements are for the purpose of justifying a liberal view in our deductions concerning mathematical attainments from the evidence which is presented. In interpreting historical evidence one is constantly in danger of reading modern ideas into the text; on the other hand some writers in discussing Egyptian mathematics have been at great pains to discount the material which we have. No one can deny that without the Ahmes papyrus, whose preservation must be regarded as a most happy accident, the Egyptians would be credited with insignificant attainments along mathematical lines. Until further mathematical documents of Egypt and Babylon are available our knowledge of the mathematical attainments in these lands will remain incomplete, but, in consequence of recent discoveries confirming ancient classical traditions concerning oriental science, the tendency of historians is to regard more seriously the allusions that ancient Greek and Roman writers made to Egypt and Babylon.

Algebraical Ideas in Egypt.

The Egyptians, even as early as 2000 B. C., attained a relatively high development in mathematics along analytical lines. This advance was made by the Egyptian priests who enjoyed that adequate leisure which is a primary essential for scientific advance. The assumption has frequently been made that the mathematics of the Egyptians was the product of their practical needs, this view being the result of a too serious regard for the statement of Herodotus that the Egyptians developed geometry in order to redistribute the lands after the periodic overflow of the Nile. The assumption is absolutely refuted by a study of their mathematical achievements. Here we have the first appearance of linear equations in one unknown quantity, a type of simultaneous quadratics, arithmetical and geometrical series discussed in problems involving the insertion of means as well as summation, and formulas for areas and volumes; taken as a whole the treatment displays an appreciation of mathematical conceptions and reasoning which has not received due recognition. Certainly the Egyptians did have the faculty of applying mathematics to practical purposes, witnessed by the construction of pyramids and obelisks; in their arithmetic problems of a vital kind, relating to the feeding of geese, the composition of bread, and other phases of domestic economy are common. However, a just view of the mathematics involved must regard these points as applications and not at all as sources of the Egyptian mathematics. Fundamentally and universally mathematics is the achievement of thinking beings, occasioned by the mind and not by the body.

The oldest mathematical textbook in existence is the Ahmes manual, an Egyptian papyrus composed in the reign of the Hiksos Kings (XVIII Dynastie; C. 1700 B. C., which is some 3,600 years old; the work is preserved now in the British museum. A German translation of the work by A. Eisenlohr appeared in 1877; there is some ground for the hope that an English translation taking advantage of the more complete present knowledge of the Egyptian language may soon appear. The manual includes a number of problems in linear equations. The solution while essentially by the method of "false position" is a definite and scientific procedure, leading to the correct value of the root of the equation.

One of these first-degree equations is the following: "Ahau (heap, mass, unknown) and its seventh, it makes 19." An arbitrary value, 7, is assumed as the root and the sum is found to be 8, instead of 19 as required; to obtain 19 from 8 the latter is doubled and multiplied by $\frac{1}{4}$ and $\frac{1}{8}$; the trial root 7 is also multiplied by $2\frac{1}{4}$ $\frac{1}{8}$, giving $16\frac{1}{2}$ $\frac{1}{8}$ as the value of the unknown; substitution of this value in the original equation follows, as a check, in accordance with the common procedure in Egyptian mathematics.

The multiplication of 8 to produce 19 reads,

$$\begin{array}{rl} 8 & 1 \\ 16 & 2' \\ 4 & \frac{1}{2} \\ 2 & \frac{1}{4}' \\ 1 & \frac{1}{8}' \end{array}$$

meaning that 8 has been doubled and multiplied by $\frac{1}{4}$ ($\frac{1}{2}$ of $\frac{1}{2}$) and $\frac{1}{8}$ to give partial summands of 19; the multiplication of 7 by $2\frac{1}{4}$ $\frac{1}{8}$ is also effected in the Egyptian manner of multiplication, by repeated doubling, reading thus:

$$\begin{array}{ll} 2\frac{1}{4}\ \frac{1}{8} & 1' \\ 4\frac{1}{2}\ \frac{1}{4} & 2' \\ 9\frac{1}{2} & 4' \\ \hline 16\frac{1}{2}\ \frac{1}{8}. & \end{array}$$

This method of obtaining the product corresponds to the fact that any number can be written in the binary system; 7 is 111, meaning $1 \times 2^2 + 1 \times 2 + 1$, while $2\frac{1}{4}$ $\frac{1}{8}$ is 10.011.

There are several of these equations of the first degree appearing in the Ahmes manual. A symbol for the unknown quantity, ahau, is found while there are also symbols for addition, a pair of legs walking in the direction of the writing, and subtraction, legs reversed, as well as an approach to an equality sign.

Other ancient Egyptian papyri contain simultaneous equations in two unknowns, leading to pure quadratics involving the Pythagorean triad $3^2 + 4^2 = 5^2$. One of the problems concerns the division of an area of 100 square ells into two squares whose sides have to each other the ratio of 3 to 4; in Euclid's *Data* entirely similar problems are discussed from the geometrical standpoint.

The incomplete state of our knowledge of Egyptian mathematics is indicated by the fact that a statement on ancient authority concerning the names given to the higher powers up to the ninth power does not correspond to any Egyptian material as yet available; an interesting probability is the suggestion that this statement, made by Michael Psellus of the eleventh century, is to be traced to Hypatia of Alexandria in her lost commentary on the arithmetic of Diophantos.

Parenthetically let us note that the Egyptian system of unit fractions, which persisted in Europe three thousand years after the time of the Ahmes manual, frequently gives a convenient method for actual computation; thus $\frac{3}{4}$ of any number is more easily obtained as $\frac{1}{2}+\frac{1}{4}$ of the number than as $\frac{3}{4}$ direct. The Egyptian method of multiplication by repeated doubling and the method of division by multiplication of the divisor to obtain the dividend are not actually as awkward as at first sight appears to be the case, particularly if the form of the numerals is taken into consideration. A consideration of these ancient methods will frequently suggest, even with our numerals, short cuts in computation processes. I mention these points to reëmphasize the fact that it is essential in a serious attempt to evaluate the contributions to science of ancient peoples to guard against attributing to the ancient precursors of modern scientists a lack of real intellectual ability.

The discussion in the Egyptian manual of arithmetical and geometrical progressions reveals an unexpected familiarity with rules which we now express by algebraical formulas, a familiarity which has not received adequate appreciation. The essential points of the two formulas which we have for the nth term and the sum of the arithmetical series, a, $a+d$, $a+2d$, $a+3d$, $\cdots$, appear from the problems to have been familiar to the Egyptians. Comparatively intricate problems are handled with the ease and intimacy born of long acquaintance.

The problem numbered 40 by Eisenlohr reads: "To distribute 100 loaves of bread among 5 people so that $\frac{1}{7}$ of the (total of the) first three equals that of the last two. What is the difference?" The solution shows that it is understood that the loaves are to be distributed in arithmetical progression.

"Following instructions, the difference $5\frac{1}{2}$," is the next somewhat cryptical suggestion of the manual; I hold that this reference implies definite rules of procedure in such problems, leading to the difference $5\frac{1}{2}$, if unity be taken as the first term, under the conditions proposed. Our common procedure, in analytical solution of this problem, leads to the result, $d = 5\frac{1}{2}a$ or $d = 5\frac{1}{2}$ if a is 1. Even if the method of arriving at this value for d be that of "false position" the procedure which, being adaptable to similar problems, arrives definitely and surely at the complete solution of the proposed problem must be regarded as scientific.

From this point the solution follows the lines of previous problems. With 1 as the first term and $5\frac{1}{2}$ as the difference the terms are 1, $6\frac{1}{2}$, 12, $17\frac{1}{2}$, and 23, having 60 as sum. To complete this to the required 100 loaves there must be added 40, or $\frac{2}{3}$ of 60. After noting that this is the case there is added to each

of the numbers in the series found $\frac{2}{3}$ of itself, giving $1\frac{2}{3}$, $10\frac{2}{3}\ \frac{1}{6}$, 20, $29\frac{1}{6}$, and $38\frac{1}{3}$ as the series fulfilling the required conditions.

A second problem involving an arithmetical series is entitled: "Instructions for the difference in distribution." The solution opens with the phrase, "If you are told," that was later adopted by Arabic mathematicians, and that is not uncommon even to-day. "If you are told, 10 measures of grain to 10 people so that the difference of each person as compared with the next one is $\frac{1}{8}$ of a measure of grain. I take the mean, one measure. I subtract 1 from 10, leaving 9. I take $\frac{1}{2}$ of the difference, $\frac{1}{16}$, and take it nine times. This gives $\frac{1}{2}\ \frac{1}{16}$ which I add to the mean. From this take away $\frac{1}{8}$ measure for each person in order to arrive at the goal. Following instructions: $1\frac{1}{2}\ \frac{1}{16}$, $1\frac{1}{4}\ \frac{1}{8}\ \frac{1}{16}$, $1\frac{1}{4}\ \frac{1}{16}$, $1\frac{1}{8}\ \frac{1}{16}$, $1\frac{1}{16}$, $\frac{1}{2}\ \frac{1}{4}\ \frac{1}{8}\ \frac{1}{16}$, $\frac{1}{2}\ \frac{1}{4}\ \frac{1}{16}$, $\frac{1}{2}\ \frac{1}{8}\ \frac{1}{16}$, $\frac{1}{2}\ \frac{1}{16}$, $\frac{1}{4}\ \frac{1}{8}\ \frac{1}{16}$, together 10." The solution of this problem as given by the Egyptian manual should be compared step by step with the solution by the ordinary procedure with the formulas of our elementary algebra; the close correspondence is too striking to be regarded as wholly accidental.

No one could ask, or even suspect, that the ancient Egyptians should have modern formulas with a literal symbolism, for this advance is not made in Europe until the end of the sixteenth century of the Christian Era. However the similarity in method is highly significant, revealing a development in analytical thinking that is not equaled for many centuries. In effect we have in these problems the first term of an arithmetical series regarded as a function of the common difference, under given conditions, and the last term as a function of the mean and the difference. This is real functional thinking whose like is hardly met again until Archimedes.

The single illustration of a geometrical series confirms the implications of the solutions found in the problems involving arithmetical series. The text is extremely concise, and possibly mutilated:

"A ladder

1	2801	Scribe	7
2	5602	Cat	49
4	11204	Mouse	343
Together	19607	Sheaf	2401
		Grain	16807
		together	19607."

At the right we have the summation of the series 7, 49, 343, 2401, and 16807 by actual addition; at the left we have the summation of the same series as 7(2801), the multiplication being effected in the usual manner. Now our formula for the summation of this series gives $7 \cdot \dfrac{7^5 - 1}{7 - 1}$ or 7 times 2801.

Some three thousand years after Ahmes an Italian mathematician of prominence, Leonard of Pisa, includes in his arithmetic the same series with one further term. He effects the solution in precisely the two ways selected by his Egyptian

predecessor. In India, too, powers of seven received special attention. The words, or illustrations, which accompany the numbers suggest the nursery rhyme concerning the old woman going to St. Ives.

While the many other problems of the Ahmes papyrus do not bear as directly upon elementary algebra as the ones which have engaged our attention, yet they do present mathematical treatment of varied phases of human activity. Rules are given for obtaining areas and volumes, including the volume of a hemisphere. Thus the area of a circle is obtained by taking from the diameter $\frac{1}{9}$ of itself and squaring the result, an early method of squaring the circle. The formula,

$$A = \left(d - \frac{d}{9}\right)^2,$$

gives the area as $0.790d^2$, instead of $0.785d^2$; the error is about $\frac{6}{10}$ of one per cent. We have no reason to assume that the Egyptians did not recognize this as an approximation. Application is made of the rule in computing the cubical contents of cylindrical granaries.

The weakest point mathematically in the Ahmes manual is the discussion of the areas of triangles and trapezoids as the leg of the figure appears to be used where the altitude is required. If these rules were found only in this place we would be justified, in view of the Egyptian renown in surveying, in assuming this error to be due to the writer of this school-book. However similar errors are reported in the discussion of areas in an inscription of about 237 B. C. on the temple walls at Edfu. It is difficult to reconcile these crude approximations with the precision of measurements found in the construction of the pyramids and with the use of a method for drawing similar figures corresponding to the use of cross-section paper. The authorities are not in full agreement concerning the interpretation of the texts in question.

Other mathematical papyri, not as complete and detailed as the manual, confirm the methods of computation as given in this text. Certain papyri from Kahun, now in London and Berlin, are of great significance since they present a type of simultaneous quadratic equations. The solution involves the number relation, $3^2 + 4^2 = 5^2$, a relation which appears to have been used by Egyptian surveyors, technically termed "rope-stretchers," in the laying out of right angles. This application of analysis to mensuration is as unexpected as it is suggestive; it is worthy of imitation to-day.

The following is a typical one of these problems:

"Another example of the distribution of a given area into squares. If you are told to distribute 100 square ells (units of area) over two squares so that the side of one shall be $\frac{3}{4}$ of the other," the relative sides are sought. One side is assumed to be unity and the other $\frac{3}{4}$ of unity. With this assumption the two areas total $\frac{25}{16}$ instead of the required 100. The square root of $\frac{25}{16}$ is $\frac{5}{4}$; the square root of 100 is 10, and 10 is to the required side as $\frac{5}{4}$ is to 1. Consequently the one side sought is 8, and the other side 6. The algebraical equivalent of

these equations is, evidently:

$$x^2 + y^2 = 100,$$

$$\tfrac{3}{4}\, y = x.$$

The six problems in Euclid's *Data* which also lead to similar simultaneous quadratics should be compared with this problem.

Egypt is, we may say, the birthplace of algebraical reasoning. Here we find first degree equations, simultaneous quadratics of a simple type, as well as some development in the formation of algebraical symbolism. In the computation of areas and volumes rules appear which are anticipations of modern formulas; in the discussion of arithmetical and geometrical series a high order of mathematical reasoning, considering the time of the work, is attained. Significant, too, is the fact that the Egyptians made the mathematics serve the life of the community, the home, the farm, the shop, and also the state in many great engineering enterprises.

Algebraical Ideas in Babylon.

Babylon in classical times was widely celebrated as the mother of astronomy and astrology, and the most recent researches confirm the ancient tradition. Aristotle suggested to some of his pupils that they should seek in Babylon ancient records of the movements of the stars. Pliny in his *Natural History* makes reference to a Chaldean authority in a discussion concerning a shadow being cast by Venus. Ptolemy, the geographer and astronomer, whose textbook on astronomy continued in active use for 1,500 years, quotes Babylonian records and mentions by name two Babylonian astronomers. The methods, the instruments, the terminology, and the observations of the Babylonians were all of direct service to the early astronomers of Greece, including the great Hysparchus.

Astral-mythology and astral-religion, the cult and worship of the stars, doubtless had their origin in Babylon; astrological tablets appear in the third millenium B. C., scientific observations are recorded in the second millenium, but the highest development was probably not attained until about 700 B. C. As the shepherds of Babylon watched their flocks by night, in the land where the stars are so wonderful in their beauty, the majestic march of the stars spoke plainly of an ordered universe. Hence it is natural that we can trace to Babylon the sun-dial, the signs of the zodiac, the astrolabe, the names of the planets, the division of the circle into degrees, minutes, and seconds, the day of twelve hours and the week of seven days, for all of these things connect naturally with the study of the stars. As the Greeks came into contact with Babylonian civilization, certainly as early as 600 B. C., astrological and astronomical ideas acquired a new meaning and received new emphasis, leading to a more intense interest in these subjects. Doubtless, too, the mysticism of numbers was suggested to Greece by the Orient. Through Greece then these ideas were transmitted to later peoples but the Oriental origin can not be denied. When we tell the time of day, and when we read an angle in degrees and minutes and seconds, we pay an unconscious tribute to the ancient Babylonians.

Concerning the number symbols and systems employed by the Babylonians we have relatively complete information. Two distinct systems, decimal and sexagesimal, seem to have flourished side by side, and three different schemes of representation, more or less closely connected, are found. The earliest system with ordinary cuneiform characters was apparently the decimal system as symbols for one and ten are fundamental in the three types of numerals. According to a principle which is almost universal the higher symbols precede the lower in the sense of the direction of the writing. In this decimal system there are symbols for 100 and 1,000 and 10,000, the latter two as 10, and ten times ten, respectively, times 100.

The sexagesimal place system of recording numbers appears as early as 3000 B. C., being found on tablets belonging to the period of the city kings. Up to 60 the symbols are the same as in the decimal plan but 60 is represented by precisely the same symbol as the unit; a second form of sexagesimal numerals (c. 3200 B. C.) uses a half-circle D for a unit, and a circle for ten, a unit to the left of a circle represents 60, e. g., DOODD = 82. The place system was first found by Hincks, an Irish astronomer, on a tablet which gives the total portion of the moon's surface which is illumined on each succeeding day from new moon to full moon. The moon's surface is conceived as constituted of 240 equal parts; the series of numbers (in cuneiform characters) are as follows:

5	10	20	40	1 : 20
1 : 36	1 : 52	2 : 8	2 : 24	2 : 40
2 : 56	3 : 12	3 : 28	3 : 44	4

Interpreted this is that on the first five nights the portion illuminated increases in geometrical ratio from 5 parts to 80 parts, written in sexagesimal notation as 1 : 20; during the following ten nights the portion illumined increases in arithmetical progression with a common difference of 16 parts out of 240. Convincing as this series of numbers with its interpretation is, definite acceptance of the remarkable discovery of Hincks was not accorded until other documents confirmed his ingenious solution. Such confirmation was soon brought by two tablets found in 1854 at Senkereh on the Euphrates by the English geologist W. K. Loftus. These tablets give in the same system of notation the squares of all integers from 1 to 59 and the cubes as far as 32, a portion being missing.

Later researches, notably those conducted under the leadership of Professor H. K. Hilprecht, formerly of the University of Pennsylvania, have brought to light Babylonian multiplication tables and tables of measures. The astronomical researches of Strassmaier, Epping, and Kugler have revealed the extension of the system in the fourth or fifth century B. C. to include a zero symbol, somewhat similar in use to the use of the letter *o* by Ptolemy to indicate a vacant place when writing series of sexagesimal fractions.

Originally the moon tablet of Hincks and the tablets of Senkereh were of prime importance as establishing definitely the use of the sexagesimal system of recording numbers. With this system established by hundreds of other tablets

these early documents retain their importance as establishing a definite interest on the part of the Babylonians in arithmetical and geometrical series as early as 700 B. C. and in square and cubic numbers. It is related by Iamblichus and Porphyry that Pythagoras took the harmonical progression from the Babylonians, but these tablets are the only evidence we have of Babylonian interest in series. Proclus, whose historical accuracy is usually not disputed, mentions that the Babylonians were the first to note that six equilateral triangles completely fill the space about a point, but again Babylonian documents to confirm the point are not available.

The contemplation of arithmetical and geometrical series is the most natural and inevitable development of the exercise of the reasoning faculty. Consideration of the number sequence 1, 2, 3, 4, 5, 6, $\cdots$ naturally leads to the sequence 1, 3, 5, 7, 9, $\cdots$ and such sequences with a known common difference lead more or less naturally, among rational beings, to series in which a common difference is not known but which can be determined by some other condition which is imposed. Among the Greeks and Babylonians the arithmetical series led to the discussion of square and cubic numbers, and to the general subject of number theory, while with the Egyptians the fruit of this contemplation was problems and developments of the kind which we have set forth.

This brief survey of algebraical developments among the Egyptians and Babylonians shows that much of the material which was developed and extended by Greek mathematicians originated, both in methods and in substance, with the scientists of the Orient. The writers of Greece did not hesitate to acknowledge the indebtedness of Greek mathematics to the mathematics of Egypt and Babylon, but nevertheless in recent years real scientific achievements have been denied as emanating from these civilizations. To measure the magnitude of the indebtedness is beyond our power, but to recognize the debt of modern mathematical science to the scientists of Egypt and Babylon is only to render that which is due.

SOME CALCULUS SUGGESTIONS BY A STUDENT.[1]

By Benjamin Graham, New York City.

Instructors in the calculus are apt to find their own thorough knowledge of their subject somewhat of an obstacle to the complete understanding of their pupils' difficulties. Where long experience has made everything equally clear, it is not easy to feel subjectively the varying degrees of obscurity that enshroud the course as it appears to the eyes of the beginner. Some interest may attach therefore to the following account by a student of his introduction to the calculus,

[1] This paper was prepared several months ago and submitted to the Monthly by the author without the knowledge of his instructor. The editors were at once interested and would have been glad to present the paper to the readers of the Monthly exactly as it came to them, but, owing to the inexperience of the author in writing for publication, it was found necessary to

including a detailed discussion of what appeared to him at the time to be a serious weakness in the development of the subject, and suggesting a possible remedy for this defect.

Ours was a typical class in a typical large university, with an excellent instructor and Osgood's *Calculus* for a textbook. We sailed quite jauntily through the first semester, finding the theory of differentiation essentially simple and the reverse process of integrating the derivative a natural and even obvious sequel. What difficulty the poorer students experienced came chiefly from an insufficient knowledge of analytical geometry.

It was with the *definite* integral that our troubles started, and it was due to this ingenious contrivance that they never afterwards ended. Instructors are by long familiarity so accustomed to regard the definite integral merely as a new development or division of the calculus that I wonder whether they realize what a logical catastrophe it precipitates in the mind of the inexperienced student.

He has learned that differentiation seeks to discover the rate of change of a given function. Conversely, and in logical sequence, the integral is a function with a given rate of change. Suddenly he is confronted with an entirely new conception. The integral is transformed into the limit of a sum—a notion which, as far as he can see, has absolutely no connection with any previous division of the subject. Connection of some sort is indeed established arbitrarily by the so-called fundamental theorem, which declares the new integral equal to the old. Although it had appeared in general that ours as an academic course aimed rather at rigor of theory than extensive practical application, we were shocked to observe that our textbook waived the proof of this vital proposition. To paraphrase Omar:

> "I must abjure the Rate of Change, I must,
> Lured by Summation Reckoning, *ta'en on trust*."

Our professor had some inkling of the disastrous consequences this complete change of viewpoint entailed, for he remarked apologetically that we were forced to "swap horses crossing a stream." But I doubt whether he realized fully what hard going we found it on our new mount. In the first place, not only was the concept of a summation entirely foreign to us logically, but its very mechanical expression was cumbersome, unfamiliar, and repellent. Our experience with series had been confined to powers of x, and as we delved more deeply into

revise it in order to put it into suitable form. After submitting it to two or three well-known professors of mathematics, all of whom recognized that this young man had hit upon a weak spot in the teaching of the calculus, but none of whom seemed willing to undertake a revision of the paper, the editors, by mere chance, referred it to the very instructor in question. He has taken it up with the author in personal conferences and by his assistance the author has been enabled to rewrite the paper in its present form.

Strangely enough, two or three other papers have, during this time, been presented to the MONTHLY proposing substitutes for, or modifications of, Duhamel's theorem. From these the editors have selected the one by Professor Huntington, printed in this issue, as representing the best treatment. Doubtless many will find "food for thought" in these papers which may lead to contributions in the Department of Discussions. EDITORS.

All functions discussed are assumed to be continuous over the interval considered and therefore have maximum and minimum values in each of such intervals.

We will assume the following definitions in which $y = f(x)$:

$$\text{(I)}\ dy = \left(\operatorname*{limit}_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x}\right) \cdot dx, \qquad \text{(II)}\ y = \int \left(\operatorname*{limit}_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x}\right) \cdot dx.$$

THEOREM 1. *If* $x_k < x_k' < x_k + \Delta x$, *then* $\operatorname*{limit}_{\Delta x \doteq 0} f(x_k') = f(x_k)$.

Proof. Let $x_k' = x_k + \Delta' x$. Then $x_k < x_k + \Delta' x < x_k + \Delta x$ by hypothesis and $0 < \Delta' x < \Delta x$. When $\Delta x \doteq 0$, $|\Delta x|$ can be made less than any assignable quantity, such as ϵ. But when $|\Delta x| < \epsilon$, then $|\Delta' x| < \epsilon$ since $\Delta' x$ is positive and $< \Delta x$. Therefore, by definition of limit, $\operatorname*{limit}_{\Delta x \doteq 0} \Delta' x = 0$, $\operatorname*{limit}_{\Delta x \doteq 0} x_k' = x_k$, $\operatorname*{limit}_{\Delta x \doteq 0} f(x_k') = f(x_k)$ by definition of continuous function.

THEOREM 2. *If* $\Delta Q = f(x_k') \cdot \phi(x_k'') \cdots \Delta x$, *where* $x_k < x_k', x_k'' \cdots < x_k + \Delta x$, *then* $Q = \int f(x) \cdot \phi(x) \cdots dx$.

Proof. $Q = \int \operatorname*{limit}_{\Delta x \doteq 0} \frac{\Delta Q}{\Delta x} \cdot dx$ Definition (II)

$$= \int \operatorname*{limit}_{\Delta x \doteq 0} \frac{f(x_k') \cdot \phi(x_k'') \cdots \Delta x}{\Delta x} \cdot dx = \int \operatorname*{limit}_{\Delta x \doteq 0} f(x_k') \cdot \phi(x_k'') \cdots dx$$

$$= \int f(x_k) \cdot \phi(x_k) \cdots dx \qquad \text{by Theorem 1.}$$

Dropping subscripts,

$$Q = \int f(x)\phi(x) \cdots dx.$$

THEOREM 3. LEMMA.—A function, $f(x)$, continuous in the interval (a, b) takes at least once every value comprised between $f(a)$ and $f(b)$ for a value of x comprised between a and b.

This Theorem is proved in Goursat's *Cours d'Analyse Mathématique*, pp. 162–3, and we shall therefore omit the demonstration here.

A special case of Theorem 2 will be found useful in practice.

THEOREM 2a. *If* $\Delta Q = f(x_k') \cdot \phi(x_k'') \cdots \Delta x$ *and* $f(x_k''') \leqq f(x_k') \leqq f(x_k^{IV})$, $\phi(x_k^{V}) \leqq \phi(x_k'') \leqq \phi(x_k^{VI}) \cdots$, *where* $f(x_k''')$ *and* $\phi(x_k^{V})$ *are minimum values, and* $f(x_k^{IV})$ *and* $\phi(x_k^{VI})$ *are maximum values of these functions over the continuous interval* $x_k < x < x_k + \Delta x$, *then*

$$Q = \int f(x) \cdot \phi(x) \cdots dx.$$

Proof. $f(x_k''') \leqq f(x_k') \leqq f(x_k^{IV})$. Then for some value of x_k' in our interval,

$$x_k''' \leqq x_k' \leqq x_k^{IV} \qquad \text{by Theorem 3.}$$

But by hypothesis,

$$x_k \leqq x_k''', \qquad x_k \leqq x_k^{IV} + \Delta x.$$

$$\therefore \quad x_k \leqq x_k' \leqq x_k + \Delta x.$$

Similarly

$$x_k \leqq x_k'' \leqq x_k + \Delta x.$$

Then by Theorem 2,

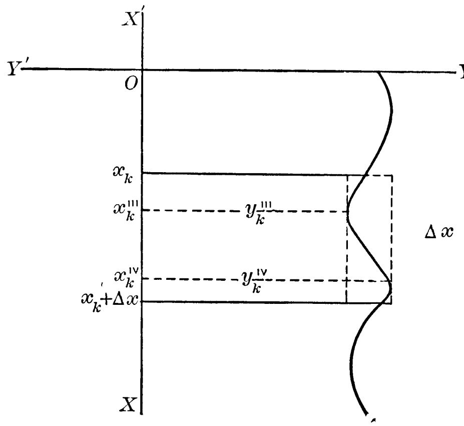
$$Q = \int f(x)\phi(x) \cdots dx.$$

By means of Theorems 2, or 2*a*, and 3 the problems usually treated by the summation process and Duhamel's Theorem may be readily solved in rigorous fashion by the indefinite integral, *i. e.*, by finding the derivative of the required function and integrating it. The application of our method to a problem involving simple integrals will serve to make it clearer.

We have selected the water-pressure example, since the solution here given can readily be compared with the current method as found, say, in Osgood's *Calculus*, and with the simplified solution worked out by Professor Bliss in the above-mentioned article.

Problem 1. To find the pressure of a liquid on a vertical wall.

Solution. Let the wall be bounded as in the figure, the Y axis lying in the surface of the liquid. At the point x_k, let the abscissa suffer an increment Δx. The area A_k and the pressure P_k will take on corresponding increments ΔA and ΔP.



By physics,

$$\Delta P = wx_k'\Delta A, \tag{1}$$

where w is the weight per unit of liquid and where

$$x_k < x_k' < x_k + \Delta x.$$

From the figure $y_k'''\Delta x < \Delta A < y^{IV}\Delta x$, where y_k''' and y_k^{IV} are respectively maximum and minimum ordinates of ΔA.

Let

$$\Delta A = y_k''\Delta x.$$

Then

$$y_k''' < y_k'' < y_k^{IV}, \tag{2}$$

$$\Delta P = wx_k'y_k''\Delta x, \tag{3}$$

whence applying Theorem 2*a* to (1), (2) and (3),

$$P = \int wxydx.$$

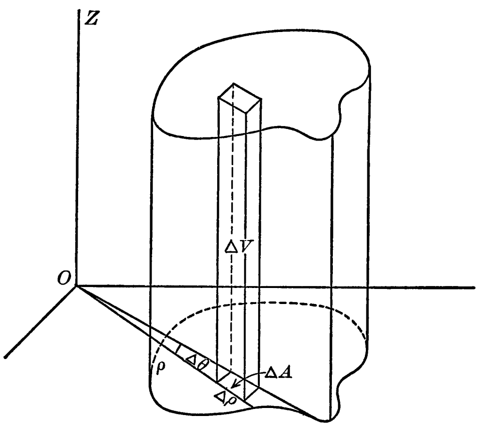
By a simple process of generalization our theorems can be extended to apply to integrals of higher order. To save space we will dispense with the demonstrations and confine ourselves to an example of their application to a problem involving a double integral.

Problem 2. To find the volume under a surface in cylindrical coördinates.

Solution. Let the equation of the surface be

$$z = f(\rho, \theta).$$

To the angle $\theta = \theta_k$ add an increment $\Delta\theta$. To the radius vector $\rho = \rho_k$ add an increment $\Delta\rho$. On the small area ΔA thus formed construct the increment of volume ΔV, in the form of a right cylinder but terminated by the given surface.



Let

$$\Delta V = z_k'\Delta A.$$

By geometry,

$$\Delta A = \frac{\Delta\theta}{2}(\rho + \Delta\rho)^2 - \frac{\Delta\theta}{2}\rho^2,$$

whence

$$\Delta A = \left(\rho + \frac{\Delta\rho}{2}\right)\Delta\theta\Delta\rho,$$

$$\rho < \rho + \frac{\Delta\rho}{2} < \rho + \Delta\rho. \tag{1}$$

From the figure, $z_k''\Delta A < \Delta V < z_k'''\Delta A$, where z_k''' and z_k'' are respectively maximum and minimum values of z in ΔV.

Then

$$z_k'' < z_k' < z_k''', \tag{2}$$

$$\Delta V = z_k'\left(\rho + \frac{\Delta\rho}{2}\right)\Delta\theta\Delta\rho, \tag{3}$$

whence, applying the generalized form of Theorem $2a$ to (1), (2) and (3),

$$V = \int d\rho \int z\rho d\theta.$$

The writer believes that the foregoing method could be used throughout the calculus, thus conserving the conception of the rate of change of a function as the basis of the entire subject. Most instructors would probably object to the elimination of the summation process on the ground that the latter is the most natural and vivid method of attacking practical problems. But is it not possible that the summation idea appears most natural to them only because they have been so long accustomed to it? To the beginner there is not very much logical clearness about the sudden jump from the inverted derivative notion to that of the integral as the limit of a sum. Certainly the former method presents no logical difficulties, and its retention throughout the calculus would at least add unity and consistency to the course.

The engineer is supposed to solve all his problems by the summation process. But after all, his method consists merely of setting an integral sign before the approximate value of a single element. Call this element an increment, and his integral reduces to the simple anti-derivative. For this intuitive process, the foregoing theorems would seem to provide a rigorous mathematical foundation.

ON SETTING UP A DEFINITE INTEGRAL WITHOUT THE USE OF DUHAMELS THEOREM.[1]

By EDWARD V. HUNTINGTON, Harvard University.

The purpose of this note is to state a simple theorem by means of which the ordinary process of "setting up an integral" may be simplified and made entirely rigorous without the use of Duhamel's Theorem or any of its modern substitutes. In view of the fundamental importance of the process of setting up an integral, it is hoped that such a simplification may be of value in both pure and applied mathematics.

To fix our ideas, let us take the familiar problem of finding the total attrac-

[1] This note contains the substance of two papers presented to the American Mathematical Society, April 29 and September 5, 1916, under the titles: (1) A simple example of the failure of Duhamel's Theorem, and (2) A simple substitute for Duhamel's Theorem.

tion, P, due to a thin rod, of length $b - a$, at a point O in line with the rod and at distance a from the nearer end. Suppose the linear density, ρ, of the rod to be variable, say $\rho = f(x)$, where $f(x)$ may be any function of x which is known for all values of x from $x = a$ to $x = b$. Also, let the law of attraction be taken in the general form, that is, suppose the attraction due to a particle to be proportional to $F(x)$ times the mass of the particle, where $F(x)$ may be any function of the distance x which is known from $x = a$ to $x = b$. [If the ordinary Newtonian law of the inverse square is adopted, then of course $F(x) = 1/x^2$.]

Now what is it that we actually do when we solve such a problem in practice? The actual steps are somewhat as follows:

First, we think of the rod as divided into small "elements," dx, where $dx = (b - a)/n$, and proceed to write down the attraction due to a typical element, say from $x = x$ to $x = x + dx$. Thus, the mass of the element is seen to be

$$f(x)dx$$

(at least approximately);[1] hence, the attraction at the point O due to the element is

$$kF(x)f(x)dx$$

(at least approximately),[2] where k is a factor of proportionality.

Next, having thus found the attraction due to a typical single element (at least approximately), we get the total attraction, P, due to all the elements, by simply writing an integral sign (with suitable limits) in front; thus:

$$P = \int_a^b kF(x)f(x)dx;$$

and in spite of the approximations used in setting up the integral, we feel assured that this final expression for P is *exact.*

Finally, we compute the value of this expression (whenever the law of attraction and the law of density are known), by the aid of a table of integrals or otherwise.

This is the simple, uncritical process of integration regarded as a method of summation. It is the process which is used probably more often than any other in the applications of the calculus.

The question that now presents itself is this: Under what conditions can this crude process be counted on to yield the correct result? Or, to put it in another way: Under what circumstances, if any, may the method be doubtful or even dangerous?

A sufficiently general answer to this question is exceedingly simple. Referring to our illustrative problem, *the naïve procedure just described will certainly be legitimate in the following case at least, namely, whenever the functions $f(x)$ and $F(x)$*

[1] This would be exact if the density throughout the element were the same as at its nearer end.

[2] This would be exact if all the attracting material in the element were concentrated at its nearer end.

are continuous. It is not necessary to consider any questions of "infinitesimals of higher order," or any questions of "uniformity"; the simple continuity of the two functions is sufficient.[1]

Stated in more general terms, we have, then, the following theorem, which appears to cover all or nearly all the cases that arise in practice:

THEOREM. *Suppose that a required quantity P is associated with a real interval, $x = a$ to $x = b$, in such a way that we are led to divide the interval into n small parts or "elements," Δx, and to regard P as the sum of n separate contributions, one from each element. Suppose also that a set of one or more functions, $F(x)$, $f(x)$, $\cdots$, can be found, such that, no matter what value of x is considered, and no matter how small Δx may be, the contribution from a typical element, $x = x$ to $x = x + \Delta x$, can be expressed "approximately" (see note 1) in the form*

$$[F(x)\, f(x) \cdots]\, \Delta x.$$

Then the required quantity P will be correctly given by the value of the definite integral

$$P = \int_a^b [F(x) f(x) \cdots]\, dx,$$

whenever the functions $F(x), f(x), \cdots$ are continuous from $x = a$ to $x = b$.

Note 1. The word "approximately" is here used in a technical sense, meaning that the exact value of the contribution in question lies between $[\overline{F} \cdot \bar{f} \cdots]\Delta x$ and $[\overline{\overline{F}} \cdot \bar{\bar{f}} \cdots]\Delta x$, where $\overline{F}, \bar{f}, \cdots$ are the smallest, and $\overline{\overline{F}}, \bar{\bar{f}}, \cdots$ the largest values of $F(x), f(x), \cdots$ in the element.

[1] The proof, which follows entirely familiar lines, is easily given, as follows. Let p be the attraction due to the part of the rod from $x = a$ to $x = x$, and let Δp be the added attraction due to the little additional element from $x = x$ to $x = x + \Delta x$. Then Δp will certainly lie between two extremes, namely:

$$k\overline{F}\bar{f}\Delta x \leqq \Delta p \leqq k\overline{\overline{F}}\bar{\bar{f}}\Delta x,$$

where $\overline{F}, \bar{f}$ are the smallest, and $\overline{\overline{F}}, \bar{\bar{f}}$ the largest values which $F(x)$ and $f(x)$ take on in the interval in question. But since $F(x)$ and $f(x)$ are continuous, each will take on at least once every value between its smallest and largest values in the interval; so that there will certainly exist values of x, say x' and x'', in the interval from $x = x$ to $x = x + \Delta x$, for which

$$\Delta p = kF(x')f(x'')\Delta x,$$

exactly. Now, keeping x fixed, divide through by Δx, and let Δx approach zero; then both x' and x'' will be squeezed down toward x as a limit, while $\Delta p/\Delta x$ approaches dp/dx, so that we have, in the limit,

$$dp/dx = kF(x)f(x), \qquad \text{whence,} \qquad p = \int kF(x)f(x)dx + C.$$

To determine the constant of integration, we have only to notice that $p = 0$ when $x = a$, so that

$$p = \int kF(x)f(x)dx - \left[\int kF(x)f(x)dx\right]_{x=a}$$

(the same value of the indefinite integral being used in each term). Finally, since P is the value of p when $x = b$, we have

$$P = \left[\int kF(x)f(x)dx\right]_{x=b} - \left[kF(x)f(x)dx\right]_{x=a} = \int_a^b kF(x)f(x)dx.$$

The proof is thus complete.

Note 2. In stating that P is to be regarded as the sum of n separate contributions, one from each element, we are assuming that P is the value, for $x = b$, of some function, $p(x)$, which has a definite value for each value of x from $x = a$ to $x = b$. It is not necessary to assume in advance that $p(x)$ is continuous, although it will, in fact, always be so whenever the other conditions of the theorem are satisfied.

Note 3. By an obvious modification, the theorem can be made to cover the case where the product $[F(x) f(x) \cdots]$ is replaced by any continuous combination of continuous functions.

With the statement of this theorem the main purpose of the present note is completed.

There are, however, a number of recent textbooks, notably Professor Osgood's *Calculus*, in which the process of setting up an integral as the limit of a sum is held to require, for complete rigor, the use of an auxiliary theorem known as Duhamel's Theorem.

In view of the simplicity and scope of the theorem proved above, the use of any auxiliary theorem such as Duhamel's would appear to be a superfluous complication. Moreover, Duhamel's Theorem in its ordinary form is known to be false.[1] Since the examples which have been adduced to prove this latter statement are rather complicated, the following simpler example may be of interest.

Duhamel's theorem in the primitive form in which it still appears in the textbooks, is as follows:

If $\alpha_1, \alpha_2, \cdots \alpha_n$ are a set of positive infinitesimals such that

$$\lim_{n=\infty} [\alpha_1 + \alpha_2 + \cdots + \alpha_n] = A;$$

and if $\beta_1, \beta_2, \cdots \beta_n$ are a second set of positive infinitesimals such that each β differs from the corresponding α by an infinitesimal of higher order, so that $\lim_{n=\infty} [\beta_i/\alpha_i] = 1$; then

$$\lim_{n=\infty} [\beta_1 + \beta_2 + \cdots + \beta_n] = A.$$

As our example, let $\alpha_i = 3/n$; and $\beta_i = (3n + 2i)/n^2$, where $i = 1, 2, \cdots n$. Then $\lim_{n=\infty} [\alpha_1 + \alpha_2 + \cdots + \alpha_n] = 3$; moreover, for any particular i, $\lim_{n=\infty} [\beta_i/\alpha_i] = \lim_{n=\infty} [1 + (2i)/(3n)] = 1$. The conditions of the theorem are therefore fulfilled, and according to the conclusion of the theorem, therefore, we should have $\lim_{n=\infty} [\beta_1 + \beta_2 + \cdots + \beta_n] = 3$. In fact, however, we have $\beta_1 + \beta_2 + \cdots + \beta_n = 3 + 2[1 + 2 + \cdots + n]/n^2 = 3 + (n + 1)/n$, so that $\lim_{n=\infty} [\beta_1 + \beta_2 + \cdots + \beta_n] = 4$.

[1] For a critical discussion of this theorem, see W. F. Osgood, "The integral as the limit of a sum and a theorem of Duhamel's," *Annals of Mathematics*, Ser. 2, Vol. 4 (1903), pp. 161–178; R. L. Moore, "On Duhamel's theorem," *ibid.*, Vol. 13 (1912), pp. 161–166; G. A. Bliss, "A substitute for Duhamel's Theorem," *ibid.*, Vol. 16 (1914), pp. 45–49. Further references may be found in the first of these papers. The present note is closely related to the third.

In view of this or similar examples,[1] it is clear that if Duhamel's Theorem is to be used at all, as a means of securing rigor, it must be taken in a modified form; interesting revised forms have in fact been proposed by Professors Osgood, R. L. Moore, and Bliss (*loc. cit.*); but, while these new forms leave nothing to be desired in point of rigor, none of them, as far as I know, has proved to be sufficiently simple to warrant its adoption in an elementary textbook.[2] The simplest plan to pursue in a first course in the calculus would therefore appear to be to omit Duhamel's Theorem altogether, substituting for it some such theorem as that suggested in the present paper.

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

Ruler and Compasses. By HILDA P. HUDSON. Longmans, Green and Company, London and New York, 1916. 143 pages.

This new volume of Longmans' Modern Mathematical Series is an attempt to collect from many sources solutions of problems and discussions of methods in which the Euclidean ruler and compasses are used as instruments; and to present them as part of a well-ordered development of the theory of such constructions. According to the author "the connecting link throughout the book is the idea of the whole set of ruler and compass constructions, its extent, its limitations, and its division."

The reader will require no more advanced mathematics than college algebra and elementary analytic geometry, although a knowledge of projective geometry and of the theory of equations in general will be helpful. The development of the theme is carefully carried out and there are no breaks in the logic, although in one or two places the author quotes a theorem which is developed later. This may prove a bit annoying to the reader with a minimum of preparation, but otherwise it is not a serious fault.

The subject matter of the text is presented as a whole in the introduction, which is rather well written, although it presupposes at times a rather full acquaintance with the material which is developed later. In Chapter II the criteria of possibility for ruler and compass constructions are established from the analytical point of view with the aid of a number of propositions from the elementary theory of equations. The chapter is divided into three parts; first, the constructions in which the ruler alone suffices, second, those in which the ruler and Euclidean compasses are required, and third, the construction of regular polygons of n sides. The cases in which n is a prime and n is composite

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[1] During the discussion of this paper at the meeting of the Society, Professor D. Jackson suggested the following even simpler example: $\alpha_i = 1/n$ when $i \leqq n/2$, $\alpha_i = 0$ when $i > n/2$; $\beta_i = 1/n$.

[2] Even in Professor Osgood's own text (1907, revised edition 1909), the original (incorrect) form of Duhamel's theorem is retained, without comment. Professor Osgood's reasons for so doing may be found in his article of 1903 (*loc. cit.*).

are discussed in the classical way and Richmond's construction for the polygon of 17 sides is given as an example. In the third chapter the question of constructions to be carried out by ruler alone is developed in detail with the separation of projective and metrical properties. Mathew's *Projective Geometry* is drawn upon extensively for the discussion of homography and its special case of involution to establish the constructions. The relation of the infinity locus to metrical constructions is unusually well presented.

In Chapter IV the fundamental theorem concerning the possibility of solution of a quadratic equation with the aid of ruler and compasses is established, two solutions being given, and, from the discussion of a pencil of conics projected from a circle through four points, a construction for the common points of projective point rows superposed on a line is obtained. A second division of this chapter treats of modern instruments, including dividers, Hilbert's Einheitsdreher, the parallel ruler, and the set-square. In Chapter V standard methods of attack for various problems are discussed. No pretense is made at developing a general method for all problems, but the data in any given case are used to point toward one of seven listed methods of solution. This is in some respects the best organized chapter of the book. In Chapter VI the methods just indicated are compared, especially as regards constructions to be accomplished in a limited space or with a minimum number of operations. The two concluding chapters are concerned with effecting all ruler and compass constructions with one fixed circle and ruler only, or with compasses only. They may be regarded as addenda to the preceding discussion, inasmuch as they are not necessary to its development. In the chapter on compasses only, the results of Mascheroni and Adler have been compared and several solutions by the methods of each given as examples.

On the whole, the teacher of analytic or projective geometry or of college algebra will find this a valuable source-book for many illustrative problems. Its development of fundamental theorems of projective geometry for the general conic starting from properties of the circle is carried out in an interesting way. The sources of many theorems are carefully indicated and a fairly complete bibliography is given facing page 1. In the way of criticism, the reviewer believes that some of the section headings are misplaced and that many of them are not illuminating. The chapter on methods of solution is an exception to this criticism.

B. M. WOODS.

THE UNIVERSITY OF CALIFORNIA.

A Treatise on the Circle and the Sphere. By JULIAN LOWELL COOLIDGE, Assistant Professor of Mathematics in Harvard University. The Clarendon Press, Oxford, 1916. 604 pages. $6.75.

Professor Coolidge has written a treatise on the circle and the sphere which may be considered as an encyclopedia of valuable information on this important subject. The successful compilation of known material of such an extent, and the incorporation of the results of his own notable investigations in this field,

in a uniform mold, could only be accomplished by a great amount of work and a full mastery of the subject by the author. The contents are too voluminous to be discussed in detail in a review of moderate size and scope. I shall therefore restrict myself, in the main, to some of the important features of the book and of the subject.

The first chapter deals with the elementary geometry of the circle, in which inversion, mutually tangent circles, circles related to a triangle, the Brocard figures, concurrent and coaxial circles, are considered. Out of the almost innumerable mass of propositions of the so-called modern geometry of the triangle and the circle, a great number of theorems have been selected and proved by the usual well-known means.

This subject is continued in a second chapter by making use of cartesian and trilinear coördinates and applying them to the nine-point circle, the Tucker circles, the Brocard circles, etc.

Next, an important step forward is made by the introduction of tetracyclic coördinates. Choosing any four mutually orthogonal circles not null as fixed, and any point P in the plane of these circles, the tetracyclic coördinates x_1, x_2, x_3, x_4 of P are defined as quantities proportional to the ratios of the powers of P to the radii of the corresponding circles. The coördinates of any point P satisfy the identity

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0,$$

are homogeneous, and do not all vanish simultaneously. The symbol (yx) stands for

$$(yx) = y_1x_1 + y_2x_2 + y_3x_3 + y_4x_4,$$

and $(yx) = 0$ is the equation of a circle. The coefficients (y) are called the coördinates of the circle. If they satisfy the foregoing identity, the point (y) is called the vertex of the circle, which is then said to be null. The angle θ between two proper circles with the coördinates (y) and (z) is defined by

$$\cos\theta = \frac{(yz)}{\sqrt{yy}\,\sqrt{zz}}.$$

The conditions for orthogonality or tangency of the two circles are clearly $(yz) = 0$, and $(yy)(zz) - (yz)^2 = 0$. A homogeneous algebraic equation, $f(x_1, x_2, x_3, x_4) = 0$, in conjunction with the given identity, defines a cyclic curve. Thus, when f is a polynomial of the second degree, it represents, in general, a bicircular quartic. It goes without saying that the important relation between the tetracyclic plane and three-space is fully explained. For example, it is shown how a one-to-one correspondence may be established between the circles of the cartesian plane and the points of cartesian space.

"The coördinates of a circle in the cartesian plane may be interpreted as the coördinates of a point in space whose polar plane with regard to a fundamental sphere cuts that sphere in a circle whose stereographic projection is the given circle."

Again: "The homogeneous coördinates (x) may be taken to represent a point in a three-dimensional space, which we shall assume has an elliptic type of measurement, the equation of the absolute quadric being

$$(xx) = 0."$$

In Chapter VI, quite in analogy with the corresponding development in the tetracyclic plane, pentaspherical coördinates and space are introduced and applied to the investigation of systems of spheres.[1] In both, the tetracyclic plane and the pentaspherical space, much use is made of the identity of Darboux and Frobenius between the coördinates of any ten circles, or any twelve spheres, which plays a fundamental rôle in the theory.

At this point I should like to say that for one familiar with the essentials of tetracyclic and pentaspheric coördinates there is, of course, no difficulty in following the author's exposition of these systems. The beginner, however, we fear, will not have very smooth sailing, if he has not the help of a teacher, or some other source for consultation. The style of Darboux's own presentation in Chapter VI, Vol. 1, of his *Leçons sur la Théorie Générale des Surfaces,* has not been surpassed, and also Bôcher's more explicit account in his *Reihenentwicklungen der Potentialtheorie* might have been successfully used as a model of clear exposition. Reference should also be made to the fact that it was Darboux who first introduced pentaspherical coördinates[2] for the investigation of certain curves and surfaces.

In the following chapters, cyclides, circle-transformations, continuous groups of transformations, sphere transformations, Laguerre transformations, oriented circles and spheres, circles orthogonal to one sphere, circle-crosses, algebraic systems, and differential geometry of circle systems are discussed. A number of topics in this series contain some of the author's own contributions to the geometry of circles and spheres. Oriented circles and spheres, based upon positive and negative radii, and the direction of the normals make it possible to define uniquely the angle included by two circles, or two spheres, or a circle and a sphere. From a purely constructive standpoint Chapter III, dealing with famous problems, is of particular interest, and it was a happy thought of the author to pay special attention to these problems. The main portions of this chapter are presented in an admirable manner. It opens with Lemoine's geometrographic criteria to which possibly too much prominence is given. They are hardly of any practical value, insofar as they do not indicate how to simplify a construction or how to make it more accurate.[3]

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[1] The foundation for a general theory of spheres in n-space was laid by G. Schlumberger in his dissertation: *Über n-dimensionale lineare und quadratische Kugelsysteme* (78 pages), Zürich, 1896.

[2] *Sur une class remarquable de courbes et de surfaces,* p. 135. Paris, 1873. See also reference given above.

[3] See in this connection Konrad Nitz: "Beiträge zu einer Fehlertheorie der geometrischen Konstruktionen," *Zeitschrift für Mathematik und Physik,* Vol. 53, pp. 1–37 (1906); also Em. Haentschel's account in *L'Enseignement Mathématique,* Vol. 9, pp. 45–51 (1907).

The author makes some remarks on the proofs of Steiner's solution of Malfattis' problem: "to construct three circles, each of which shall touch the other two, and two sides of a given triangle," which are misleading. Coolidge says with reference to Hart's proof: "There is a suspicion which naturally arises that, if the first discoverer of a proof had been of Steiner's own nationality, less trouble would have been given to disparaging his work." I doubt that there was any national feeling which induced criticism of Hart's proof. In the first place, Steiner was a Swiss. In the second place, Steiner, the discoverer of inversion, undoubtedly used this principle in the solution of this and similar problems. Criticism was due solely to the demand that, in the proofs, the principles discovered and used by Steiner, which form the natural foundation for a discussion of these problems, should be followed. In fact, Plücker, a German, who was not able to understand and appreciate Steiner's powerful methods, made the insulting remark, in print, that Steiner presumably had no proof at all for his construction. As indicated above, Schröter's remarks on Hart's proof were, therefore, certainly dictated by purely scientific motives only.

Malfattis's problem has an equally famous counterpart in Apollonius's problem: "To construct a circle tangent to three given circles." Also the solution of this problem caused many controversies. Gergonne, whose solution is given, claimed that analytic methods were superior to synthetic methods, whereupon Poncelet[1] published a purely geometric solution. No reference is given to Poncelet in this connection. Fiedler's cyclographic method of solving certain circle-problems is fully discussed. Here we see again, for example in the solution of Apollonius's problem, that for certain classes of problems there are certain "natural" methods leading to their solutions.

The general make-up and the typography of the book are excellent. In a number of places the author makes rather abrupt statements; for example, on page 130: "We distinguish the following types of circles:

$$\sum_{i=0}^{i=3} y_i z_i = (yz).$$

(*a*) Proper circles $(xx) \neq 0$, $ix_0 + x_1 \neq 0$, etc."

Considered all in all, Coolidge's treatise is a great and most up to date work of reference and important information on the theory of the circle and the sphere. It is indispensable for anyone who intends to take up this field as a student, or as an investigator.

ARNOLD EMCH.

UNIVERSITY OF ILLINOIS.

[1] *Traite des propriétés projéctives des figures*, Vol. 1, pp. 136–148, Paris, 1865. Also *Applications d'analyse et de géométrie*, Vol. 1, pp. 30–41 (1862).

FRÈRE GABRIEL MARIE.

For forty years elementary mathematical texts issued under the auspices of the Institut des Frères des Écoles Chrétiennes have been widely used in France and Belgium. While no author's name ever appeared on the title pages there was usually some such indication as "par F. J.," "par F. I. C.," and "par F. G. M." In recent years it has become generally known that "F. G. M."[1] stood for Frère Gabriel Marie who was appointed Superior General of his Order in 1897.[2] As a result of legislation in 1904 more than half of the 2,000 schools of the Order in France were closed during 1904–1906. Frère Gabriel Marie then continued his work in Lembecq-lez-Hal, Belgium. With the opening of the war he returned once more to Paris. There he died on October 25, 1916, in the eighty-second year of his age.[3] Of the many mathematical works which have come from his pen there are two which, even though of extraordinary interest, are almost unknown in America. I refer to: (1) *Exercices de géométrie,* of which the fifth edition was published in 1912 (4 supplementary pages in 1913), and (2) *Exercices de géométrie descriptive* (fourth edition, 1909, 61 supplementary pages, 1912). Both of these works are well worth placing in every college library of the country, and few high-school teachers of mathematics could fail to find the first of them a veritable encyclopædia of interesting methods, solutions, and historical notes.

Exercices de géométrie is a book of over 1,300 pages. In the first section of 210 pages, methods for solving geometrical problems are discussed. On pages 211–1259 we find detailed solutions of all the exercises in *Éléments de Géométrie,* par F. J. Throughout the work there is an immense amount of bibliographical and historical information. On account of the difficulties under which the last edition was published it contains a good many inaccuracies and misprints. These are very few, however, in comparison with innumerable accurately stated facts and references difficult of access in any other work. Indexes covering nearly 40 pages render the volume most convenient as a work of reference. The only work in recent times to compare with it, is the one by Antoine Dalle.[4]

Exercices de géométrie descriptive has similar characteristics. The most recent edition contains nearly 1,200 pages, with the solutions of all exercises in *Éléments de géométrie descriptive avec de nombreux exercices,* of the same series.

In sending his recently published *Manuel de Géométrie* to the writer early in 1916 Frère Gabriel Marie stated that he expected to end his career with this new edition of his first work which had been published forty years before. Nevertheless within a month of his death yet another book had been added to the long list of his productions.[5]

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[1] As "F. I. C." and "F. J." occur on the title pages of the second and third editions of a work of which the fourth and fifth editions have "F. G. M." I hazard the guess that these also refer to Frère Gabriel Marie; the first indicating the Order in which he was a brother, the latter that he was a Jesuit.

[2] *Cf.* the article on "Institute of Brothers of Christian Schools" in *The Catholic Encyclopedia,* Vol. 8, New York, 1911.

[3] *Cf. L'Enseignement Mathématique,* Novembre, 1916, XVIIIe année, pp. 445–446.

[4] *2000 théorémes et problèmes de géométrie avec solutions.* Namur, 1912. 8 + 825 pp.

[5] *Manuel de mécanique.* Paris, 1916. 432 pp.

It is with more than passing pleasure that I recall the fine courtesy, the generosity, the extreme modesty and the enthusiasm exhibited by Frère Gabriel Marie in occasional correspondence during the past decade.

R. C. ARCHIBALD.

BROWN UNIVERSITY,
March 23, 1917.

FIRST REGULAR MEETING OF THE IOWA SECTION.

The first regular meeting of the Iowa Section of The Mathematical Association of America was held at Grinnell College, Grinnell, Iowa, on April 28, 1917, and the following program given:

(1) "A unified course for Freshman mathematics:" by Professor R. B. McCLENON, Grinnell College. Leader of the discussion: Professor JULIA COLPITTS, Iowa State College.

(2) "The foundation of Freshman mathematics in technical schools:" by Dean E. W. STANTON, Iowa State College. In his absence, the paper was read by Professor MARIA ROBERTS, Iowa State College. Leaders of the discussion: Professors J. F. REILLY, Iowa State University, and C. W. EMMONS, Simpson College.

(3) "Putting life into dry bones:" by Professor F. M. McGAW, Cornell College. Leaders of the discussion: Professors W. J. RUSK, Grinnell College, and W. E. BECK, Iowa State University.

The following also took part in the discussions: Professors Weston, Trowbridge, Stewart and Neff. All the papers were good and the discussions were to the point showing a keen interest in the sort of a program offered. The action at the business session in planning two meetings each year also indicates something of the interest taken in the Iowa Section. The attendance included some twenty members of the Association and others who will become members in due course.

The following officers were elected for the ensuing year: I. F. NEFF, Drake University, Chairman; R. B. McCLENON, Grinnell College, Vice-Chairman; W. E. BECK, Iowa State University, Secretary.

G. A. CHANEY, *Chairman*, I. F. NEFF, *Secretary-Treasurer*.

THE ROCKY MOUNTAIN SECTION OF THE ASSOCIATION.

In September, 1916, it was suggested to Dr. G. H. Light, of the University of Colorado, that a section of The Mathematical Association of America be formed to include the states of Wyoming and Colorado. The suggestion was acted upon and as a result a meeting was called at the University of Colorado on April 7, 1917.

The meeting was a great success and the Rocky Mountain Section of the Association was formed with the following officers: C. B. Ridgaway, Professor of Mathematics, University of Wyoming, Chairman. C. C. VanNuys, Professor of Physics, Colorado School of Mines, Vice-Chairman. G. H. Light, Assistant Professor of Mathematics, University of Colorado, Secretary-Treasurer.

Papers were presented by O. C. Lester, Professor of Physics, University of Colorado, on "The Solid Angle," and Florian Cajori, Professor of Mathematics, Colorado College, on "Fluxions." Discussion of these papers was general.

There were twenty-one present at the meeting, fifteen of whom are already members of the Association and the others will join at once: C. B. RIDGAWAY, Professor of Mathematics, C. E. STROMQUIST, Professor of Mathematics, J. C. FITTERER, Professor of Civil Engineering, University of Wyoming; C. R. BURGER, Professor of Mathematics, G. E. F. SHERWOOD, Assistant Professor of Mathematics, C. C. VANNUYS, Professor of Physics, H. M. SHOWMAN, Assistant Professor of Civil Engineering, F. W. LUCHT, Assistant Professor of Mechanical Engineering, W. J. HAZARD, Assistant Professor of Mechanical Engineering, Colorado School of Mines; S. L. MACDONALD, Professor of Mathematics, Colorado A. & M. College; G. W. FINLEY, Professor of Mathematics, Colorado State Teacher's College; FLORIAN CAJORI, Professor of Mathematics, Colorado College; W. H. HILL, Greeley High School; E. L. BROWN, East Denver High School; I. M. DELONG, Professor of Mathematics, G. H. LIGHT, Assistant Professor of Mathematics, CLARIBEL KENDALL, Instructor in Mathematics, O. C. LESTER, Professor of Physics, J. W. WOODROW, Assistant Professor of Physics, DR. O. A. RANDOLPH, Instructor in Physics, C. E. SPERRY, Assistant Professor of Mathematics, University of Colorado.

G. H. LIGHT,
Secretary-Treasurer.

THE KENTUCKY SECTION OF THE ASSOCIATION.

The Mathematics Section of the Association of Kentucky Colleges and Universities (now the Kentucky Section of the Mathematical Association of America) was organized in April, 1909, and since then has met regularly twice a year.

This organization has directed most of its attention to a consideration of problems peculiar to collegiate work, one result of which has been a tendency toward a standardization of the mathematical courses in the colleges of the state.

Another feature of the work of this organization has been the consistent efforts put forth to strengthen and improve the teaching of mathematics in the high schools of the state. It was at first planned to work out a correlated course in mathematics for the high schools but this was later abandoned. In 1910 it was decided to test the degree of preparation of all students entering the colleges of the state by setting examinations covering algebra and plane and solid geometry.

The results of these examinations were collected for a number of years. The grades of the students of each preparatory school of the state were then averaged, and the results tabulated and sent to all of the schools concerned. In this way an appeal was made to the pride of the best schools, which led them, in a spirit of friendly rivalry, to compete for the places of honor, while it acted as a spur, though not in an offensive way, to those who did not show up well in the report. It was also possible by this means to point out the parts of the different subjects in which the students were most poorly prepared, and to make helpful suggestions in a friendly and coöperative spirit. This was accomplished through the medium of the Kentucky Educational Association.

The pleasant social events connected with the meetings, the stimulating and helpful results of the pooling of ideas, and the inspiration of professional contact have led to the development of a fraternal spirit among the mathematicians of the state.

When the Mathematical Association of America was organized, the Mathematics Section of the Association of Kentucky Colleges felt that, as it had been a pioneer organization in the collegiate field, it was in a peculiarly fortunate position to be immediately incorporated as a section of the national association. Accordingly, application was made in May, 1916.[1] Admission was granted in February of this year.

The first meeting as a Section of the Mathematical Association of America (ninth annual meeting as an organization) was held at Berea College, Berea, Ky., May 11–12, 1917.

Friday afternoon preceding the meeting was given over to entertainment features, including luncheon at Boone Tavern, horseback trip into the mountains, supper at the college dining hall. In the evening there was a business meeting, and the members were guests at a college faculty meeting.

On Saturday morning there was a tour of inspection of college grounds and buildings, and attendance at the college chapel service, and luncheon at the home of Professor Phalen.

Following is the program of the Friday meetings:

Morning Session.

Remarks by President of the Section. By Professor H. R. PHALEN, Berea College.

"The status of mathematics." By Professor J. M. DAVIS, University of Kentucky.

"The case against mathematics." By Professor A. L. RHOTON, Georgetown College.

"Demonstration of an equation balance." By Professor E. L. REES, University of Kentucky.

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[1] The delay in completing the arrangements for the admission of the Kentucky Section was due to an effort on the part of the Committee of the Association to make a combination with another state, which plan finally proved not to be feasible. EDITORS.

Afternoon Session.

"Pedagogical problems of the Freshman course." By Professor P. P. Boyd, University of Kentucky.

"The best mathematical library for one hundred dollars." By Professor W. H. Garnett, Kentucky Wesleyan University.

"Helmholtz's theory of contraction." By Professor H. H. Downing, University of Kentucky.

"Three methods of presenting subject matter." By Professor Henry Lloyd, Transylvania University.

At each session the formal papers were followed by an open discussion.

Those present at the meeting were: H. R. Phalen, J. N. Peck, J. Van Hook, Miss L. J. Ritscher, Miss E. C. DeBord, and Miss L. J. Harris, Berea College; A. L. Rhoton, Georgetown College; W. H. Garnett, Kentucky Wesleyan College; Henry Lloyd, Transylvania University; J. M. Davis, E. L. Rees, P. P. Boyd, and H. H. Downing, University of Kentucky.

Not all of those in attendance are now members of the Association but all will be given an opportunity to join at once.

The following officers were elected for the ensuing year: A. L. Rhoton, Chairman; H. H. Downing, Secretary-Treasurer.

The next meeting will be held in December at Lexington, Ky.

E. L. Rees,
Secretary-Treasurer.

FOURTH MEETING OF THE KANSAS SECTION.

The fourth meeting of the Kansas Section of The Mathematical Association of America was held at the University of Kansas on Saturday, March 17. The Kansas Section is now following the plan of holding two meetings a year, one in November, at the time of the State Teachers' Association and the other in March. The meeting was given over entirely to discussing college algebra, and the time proved too short for the following program:

1. "The content of Freshman algebra:" Professor W. H. Andrews, Kansas State Agricultural College.

2. "Algebra courses for college Juniors and Seniors:"

(*a*) For students preparing to teach: Professor U. G. Mitchell, University of Kansas.

(*b*) For students preparing to do research work: Professor W. H. Garrett, Baker University.

(*c*) For students preparing to enter applied sciences: Professor A. R. Crathorne, University of Illinois.

These papers were followed by general discussion and a business session.

The first topic, "The content of Freshman algebra," is of special interest to Kansas Colleges just now because recent legislative action has brought it about

that a considerable part of next year's freshman class will enter college with only one year of high school algebra, a larger number will have had a year and a half, while a few will offer two years of preparatory algebra. The second number received independent consideration from three distinct viewpoints and much interest was manifested in noting the topics common to all and the relative emphasis placed upon them. The general discussion was enthusiastic and, in addition to covering points involved in the papers, raised the question of giving college students at some time in their course an acquaintance with building and loan investments and life insurance problems in so far as interest returns are concerned.

The following persons were in attendance, representing ten colleges and universities and five high schools: (*A*) Members—W. H. Andrews, State Agricultural College, C. H. Ashton, University of Kansas, A. R. Crathorne, University of Illinois, Lucy F. Dougherty, Kansas City High School, Elizabeth G. Flagg, Kansas City High School, W. H. Garrett, Baker University, Emma Hyde, Kansas City High School, S. Lefschetz, University of Kansas, Theodore Lindquist, State Normal School, Emporia, A. W. Larsen, University of Kansas, W. A. Luby, Polytechnic Institute, Kansas City, Mo., U. G. Mitchell, University of Kansas, Mary W. Newson, Washburn College, B. L. Remick, State Agricultural College, J. A. G. Shirk, State Normal School, Pittsburg, E. B. Stouffer, University of Kansas, E. M. Stahl, Midland College, L. L. Steimley, University of Kansas, J. N. Van der Vries, University of Kansas, J. J. Wheeler, University of Kansas, Ella Woodyard, High School, Kansas City. (*B*) Non-members—O. W. Dueker, Friends University, Eleanora Harris, Hutchinson High School, Mary Lloyd, Atchison High School, Stella M. Olcott, Topeka High School.

J. J. Wheeler,
Secretary-Treasurer.

THE SECOND REGULAR MEETING OF THE MINNESOTA SECTION.

The second regular meeting of the Minnesota Section of the Mathematical Association of America was held at the University of Minnesota, Minneapolis, Minnesota, on Monday, April 9, 1917. There were 22 persons present including the following members of the Association:

R. M. Barton, University of Minnesota, Minneapolis.
George N. Bauer, University of Minnesota, Minneapolis.
W. H. Bussey, University of Minnesota, Minneapolis.
Edla M. Berger, St. Catherine's College, St. Paul.
Father William Earnshaw Etzel, College of St. Thomas, St. Paul.
G. H. Hartwell, Hamline College, St. Paul.
J. S. Mikesh, Hibbing Junior College, Hibbing.
Clarence McCormick, University of Minnesota, Minneapolis.
Sister M. Magna, O.S.B., St. Benedict's College, St. Joseph.
R. R. Shumway, University of Minnesota, Minneapolis.

H. L. Slobin, University of Minnesota, Minneapolis.
A. L. Underhill, University of Minnesota, Minneapolis.
Vera L. Wright, University of Minnesota, Minneapolis.
And the following persons not yet members:
J. B. Frear, University of Minnesota, Minneapolis.
Orville A. George, University of Minnesota, Minneapolis.
C. H. Gingrich, Carleton College, Northfield.
Sister Mary John, St. Catherine's College, St. Paul.
Sister Patricia, St. Catherine's College, St. Paul.
Ella A. M. Thorp, University of Minnesota, Minneapolis.

The program of the morning session consisted of a business meeting and the two following formal papers:

1. *The teaching of mathematics in the French high schools, colleges and universities.* By Father William Earnshaw Etzel, College of St. Thomas, St. Paul.

Father Etzel, for some years a teacher in France, very pleasantly and clearly outlined the mathematical curriculum for the high schools, colleges and universities of France in arrangement and content, drawing frequent parallels and contrasts with the corresponding courses in this country, and he gave problems illustrative of the work in different years of the programs.

2. *The origin of mathematical induction.* By Professor W. H. Bussey, University of Minnesota.

This paper appeared in full in the May number of the MONTHLY.

The program of the afternoon session consisted of the following voluntary informal reports:

1. *Review of Cajori's "Revised and enlarged edition of the history of mathematics."* By Professor R. R. Shumway, University of Minnesota.
2. *An extension of the problem of orthogonal trajectories of* $y = e^x$. By Professor C. H. Gingrich, Carleton College.
3. *The present status of mathematics in the junior colleges of the state.* By Director J. S. Mikesh, Hibbing Junior College.
4. *Radial Curves.* By Mr. O. A. George, University of Minnesota.
5. *A note on symbolic notation for circular functions and its application in trigonometric investigations.* By Professor H. L. Slobin, University of Minnesota.
6. *A note on infinite series.* By Professor G. N. Bauer, University of Minnesota.

R. M. BARTON,
Secretary pro tem.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

ALGEBRA.

483. Proposed by C. R. DUNCAN, Amherst College.

Prove or disprove the following theorem: An infinite series, $A_1 + A_2 + A_3 + \cdots + A_n + \cdots$ is convergent or divergent according as

$$\lim_{n \doteq \infty} \frac{A_n}{1 - \dfrac{A_n}{A_{n-1}}} = 0 \qquad \text{or} \qquad \neq 0.$$

484. Proposed by E. V. HUNTINGTON, Cambridge, Massachusetts.

Show that

$$\frac{\dfrac{1}{m^2} - \dfrac{k_1}{(m+1)^2} + \dfrac{k_2}{(m+2)^2} - \cdots + \dfrac{(-1)^k}{(m+k)^2}}{\dfrac{1}{m} - \dfrac{k_1}{m+1} + \dfrac{k_2}{m+2} - \cdots + \dfrac{(-1)^k}{m+k}} = \frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{m+k}$$

for all positive integral values of m and k. Here

$$k_1 = \frac{k}{1}, \qquad k_2 = \frac{k(k-1)}{1\cdot 2}, \qquad k_3 = \frac{k(k-1)(k-2)}{1\cdot 2\cdot 3}, \quad \text{etc.}$$

This equation was suggested to the proposer by a professor of chemistry who wishes to make use of the equation, if correct, in an actual problem in bacteriology.

GEOMETRY.

516. Proposed by R. M. MATHEWS, Riverside, California.

Through the edges of a trihedral angle planes are passed orthogonal to the opposite faces. Prove them coaxial.

517. Proposed by R. P. BAKER, University of Iowa.

The coördinates of the vertices of a regular icosahedron can be expressed rationally in terms of $\dfrac{\sqrt{5}-1}{4}$ and $\sqrt{\dfrac{5+\sqrt{5}}{8}}$, that is, $\cos\dfrac{2\pi}{5}$ and $\sin\dfrac{2\pi}{5}$. Prove (1) that the cosine only is sufficient; (2) that the irrationalities cannot be reduced further. (The theorem that they cannot be rational is proved in books on crystal theory.)

CALCULUS.

431. Proposed by J. W. LASLEY, University of North Carolina.

Explain Bertrand's fallacy:

$$\int_{x=0}^{x=1}\int_{y=0}^{y=1} \frac{x^2-y^2}{(x^2+y^2)^2}\,dydx = \int_{y=0}^{y=1}\int_{x=0}^{x=1} \frac{(x^2-y^2)}{(x^2+y^2)^2}\,dxdy;$$

$$\tfrac{1}{4}\pi = -\tfrac{1}{4}\pi, \quad 1 = -1.$$

432. Proposed by R. P. BAKER, University of Iowa.

The expressions

$$x^{i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^i \left(\frac{c_1e^{ax} + c_2e^{-ax}}{x}\right)$$

and

$$x^{-(i+1)}\left(x^3\frac{d}{dx}\right)^i \left(\frac{c_1e^{ax} + c_2e^{-ax}}{x^{2i-1}}\right)$$

are formally equivalent for every integral value of i.

MECHANICS.

348. Proposed by ALTON L. MILLER, Ann Arbor, Michigan.

If equilateral triangles be constructed on the sides of any triangle, their centers are the vertices of a new equilateral triangle. Show that the center of gravity of this new equilateral triangle coincides with the center of gravity of the original triangle.

349. Proposed by S. A. COREY, Albia, Iowa.

A 9 pound weight is attached to a string which passes over a smooth fixed pulley. The other end of the string is fastened to and supports a smooth pulley P_1 of weight 1 pound, over which passes a second string to one end of which is attached a 3 pound weight, and the other end of which is attached to and supports another smooth pulley P_2 of weight 1 pound. Over the pulley P_2 passes a third string supporting weights, 2 pounds and $3\frac{1}{3}$ pounds.

If the system is acted on by gravity alone show that the accelerations of the 9 pound weight, $3\frac{1}{3}$ pound weight, and pulley P_2 are 0, $\frac{1}{2}g$, and $\frac{1}{8}g$, respectively.

Determine the motion of the weights when pulleys are not smooth, that is, when friction is present.

NUMBER THEORY.

266. Proposed by J. L. RILEY, Tahlequah, Oklahoma.

In how many ways can a given number be polygonal?

267. Proposed by C. C. YEN, Tangshan, North China.

A number theory function $\phi(n)$ is defined for every positive integer n, and for every such number n, it satisfies the relation $\phi(d_1) + \phi(d_2) + \cdots + \phi(d_r) = n$. From this property alone show that

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right),$$

where $p_1, p_2, p_3, \cdots, p_k$ are the different prime factors of n.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

471. Proposed by E. T. BELL, University of Washington.

If there is an infinite number of positive integers r for which the equation $\sum_{i=1}^{n} a_i^r = \sum_{i=1}^{m} b_i^r$ holds, where the a_i and b_i are given positive integers, prove that $m = n$, and that in some order the a_i are identical with the b_i.

472. Proposed by E. T. BELL, University of Washington.

If a_i and b_i $(i = 1, \cdots, n;\ j = 1, \cdots, m)$ denote positive integers, and if $\sum_{i=1}^{n} a_i^r = \sum_{j=1}^{m} b_j^r$ for all odd positive integral values of r, prove that $m = n$, and that in some order the a_i are identical with the b_j.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

As the proof which I shall give covers both these problems, I shall not treat them separately.

Let a_1 be that a which is the largest of the a's and let there be k a's equal in value to a_1. Similarly b_1 is the largest b, and there are k' b's of the same value. (These exist since there is only a finite number of the a's and of the b's.) Now it is a well-known theorem that

$$\lim_{r \doteq \infty} \Sigma a_i^r / k a_1^r = 1.$$

Since there is an infinite number of positive integers r for which the given equations hold, we can find an r as large as we please for which they hold. For such an r, $\Sigma a_i^r = k a_1^r + \delta k a_1^r$ and

$\Sigma b_i^r = k'b_1^r + \delta'k'b_1^r$, where δ and δ' can be made as small in absolute value as we please by taking r sufficiently large. Since the left-hand sides of these equations are equal, the right-hand sides must be equal also. Equating them we obtain

$$\frac{k}{k'} = \frac{b_1^r}{a_1^r}\frac{(1+\delta')}{(1+\delta)} = \frac{b_1^r}{a_1^r}(1+n),$$

where n is an infinitesimal. If, now, a_1 and b_1 are not equal we can make the right-hand side as small (or as large) as we please by taking r sufficiently large. This is impossible for the ratio $k \div k'$ is a definite number. Consequently $a_1 = b_1$, and $k = k'$. In each equation we may now subtract from each side the a's and b's proven equal and proceed as before to prove the largest remaining a's and b's to be equal in value and number. We may continue in this way until there are no a's (or b's) left, when the b's (a's) must also be exhausted.

471 was also solved by C. F. Gummer and Frank Irwin.

473. Proposed by J. I. GINSBURG, Student, Cooper Union, New York.

Factor the expression $x^{30} + x^{25} + x^{20} + x^{15} + x^{10} + x^5 + 1$.

Solution by Claribel Kendall, University of Colorado.

Multiplying the given expression by $x^5 - 1$, we obtain $x^{35} - 1$. This may be factored in the following way:

First, consider

$$(1)\qquad \begin{aligned} x^{35} - 1 &= (x^5)^7 - 1 = (x^5 - 1)(x^{30} + x^{25} + x^{20} + x^{15} + x^{10} + x^5 + 1) \\ &= (x-1)(x^4 + x^3 + x^2 + x + 1)(x^{30} + x^{25} + x^{20} + x^{15} + x^{10} + x^5 + 1). \end{aligned}$$

Next consider

$$(2)\qquad \begin{aligned} x^{35} - 1 &= (x^7)^5 - 1 = (x^7 - 1)(x^{28} + x^{21} + x^{14} + x^7 + 1) \\ &= (x-1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)(x^{28} + x^{21} + x^{14} + x^7 + 1). \end{aligned}$$

The factors of $x^{35} - 1$ must be the same in both (1) and (2). $x - 1$ occurs in both. The prime factor $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ in (2) must also be found in (1). It cannot be contained in the prime factor $x^4 + x^3 + x^2 + x + 1$ and hence must be found in the third factor. By division the factors of

$$x^{30} + x^{25} + x^{20} + x^{15} + x^{10} + x^5 + 1$$

are found to be

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$$

and

$$x^{24} - x^{23} + x^{19} - x^{18} + x^{17} - x^{16} + x^{14} - x^{13} + x^{12} - x^{11} + x^{10} - x^8 + x^7 - x^6 + x^5 - x + 1.$$

Also variously solved by H. C. Feemster, Horace Olson, O. S. Adams, W. F. Shelton, J. W. Baldwin, A. W. Smith, E. B. Escott, L. C. Mathewson, Paul Capron, E. J. Oglesby, and the Proposer.

GEOMETRY.

493. Proposed by FLORENCE P. LEWIS, Goucher College, Baltimore, Md.

Construct three circles each of which shall be tangent to the other two and to two sides of a given triangle.

Note by J. L. Coolidge, Harvard University.

It is always a pleasure to see this old friend. Few problems in elementary geometry have a longer or more distinguished pedigree. At least forty articles dealing with it were published in the nineteenth century and the twentieth is doing its share.[1] The history is briefly this. The

[1] Simon, *Die Entwickelung der Elementargeometrie im XIX Jahrhundert*, Leipzig, 1906, pp. 147ff.

problem was first proposed and solved in 1803 by Malfatti, and is generally known by his name. A very beautiful construction was given by Steiner in 1826 with no proof, but the appended remark that this construction showed how powerful were the methods which the writer had developed.[1] Steiner's construction served as a challenge to geometers, and many proofs were attempted. By far the simplest and most elegant was given in 1856 by Hart.[2] The German mathematicians were ill pleased that the first good proof should be in English; perhaps they failed to appreciate the strategic significance of the fact that Hart was an Irishman. At any rate they objected that Hart had used methods unknown to Steiner.[3] This criticism seems to us trivial. Steiner was a sly old fox, who probably knew a good deal more mathematics than he ever put on paper. For instance, there seems good reason to believe that he was familiar with inversion in a circle, though he did not give it as one of his working methods.[4] As for simplicity, one has but to compare Hart's proof as given, let us say, in Casey's *Sequel to Euclid*,[5] with the proof in Petersen's little classic on geometrical constructions[6] to see how immeasurably superior the former is. Analytic determinations of the radii and points of contact of the circles have not been wanting; that given by Professor Gummer in the MONTHLY, Vol. XXIV, No. 3, being quite as simple as any. On the other hand the Lemoine geometrographic numbers called for by Steiner's construction can be reduced to the remarkably small proportions of Simplicity 66, Exactitude 42.[7]

502. Proposed by R. P. BAKER, University of Iowa.

A designer of machinery requires a curve having the following properties:

(1) A closed curve touching a given circle at two diametral points and enclosing it.

(2) The sum of the three radii from the center of this circle to the curve which make with each other angles of 120° is constant.

(3) The locus of a point which lies at some constant distance from the curve on its inner normal must be such that it is also the locus of a point fixed on a bar of some simple linkage. In estimating the value of the word "simple" pivoted bars are preferred to slides and the total number should be as small as possible.

Condition (3) is needed to enable a cylinder to be ground accurately to the curve.

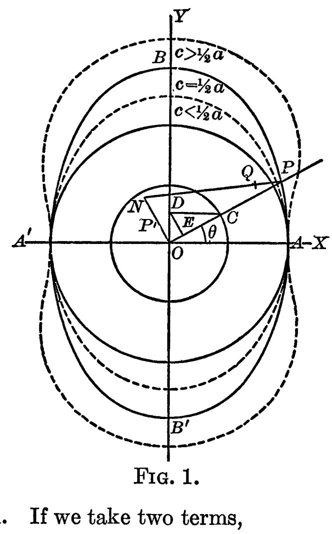


FIG. 1.

SOLUTION BY TOBIAS DANTZIG, Indiana University.

Let (Fig. 1) AOA' be the diameter of contact and BOB' a perpendicular diameter. I shall seek a continuous algebraic curve symmetric with respect to the two diameters satisfying the conditions of the problem. If $\rho = f(\theta)$ is the equation of the curve, f is a function with a period π in θ and consequently of the form

$$\rho = a + c\sin^2\theta + e\sin^4\theta + \cdots, \tag{1}$$

where a is evidently the radius of the circle. If we stop at the first term, we obtain the trivial solution of the circle itself, which evidently satisfies all the conditions of the problem. If we take two terms,

$$\rho = a + c\sin^2\theta, \tag{2}$$

we obtain a circular sextic whose cartesian equation is

$$(x^2 + y^2)^3 = (ax^2 + by^2)^2, \tag{3}$$

[1] "Einige geometrische Betrachtungen," *Crelle*, Vol. 1, 1826.

[2] "Geometrical Investigation of Steiner's Solution of Malfatti's Problem," *Quarterly Journal of Mathematics*, Vol. 1, 1856.

[3] SCHROETER, *Crelle*, Vol. LXXVII, 1874, p. 232.

[4] BÜTZBERGER, *Ueber bizentrische Polygone*, Leipzig, 1913, pp. 50ff.

[5] P. 149 in the first ed., Dublin, 1881.

[6] Fourth edition of the French translation, *Méthodes et théories pour la résolution des problèmes de constructions geometriques*, Paris, 1908, pp. 103ff.

[7] HAGGE, "Zur Konstruktion der Malfattiscen Kreise," *Zeitschrift für mathematische Unterricht*, Vol. XXXIX, 1908, p. 588.

where $b = a + c$. We have here

$$x^2 = \frac{\rho^2(b-\rho)}{c}, \quad y^2 = \frac{\rho^2(\rho - a)}{c}$$

and the maximum value for x occurs for $\rho = \frac{2}{3}b$. According as $c \gtreqless a/2$, this maximum value really occurs, is coincident with A, or is imaginary. The three cases are illustrated in Fig. 1. That condition (2) is satisfied for the curve in question is easy to verify. Indeed, let ρ_1, ρ_2, ρ_3 be three radii inclined 120° to each other. Then

$$\rho_1 + \rho_2 + \rho_3 = 3a + c\left[\sin^2\theta + \sin^2\left(\theta + \frac{2\pi}{3}\right) + \sin^2\left(\theta - \frac{2\pi}{3}\right)\right] = 3a + \frac{3}{2}c = \frac{3(2a+c)}{2}.$$

The angle φ which the normal makes with the radius vector has for tangent, ρ'/ρ, and $\rho' = 2c\sin\theta\cos\theta$. The curve can be constructed by points in the following manner. Let c be the point where the circle of radius c meets an arbitrary radius. Project C on OY in D, and back on the radius in E. Then $OE = c\sin^2\theta$ and if we carry $EP = a$, the point P is on the curve. If we now draw at O a perpendicular to OP and carry $ON = 2DE = 2c\sin\theta\cos\theta$, then NP is the normal at P.

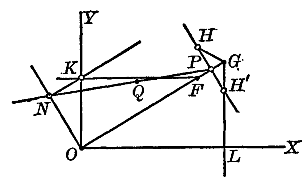


FIG. 2.

The linkage of Fig. 2 enables one to construct the locus of P as well as that of any point Q on its normal by a continuous motion. O, F, and G are fixed pivots. $OF = a$. $FG = GH = GH' = \frac{1}{2}c$. Through H and H' slides the bar HPH' and GH' is kept parallel to OY. The point P, intersection of OG with HH', describes the curve.

In the second part of the linkage, FK is kept parallel to OX, ON is rigidly fixed at right angles to OG and K is a loose pivot, while N slides on ON. If, in the motion of the radius OG, K is describing OY, the bar NP is enveloping the evolute of the curve P and any point Q at a constant distance $PQ = d$ will describe the locus sought in the problem.

503. Proposed by J. W. CLAWSON, Ursinus College, Penn.

If two points A and B invert with respect to a third point O as center of inversion into A' and B', the middle point of the segment AB inverts into the point other than O where the circle of Apollonius (the locus of a point P moving so that $A'P/PB = A'O/OB'$) cuts the circle $OA'B$.

SOLUTION BY HORACE OLSON, Chicago, Illinois.

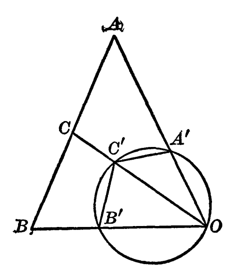
By elementary geometry it is evident that the line AB inverts into the circle passing through O, A', and B'. Let C be the middle point of AB, and C' the point into which it inverts. Draw the lines $A'C'$ and $C'B'$. Triangle ACO is similar to triangle $C'A'O$, and triangle BCO to triangle $C'B'O$. Hence,

$$\frac{A'C'}{AC} = \frac{A'O}{CO}, \quad \text{and} \quad \frac{C'B}{CB} = \frac{B'O}{CO};$$

whence (since $AC = CB$)

$$\frac{A'C'}{C'B'} = \frac{A'O}{B'O},$$

and the proposition is proved.



Also solved by FRANK IRWIN and R. A. JOHNSON.

CALCULUS.

420. Proposed by W. J. GREENSTREET, Stroud, England.

The join of the center of curvature of a curve to the origin is at an angle α to the initial line. Prove that with the usual notation,

$$\frac{d\alpha}{d\psi}\left[\left(\frac{dp}{d\psi}\right)^2 + \left(\frac{d^2p}{d\psi^2}\right)^2\right] = \frac{dp}{d\psi}\cdot\frac{d\rho}{d\psi}.$$

where $b = a + c$. We have here

$$x^2 = \frac{\rho^2(b - \rho)}{c}, \quad y^2 = \frac{\rho^2(\rho - a)}{c}$$

and the maximum value for x occurs for $\rho = \frac{2}{3}b$. According as $c \gtreqless a/2$, this maximum value really occurs, is coincident with A, or is imaginary. The three cases are illustrated in Fig. 1. That condition (2) is satisfied for the curve in question is easy to verify. Indeed, let ρ_1, ρ_2, ρ_3 be three radii inclined 120° to each other. Then

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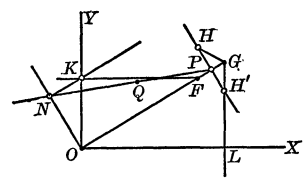

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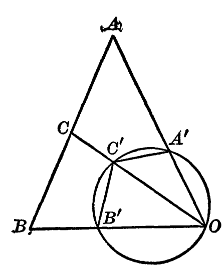

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$$\frac{d\alpha}{d\psi}\left[\left(\frac{dp}{d\psi}\right)^2 + \left(\frac{d^2p}{d\psi^2}\right)^2\right] = \frac{dp}{d\psi} \cdot \frac{d\rho}{d\psi}.$$

421. Proposed by E. H. MOORE, The University of Chicago.

Given n continuous real-valued functions $\phi_g(x)$ $(g = 1, 2, \cdots, n)$ of the real variable x on the interval (01) and set exp. $\int_0^1 \phi_g(x)\phi_h(x) = w_{gh}$ $(g, h = 1, 2, \cdots, n)$. Prove that the determinant $| w_{gh} |$ of the matrix (w_{gh}) is always $\geqq 0$ and that it is > 0 if no two of the functions $\phi_1, \cdots, \phi_n$ are identically equal on (01).

Solution by C. F. Gummer, Kingston, Ont.

(1) *Proof that* $| w_{gh} | \geqq 0$.

$$w_{gh} = \lim_{m=\infty} w_{gh}^{(m)}, \qquad \text{where} \qquad w_{gh}^{(m)} = 1/m \sum_{i=1}^{m} \phi_g(i/m)\phi_h(i/m);$$

and since $| w_{gh} |$ is a continuous function of the w's, it follows that

$$| w_{gh} | = | \lim w_{gh}^{(m)} | = \lim | w_{gh}^{(m)} | .$$

Now

$$| w_{gh}^{(m)} | = 1/m^n \, | \Sigma\phi_g(i/m)\phi_h(i/m) | ,$$

which, by the rule for a minor of a product matrix, is equal to $(1/m^n) \times$ the sum of the squares of the nth order determinants of the matrix $(\phi_g(i/m))_{\substack{i=1, \dots m. \\ g=1, \dots n}}$

$$\therefore \quad | w_{gh}^{(m)} | \geqq 0.$$

$$\therefore \quad | w_{gh} | = \lim | w_{gh}^{(m)} | \geqq 0.$$

(2) *The condition that* $| w_{gh} | = 0$.

Suppose $| w_{gh} | = 0$. Then there is a real linear relation

$$\sum_{g=1}^{n} a_g w_{gh} = 0 \qquad (h = 1, 2, \cdots, n). \tag{A}$$

$$\therefore \quad \int_0^1 \{\Sigma a_g \phi_g(x)\}\phi_h(x)dx = 0 \qquad (h = 1, \cdots n).$$

Multiplying by a_h and adding, we have

$$\int_0^1 \{\Sigma a_g \phi_g(x)\}^2 dx = 0.$$

Hence, the integrand being continuous,

$$\Sigma a_g \phi_g(x) = 0 \text{ on } (01). \tag{B}$$

Conversely, if (B) is true, so is (A), and $| w_{gh} | = 0$. $\therefore$ the necessary and sufficient condition for the vanishing of $| w_{gh} |$ is that the ϕ's be linearly dependent, and the problem as stated is incorrect. (Consider for example the case $\phi_1 = 1$, $\phi_2 = x$, $\phi_3 = 1 + x$.)

MECHANICS.

330. Proposed by PAUL CAPRON, U. S. Naval Academy.

A Barker's Mill operates under a head of h ft.; the linear speed of the orifices is $u(f/s)$, the speed of the water relative to the orifices is $v(f/s)$, the coefficient of discharge is c, so that $v^2 = c^2(2gh + u^2)$. Given that the work done by the water on the mill is $u/g(v - u)$ ft. lbs. per sec. per lb. of water used, find the values of u and v such that the water-power may be most economically used, and find what part of the power is so used.

Solution by the Proposer.

It is required to make $f(u) = u(c\sqrt{k^2 + u^2} - u)$ a maximum. $(k^2 = 2gh)$.

$$f'(u) = \frac{cu^2}{\sqrt{k^2 + u^2}} + c\sqrt{k^2 + u^2} - 2u = 0$$

when $c(u^2 + k^2 + u^2) = 2u\sqrt{k^2 + u^2}$, or $4(1 - c^2)(u^4 + k^2u^2) = c^2k^4$.

Whence,

$$u^2 = \frac{k^2}{2}\left(\frac{1-\sqrt{1-c^2}}{\sqrt{1-c^2}}\right),$$

and

$$v^2 = c^2(k^2+u^2) = \frac{c^2k^2}{2}\left(\frac{1+\sqrt{1-c^2}}{\sqrt{1-c^2}}\right).$$

Hence,

$$u = \frac{k}{2\sqrt[4]{1-c^2}}(\sqrt{1+c}-\sqrt{1-c}), \qquad v = \frac{ck}{2\sqrt[4]{1-c^2}}(\sqrt{1+c}+\sqrt{1-c}),$$

and

$$u(v-u) = \frac{k^2}{2}(1-\sqrt{1-c^2}).$$

The available work is h ft. lbs. per sec. per lb. of water used; the work utilized is

$$\frac{u}{g}(v-u) = h(1-\sqrt{1-c^2});$$

the proportion used is $(1-\sqrt{1-c^2})$.

If we let $c = \sin\alpha$, we have $u^2 = gh\tan\alpha\tan\alpha/2$, $v = 2u\cos^2\alpha/2$, and efficiency $=$ vers α.

331. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

A cyclist is riding due west at a speed of 12 miles per hr., and the wind is at the same time blowing from the southeast with a speed of $5\frac{1}{2}$ miles per hour. If the cyclist carries a small flag, in what direction will this flag fly? At what speed would the cyclist need to ride if the flag is to fly due north?

SOLUTION BY B. J. BROWN, Victor, Colo.

The component of the wind north is $11\sqrt{2}/4$ miles per hr., while that west is $11\sqrt{2}/4$ miles per hr. The resistance offered to the machine traveling west is $[12-(11\sqrt{2}/4)]$ and the reaction is toward the east. Hence the forces affecting the flag are $[12-(11\sqrt{2}/4)]$ *toward east*, and $11\sqrt{2}/4$ *toward north* and the direction θ, which the resultant makes with the east-west line $=$ arc tan $11\sqrt{2}/(48-11\sqrt{2}) = 25° 36' 55''$ N. of E. In order for the flag to fly due N. the rider must travel W. at rate of $11\sqrt{2}/4$ miles per hr. Then he counteracts the resistance E. and *only* the component north is effective.

Also solved by PAUL CAPRON, W. J. THOME, and G. W. HARTWELL.

NUMBER THEORY.

236. Proposed by V. M. SPUNAR, Chicago, Illinois.

Find integral values of x, y, z, such that

$$xy+z=\square,\quad yz+x=\square,\quad xz+y=\square.$$

NOTE BY G. H. LING, University of Saskatchewan.

The following proposition generalizes somewhat the solution of this problem which was given in the February, 1917, MONTHLY.

THEOREM. *If* (1) a *is any integer,* (2) N_1, N_2 *are integers such that* $N_1\cdot N_2 = a^2+1$, (3) $x = N_1-1$, $y = N_2-1$, $z = N_1+N_2-1+2a$, *then*

(I) $$xy+z=(a+1)^2;\quad yz+x=(N_2+a-1)^2;\quad zx+y=(N_1+a-1)^2,$$

(II) $$xy+x+y=a^2;\quad yz+y+z=(N_2+a)^2;\quad zx+x+z=(N_1+a)^2.$$

The solution given in the MONTHLY is the special case of this one in which

$$a = n^2+n+1.$$

248. Proposed by E. T. BELL, Seattle, Washington.

If $u_{n+2} = 4u_{n+1} - u_n$, with $u_0 = 2$, $u_1 = 4$, prove that the $\triangle_n$, whose sides are $u_n - 1$, u_n, $u_n + 1$, has an integral area; also that all triangles, $\triangle_n$, whose areas are integers, and whose sides are consecutive integers, are given by this process. Hence show that, as n increases, the area $\triangle_n$ approximates $(\sqrt{3}/4)u_n^2$, and find the degree of approximation.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

Integrating,

$$u_n = C_1(2 + \sqrt{3})^n + C_2(2 - \sqrt{3})^n. \tag{1}$$

When $n = 0$, $u_0 = 2$, and when $n = 1$, $u_1 = 4$; then

$$2 = C_1 + C_2 \cdots \tag{2}$$

$$4 = C_1(2 + \sqrt{3}) + C_2(2 - \sqrt{3}). \tag{3}$$

(2) and (3) give $C_1 = 1$, $C_2 = 1$; so (1) becomes

$$u_n = (2 + \sqrt{3})^n + (2 - \sqrt{3})^n \tag{4}$$

Taking u_n as the second of three consecutive numbers, $u_n - 1$, $u_n + 1$ are the others, and the area

$$\begin{aligned}\triangle_n &= \tfrac{1}{4}\sqrt{3\{(2 + \sqrt{3})^n + (2 - \sqrt{3})^n\}^2[\{(2 + \sqrt{3})^n + (2 - \sqrt{3})^n\}^2 - 2^2]} \\ &= \tfrac{1}{4}\{(2 + \sqrt{3})^n + (2 - \sqrt{3})^n\}\sqrt{3[\{(7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n\} - 2]} \cdots. \end{aligned} \tag{5}$$

The expression under the radical sign must be shown to be a perfect square. For $n = 1, 2$, etc., this is the case. For the extreme case,

$$\lim_{n \doteq \infty} \triangle_n = \lim_{n \doteq \infty} \tfrac{1}{4}\{(2 + \sqrt{3})^n + (2 - \sqrt{3})^n\}^2\sqrt{3}\sqrt{1 - \frac{4}{(2 + \sqrt{3})^n + (2 - \sqrt{3})^n}}$$

$$= \tfrac{1}{4}\{(2 + \sqrt{3})^n + (2 - \sqrt{3})^n\}^2\sqrt{3} = \tfrac{1}{4}\sqrt{3}u_n^2.$$

249. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

A perfect number is a number which is equal to the sum of all its different divisors. In an old book on mathematics, the following method is given without proof for determining perfect numbers. The number $2^{n-1}(2^n - 1)$ is a perfect number, if $2^n - 1$ is a prime number. Prove the formula.

SOLUTION BY MRS. ELIZABETH BROWN DAVIS, U. S. Naval Observatory, Washington, D. C.

The sum of all the divisors of 2^{n-1}, including unity, is

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-3} + 2^{n-2} = 2^{n-1} - 1.$$

If $2^n - 1$ is prime, the sum of all the divisors of $2^{n-1}(2^n - 1)$ is equal to the sum of all the divisors of 2^{n-1}, including unity, plus the product of the sum of these divisors into $(2^n - 1)$, plus 2^{n-1}. Hence, the sum of all the divisors of

$$\begin{aligned}2^{n-1}(2^n - 1) &= 2^{n-1} - 1 + (2^{n-1} - 1)(2^n - 1) + 2^{n-1} \\ &= 2.2^{n-1} - 1 + (2^{n-1} - 1)(2^n - 1) \\ &= 2^n - 1 + (2^{n-1} - 1)(2^n - 1) \\ &= 2^{n-1}(2^n - 1).\end{aligned}$$

Therefore, $2^{n-1}(2^n - 1)$ is a perfect number. Q. E. D.

Also solved by H. N. CARLETON, ELIJAH SWIFT, J. H. WEAVER, J. W. BALDWIN, E. B. ESCOTT, H. C. FEEMSTER, and J. W. CLAWSON.

248. Proposed by E. T. BELL, Seattle, Washington.

If $u_{n+2} = 4u_{n+1} - u_n$, with $u_0 = 2$, $u_1 = 4$, prove that the $\triangle_n$, whose sides are $u_n - 1$, u_n, $u_n + 1$, has an integral area; also that all triangles, $\triangle_n$, whose areas are integers, and whose sides are consecutive integers, are given by this process. Hence show that, as n increases, the area $\triangle_n$ approximates $(\sqrt{3}/4)u_n^2$, and find the degree of approximation.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

Integrating,

$$u_n = C_1(2 + \sqrt{3})^n + C_2(2 - \sqrt{3})^n. \tag{1}$$

When $n = 0$, $u_0 = 2$, and when $n = 1$, $u_1 = 4$; then

$$2 = C_1 + C_2 \cdots \tag{2}$$

$$4 = C_1(2 + \sqrt{3}) + C_2(2 - \sqrt{3}). \tag{3}$$

(2) and (3) give $C_1 = 1$, $C_2 = 1$; so (1) becomes

$$u_n = (2 + \sqrt{3})^n + (2 - \sqrt{3})^n \tag{4}$$

Taking u_n as the second of three consecutive numbers, $u_n - 1$, $u_n + 1$ are the others, and the area

$$\begin{aligned}\triangle_n &= \tfrac{1}{4}\sqrt{3\{(2 + \sqrt{3})^n + (2 - \sqrt{3})^n\}^2[\{(2 + \sqrt{3})^n + (2 - \sqrt{3})^n\}^2 - 2^2]} \\ &= \tfrac{1}{4}\{(2 + \sqrt{3})^n + (2 - \sqrt{3})^n\}\sqrt{3[\{(7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n\} - 2]} \cdots.\end{aligned} \tag{5}$$

The expression under the radical sign must be shown to be a perfect square. For $n = 1, 2$, etc., this is the case. For the extreme case,

$$\begin{aligned}\lim_{n \doteq \infty} \triangle_n &= \lim_{n \doteq \infty} \tfrac{1}{4}\{(2 + \sqrt{3})^n + (2 - \sqrt{3})^n\}^2\sqrt{3}\sqrt{1 - \frac{4}{(2 + \sqrt{3})^n + (2 - \sqrt{3})^n}} \\ &= \tfrac{1}{4}\{(2 + \sqrt{3})^n + (2 - \sqrt{3})^n\}^2\sqrt{3} = \tfrac{1}{4}\sqrt{3}u_n^2.\end{aligned}$$

249. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

A perfect number is a number which is equal to the sum of all its different divisors. In an old book on mathematics, the following method is given without proof for determining perfect numbers. The number $2^{n-1}(2^n - 1)$ is a perfect number, if $2^n - 1$ is a prime number. Prove the formula.

SOLUTION BY MRS. ELIZABETH BROWN DAVIS, U. S. Naval Observatory, Washington, D. C.

The sum of all the divisors of 2^{n-1}, including unity, is

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-3} + 2^{n-2} = 2^{n-1} - 1.$$

If $2^n - 1$ is prime, the sum of all the divisors of $2^{n-1}(2^n - 1)$ is equal to the sum of all the divisors of 2^{n-1}, including unity, plus the product of the sum of these divisors into $(2^n - 1)$, plus 2^{n-1}. Hence, the sum of all the divisors of

$$\begin{aligned}2^{n-1}(2^n - 1) &= 2^{n-1} - 1 + (2^{n-1} - 1)(2^n - 1) + 2^{n-1} \\ &= 2.2^{n-1} - 1 + (2^{n-1} - 1)(2^n - 1) \\ &= 2^n - 1 + (2^{n-1} - 1)(2^n - 1) \\ &= 2^{n-1}(2^n - 1).\end{aligned}$$

Therefore, $2^{n-1}(2^n - 1)$ is a perfect number. Q. E. D.

Also solved by H. N. CARLETON, ELIJAH SWIFT, J. H. WEAVER, J. W. BALDWIN, E. B. ESCOTT, H. C. FEEMSTER, and J. W. CLAWSON.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence.

DISCUSSIONS.

I. SOME REFLECTIONS ON THE TEACHING OF MECHANICS, SUGGESTED BY PROFESSOR E. V. HUNTINGTON'S ARTICLE "A LOGICAL SKELETON OF ELEMENTARY DYNAMICS."[1]

By ALEXANDER ZIWET, University of Michigan.

1. Even those who do not agree with Professor Huntington in all particulars, or even in his main idea, will readily admit that his article is a model of really serviceable pedagogical discussion. We have here a definite concrete scheme, carefully set out in its essential features, unassailable from the scientific point of view, and based on ripe experience in teaching. The question however remains whether Professor Huntington's "skeleton" is the only admissible scheme for teaching mechanics, and if not, whether a better scheme can be devised.

2. There is certainly something tempting and alluring in what seems to be our author's main contention that to *force* as the active, aggressive, not to say vital, principle we should concede a "logical priority" (whatever that may mean) over "inert," "dead," "passive" matter or *mass*. But we can hardly think of force otherwise than exerted by matter on matter, by one body on another body (dead or alive). As this point has been very ably discussed by Professor Hoskins it is here unnecessary to say more about it. It may suffice to say that in the "principle of force and acceleration" (p. 4) the idea of mass, here called particle, is presupposed.

I wish to state explicitly that I do not object in the least to the statement and explanation of the fundamental principles of dynamics as given in section II (pp. 3–5). The question of the "logical priority of mass over force," or vice versa, appears to me of slight importance, except for the fact that mass is a scalar and in so far more simple than the vector force.

3. This brings me to the main objection that I have to the whole trend of Professor Huntington's article: I believe that the vector idea should be emphasized more strongly, and the fundamental theorems should be stated for three (not two) dimensions. It is true that in sections VI and VII three dimensions are used. But right after this, in section VIII (top of p. 14), we find the startling statement that "any set of forces acting on a rigid body can be 'boiled down' either to a single force, or else to a single couple." Of course, the author had in mind the case of forces in a plane; but then a "rigid body" in a plane is, without further explanation, a rather artificial thing.

The use of vectors simplifies the case of three dimensions very essentially. But this is not the only advantage. Professor Huntington complains of the difficulty that students find in the notion of acceleration and proposes a remedy

[1] See AMERICAN MATHEMATICAL MONTHLY, Vol. 24, pp. 1–16.

(p. 2). But, however ingenious and striking the "snap shot" illustration, does it hit the vital point of the difficulty?—That the velocity in rectilinear motion may vary (and that is all the snap shots show) is hardly an unfamiliar notion to a junior in college who has studied the calculus. The difficulty arises only in curvilinear motion where velocity, and hence also acceleration, must be regarded as a vector.

In the first paragraph of section IV, where this question is discussed, it would have been far better not to mention any axes of reference which "remain fixed throughout the discussion." The formulae for tangential and normal acceleration are quite independent of any fixed axes of reference.

In sections VI and VII the misleading statements "Work = Force $\times$ Distance" and "Impulse = Force $\times$ Time," which are as mischievous as statements like "Velocity = Space/Time," etc., could be made correct by using vector notions.

4. Professor Huntington seems to base his main argument in favor of the "logical priority" of force over mass on the tables of dimensions and units on pp. 15, 16. I must confess that the inspection of these tables does not convince me at all. Such tables are a nuisance, anyhow. They may have a place in an encyclopedic handbook; I do not like to see them in a textbook; the student should not be encouraged to consult them.

But if we must have such tables, they should be arranged less artificially. The units of space and time should certainly come first, to be followed by those of velocity and acceleration. Then should come either mass and force or force and mass; and these can then both be used in defining momentum, kinetic energy, etc. It appears just as odd to me to say that momentum is force $\times$ time (by p. 15) as that impulse is mass $\times$ velocity (p. 16), and similarly for work and energy.

5. In the present state of science it would seem best, indeed necessary, to tell the student that both systems are in use, and that both are equally justifiable. It is a matter of personal opinion whether force or mass is regarded as more simple or more fundamental. Why not introduce both these notions from the beginning and use both? Is there really anything gained by always writing w/g instead of m? Why should we carefully avoid the use of the simple term mass after it has been introduced, and the simple symbol m, and use the more complicated symbol w/g and any number of terms such as matter, lump of matter, body, particle, and especially inertia (which is liable to suggest the objectionable force of inertia)?

When a past-master in the art of devising sets of axioms takes the trouble to give an exposition of the fundamental dynamic concepts and theorems, we are surely greatly indebted to him; and the work cannot fail to be clarifying and stimulating. But it is certainly only fair to the student to let him know that the system of units here advocated is not used by a single writer on higher mechanics.

II. Mass and Force in Elementary Dynamics.

By Dunham Jackson, Harvard University.

In a series of articles published during the last few years, Professor Huntington has contended for the use of the equation

$$F/F' = a/a',$$

instead of the equation

$$ma = \lambda F,$$

as the fundamental equation of dynamics. A reader of his paper entitled "The logical skeleton of elementary dynamics," recently published in the Monthly,[1] can scarcely question the correctness and adequacy of his treatment. Controversy can have to do only with matters of arrangement and emphasis, on which anyone may naturally have an opinion. It has been my peculiar privilege to discuss the subject informally with Professor Huntington at frequent intervals, and I may perhaps be allowed to acknowledge my indebtedness to him for a very great clarification of my own ideas, and at the same time to say a few words with regard to some of the points on which I have remained unconvinced. To novelty in the views presented I can make no claim; I offer them here merely for purposes of ready comparison.

It is characteristic of Professor Huntington's presentation that the notion of *mass* is subordinated to the notion of *force*, and is not merely assigned a secondary place at the beginning, but is regarded as of secondary importance throughout. It manifests itself as *inertia*, through the constant ratio F/a of the fundamental equation, and can be measured in terms of *standard weight*. All that is needed for the solution of problems is contained in these two ideas. The reference of all measurements to fundamental units of force, length, and time, does away with the difficulties that arise when units of mass and units of force are used together in such a way that it is necessary to consider the relations between them. The whole approach to dynamical theory is extraordinarily simplified.

It may seem that with this acknowledgement, discussion should stop. And yet, many teachers and students of mechanics feel that the whole story has not been told, that there was something more in the old idea of mass, which was altogether worth while when you understood it. Granted that it is best to begin with the equation in Professor Huntington's form, and not to pile up difficulties at the threshold of the subject, there ought to come a time when you are ready to search the concept of mass for all that there is in it.[2] The skeleton of elementary dynamics may have every bone in place, but, like other

[1] American Mathematical Monthly, Vol. 24 (1917), pp. 1–16. See also references to controversial articles in *Science*, at the close of the paper.

[2] The writer, who has had occasion to teach an elementary course in mechanics for several years, has tried two or three times the experiment of using the equation $F/F' = a/a'$ for perhaps the first three weeks of kinetics, and then introducing the equation $ma = \lambda F$, with a discussion of units of force and mass; and he is at present disposed to think that any lost motion that results is more than compensated by the advantages of the arrangement.

skeletons, it loses in richness of outline something of what it gains in transparency. So far as I can judge my own sentiments by introspection, the pleasure that I take in the use of the equation $F = ma$, with all its pounds and poundals, has nothing of pedantry about it; it is as genuine and healthy as that which belongs to the use of any improved tool in analysis.

The study of mechanics in college serves two purposes. It leads to the solution of practical or conceivably practical problems, and it stimulates the imagination by an insight into the orderly working of the universe. Few students will ever have any "use" for any but the terrestrial applications of the general laws; but neither the historical development of the science nor its far-reaching significance can be understood without reference to the working of its laws among the heavenly bodies. While the equation $F/F' = a/a'$ can with entire justification be taken as the fundamental one, there as elsewhere, still there is reason for regarding something else as equally fundamental at the same time.

Two things are characteristic of material bodies throughout the universe. They are disposed to disturb other bodies, and they are reluctant to be disturbed by them. It is highly remarkable that each body, wherever it goes, and to whatever physical or chemical changes it is subjected, has a characteristic quantity invariably associated with it, which measures its power of attracting other bodies, in accordance with the law of universal gravitation. It is highly remarkable that each body has a quantity invariably associated with it which measures its inertia, in accordance with the equation $F/F' = a/a'$. Most remarkable of all, however, is the circumstance that for different bodies these two "body-constants" stand in an invariable relation to each other. Every material body has the property of attractiveness and the property of inertia, and these are somehow not two properties, but one. This, it seems to me, is the significance of the word *mass*—it is a name for the one fundamental property which manifests itself in two aspects. The equation $F/F' = a/a'$ describes one aspect; the equation $ma = \lambda f$ describes one, implies the other, and asserts the essential identity of the two. The second equation, admittedly more difficult, is richer in content and more suggestive. Even to supplement the other by the statement that inertia is proportional to standard weight, does not produce quite the same effect, for it is more impressive to invoke the law of universal gravitation at the start than merely to observe that at a particular station on the earth's surface different bodies fall with the same acceleration, and mention the general law only incidentally later. The idea of mass, and the equation containing it, are worthy of all the emphasis that it is possible to give them.

The contention, sometimes advanced, that the idea of inertia is more fundamental than the other part of the mass-concept, that without bringing in acceleration it is impossible to form any precise quantitative notion of mass at all, does not seem to me well sustained. Of course the first rough idea of a beam-balance for measuring mass in terms of its gravitational power has to be refined, but so does the naïve conception of a spring-balance and a fixed frame of reference for measuring force and acceleration. There is no difficulty in either case if you are

not too critical; and if you are, the difficulties in either case are great, if not for the present insurmountable. The two are very much on a par.

So far, nothing has been said about the interpretation of mass as quantity of matter. As Professor Huntington has insisted, this attribute can hardly be made the basis for a satisfactory *definition* of mass, for the purposes of dynamical theory. But as a bond between that theory, founded on suitable definitions, and the facts of every-day experience, it is undoubtedly important to recognize that mass *is* quantity of matter, so far as quantity of matter means anything at all, and this recognition adds very much to the significance of mass. Not only does a quart of water have twice the mass of a pint, but if you take a quart of water and compress it under great pressure, or boil it, or freeze it, or decompose it by electrolysis, it still has the same mass as before. The qualities of mass seem somehow inherent in the matter itself, inseparable from it when it is otherwise changed in almost every conceivable way. It is to be said that the primary appeal of the identification of mass with quantity of matter is perhaps cruder still: you measure mass on a beam-balance, and a beam-balance, in real life, is an instrument for finding out how much of a substance you have got.

Leaving quantity of matter aside, we have made the concept of mass, fundamental as it is, depend logically on the concept of force; for the latter enters both into the statement of the law of gravitation and into the statement of the law of inertia. But it is not to be granted that the former concept is even to this extent essentially a subordinate one. It is perfectly possible to introduce the two side by side. For the measurement of masses by means of a beam-balance does not necessarily involve the measurement of force—quantitative determination of the relative magnitudes of different forces—nor even the concept of force as a measurable quantity. If it seems necessary to make the measurement of mass depend on the existence of force as a physical agency, the difference is that force is recognizably homogeneous, and for that reason capable of independent measurement, while matter is not; quantity of force has just the *a priori* significance which it is hard to attach to quantity of matter.

Finally, as to the choice of fundamental units for a systematic table, it is certainly allowable to start with mass, length, and time, and define force in terms of them, and then to define anything you please in terms of force, length, and time, and write its dimensions accordingly. The dimensions of work are FL, whether you remember the equation $F = MLT^{-2}$ or forget it or never had it to forget. There is a certain loss of formal simplicity in permitting a derived unit to act as deputy for a fundamental one, but it is not such a loss as to put any burden on the understanding; and it is merely a question whether it is worth while to make so slight a sacrifice for the sake of commemorating in the table the fact that the unit of force is derived from the unit of mass in practice.

III. Approximate Construction for an Ellipse.

By J. W. Bradshaw, University of Michigan.

In view of the approximate construction for an ellipse given by Mr. Richard Iwerson in the November number of the Monthly, 1916, pp. 354–355, and discussed by Mr. Paul Capron in the number for February, 1917, pp. 90–93, readers of the Monthly may be interested in a few observations on other solutions of this problem. Several such have been published, to six of which I shall refer as French's[1] first, second, and third, Richter's,[2] Honey's,[3] and Clark's[4] constructions. French's first and third and Richter's replace the quadrant of the ellipse by two circular arcs; French's second, Honey's, and Clark's by three. In all of these the approximate quadrant passes through the ends of the axes and the circular arcs are tangent to each other where they meet.

Two-arc constructions of the quadrant. We may think of the two-arc approximation as arising from an alteration of the major and minor circles of curvature until they become tangent to each other. In Fig. 1 let $OA = a$ and $OB = b$ be the semi-axes of the ellipse, which for the sake of brevity of statement we refer to coördinate axes through its center. Let C_1 and C_2 be the centers of curvature for the points A and B; they lie on the perpendicular to the line AB through the point $K = (a, b)$. Let ρ_1 and ρ_2 be the corresponding radii of curvature, σ_1 and σ_2 their altered values, and D_1 and D_2 the corresponding centers. The condition that the two arcs where they meet shall be tangent requires that the centers D_1 and D_2 and the point of meeting M shall be collinear. The right triangle OD_1D_2 gives

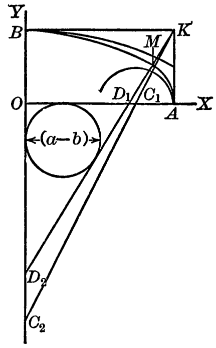

Fig. 1.

$$(\sigma_2 - b)^2 + (a - \sigma_1)^2 = (\sigma_2 - \sigma_1)^2,$$

From this it is easy to show that the line D_1D_2 is a tangent of the circle whose equation is

$$\left(x - \frac{a-b}{2}\right)^2 + \left(y + \frac{a-b}{2}\right)^2 = \left(\frac{a-b}{2}\right)^2.$$

French's first and third and Richter's constructions use different tangents of this circle. The last named possesses the advantage that the point M lies on the true ellipse. The condition for this is that the line D_1D_2 shall pass through the point K. We have then the following very simple statement for the construction of this approximation:

[1] Thomas E. French, *Engineering Drawing*, pp. 53, 54.

[2] Otto Richter, Der Ellipsenreif, *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, Vol. 42 (1911), p. 583.

[3] Frederic R. Honey, "A Method of Constructing an Ellipse and Measuring the Curved Length," *Engineering News*, April, 1907, p. 388.

[4] J. J. Clark, "A Method of Constructing an Approximate Ellipse with Three Radii," *Engineering News*, Nov., 1911, p. 627.

From the point (a, b) *draw that one of the two tangents of the fourth-quadrant circle of diameter* $a-b$ *touching the axes which is farther from the origin. This tangent cuts the axes in the centers* D_1 *and* D_2 *sought.*

While Richter's construction yields the same approximation, its execution is less simple.

Three-arc Constructions of the Quadrant.—The three-arc constructions may be thought of as derived from three circles of curvature, the major, minor, and some intermediate one. There is considerable latitude in altering these so that the intermediate circle shall be tangent to the others. French's second construction uses the major and minor circles of curvature without alteration, the radius of the intermediate circle being the mean proportional between a and b, Clark's uses the major circle, Honey's alters both the major and the minor circles and uses the arithmetic mean between a and b as the radius of the intermediate circle. For most purposes a construction would seem best which alters all three circles. The following is suggested as typical:

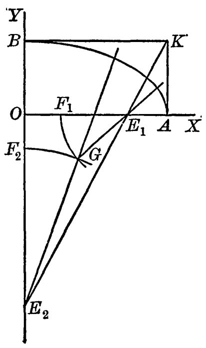

FIG. 2.

In Fig. 2 let E_1E_2 be some line between the lines C_1C_2 and D_1D_2 of Fig. 1—*e. g.*, the bisector of the angle formed by these lines. With a convenient radius—*e. g.*, the mean proportional between a and b—lay off AF_1 and BF_2. With E_1 and E_2 as centers and radii E_1F_1 and E_2F_2 strike arcs intersecting in G. E_1, G, and E_2 are the desired centers and the radii are E_1A, $F_1A = F_2B$, and E_2B.

It would be difficult to give a categorical answer to the question, "What criterion best measures the degree of approximation attained?" The purpose for which the curve is to be used would influence the choice of such a criterion. Four criteria suggest themselves to me: the total area between the curve and the approximation, the difference in length of the two curves, the maximum distance between corresponding points, where by corresponding points I mean those lying on a normal to the ellipse, the maximum angle between the tangents at corresponding points. Doubtless many others might be suggested.

A CORRECTION BY P. J. DANIELL.

TO THE EDITORS: Will you oblige me by publishing this notice of an error in my paper on "New Rules of Quadrature" in the March issue?

Rule 3 is not new but was given by S. A. Corey, in the MONTHLY for June, 1912, formula 25*s*. His error term is too large but mine was incorrect. In my paper should be read on page 112:

$$\psi_1(x) = z^2(z + \tfrac{1}{2}h^2), \quad \psi_2(x) = z_2{}^2(z_2 + 2h^2),$$

$$\psi_2(x) - 16\psi_1(x) = t^4(15t^2 - 42t + 30),$$

$$| S' | \leq \frac{8}{15}\frac{h^6}{7!}(b - a) \max | y^{\mathrm{VI}}(x) | .$$

SECOND SUMMER MEETING OF THE ASSOCIATION.

The second summer meeting of the Mathematical Association of America will be held at Adelbert College and Case School of Applied Science, Cleveland, Ohio, on Thursday and Friday, September 6–7, 1917. The meeting will be preceded by that of the American Mathematical Society on September 4–5. A joint meeting of the two organizations will be held at 9 A. M. on Thursday morning at which Professor L. P. Eisenhart, of Princeton University, will give an address on the "Life and Work of Darboux."

On Wednesday evening there will be a joint dinner of the Association and the Society.

The sessions of the Association will extend through Thursday and Friday, and the Program Committee, Professors L. S. Hulburt, E. J. Wilczynski, and C. S. Slichter, chairman, make the following preliminary announcement of topics to be discussed:

(1) "Geometry for Juniors and Seniors." By Professor E. B. STOUFFER, University of Kansas. Discussion led by Professors FRANK MORLEY, Johns Hopkins University, and L. W. DOWLING, University of Wisconsin.

(2) "The Treatment of the Applications in College Courses in Mathematics." By Professor L. C. PLANT, Michigan Agricultural College. Discussion led by Professors W. B. CARVER, Cornell University, and G. H. LING, University of Saskatchewan.

(3) Presidential Retiring Address: "The Significance of Mathematics." By Professor E. R. HEDRICK, University of Missouri.

(4) "Organization and Work of Undergraduate Mathematical Clubs." By Professor H. E. HAWKES, Columbia University. Discussion led by Professors R. C. ARCHIBALD, Brown University, and D. A. ROTHROCK, Indiana University.

There will be important matters for the consideration of the Council and a business session of the Association.

The Joint Committee on Arrangements, Professors F. N. Cole, W. D. Cairns, E. V. Huntington, A. D. Pitcher, D. T. Wilson, and T. M. Focke, Chairman, make the following preliminary announcement:

The meetings will be held in Case School of Applied Science which is on Euclid Avenue about five miles from the center of the city, and may be reached by any Euclid Avenue car to 109th Street or East Boulevard.

Luncheons, and possibly other meals, will be served at the Students' Club of the Case School, which will be thrown open for the use of the members. The hotel headquarters will be at the Hotel Statler. Further details as to hotel rates, local excursions, etc., will be given in the final announcement.

This is the only announcement of the meeting which will be made until the final program is mailed to all members about the middle of August. These will be mailed under second class postage (not forwardable) to the address of each member on the regular MONTHLY mailing list unless a different address is sent to the Secretary before August 10th. In sending a change of address, please state whether it is *for this purpose only* or also for mailing the September MONTHLY.

REPORT OF THE COMMITTEE ON MEMBERSHIP.

The Committee on Membership, Professor E. R. Hedrick, Chairman, had expected to a carry on a campaign among foreign countries in some of which the Association already has representative members, but the war has interfered with their plans. However, there are still many in this country engaged in the field of collegiate mathematics who have not yet cast in their influence with this forward movement. The slogan of the Association is to include in its membership every teacher of collegiate mathematics in the United States and Canada, and to make such membership worth while. Although the membership, now over eleven hundred, is by far the largest in any mathematical organization in this country, if not in the world, still there are many others who should be included and who doubtless will come in as soon as the importance of the matter is sufficiently understood by them.

It is believed that the Mathematical Association of America is destined to hold as important a relationship to the teaching profession in the field of collegiate mathematics as the American Bar Association and the American Medical Association now hold to the professions which they represent,—a relationship wherein membership is a mark of professional standing.

It is probably true that every eligible non-member is known to some one or more of the present members, and that the most effective presentation to these non-members would be a personal appeal from one who speaks from experience, and hence, with assurance, of the aims and work of the Association.

The Committee would, therefore, appeal to every member of the Association to act as a sub-committee of one for the purpose of reaching every non-member within your sphere of influence. Will you not by word of mouth or by personal letter present the claims of the Association to any such person whom you may know, and at the same time send the name and address to the Chairman, Professor E. R. Hedrick, Columbia, Missouri, who will see to it that a sample copy of the MONTHLY and an application blank are at once forwarded, the former directly to the prospective candidate for membership, the latter to you in order that you may endorse it and then transmit it as a second communication from you on the subject. You are likewise requested to use your influence in securing new institutional members. Special opportunity will be afforded for activities along both these lines at the many summer sessions throughout the country. In this way it should be possible to have a long list of new members to be acted on by the Council at the Cleveland meeting.

SPECIAL NOTICE.

There is a constant demand for back numbers of the MONTHLY, especially previous to 1905. The most pressing requests just now are from Sir Thomas Muir, former Superintendent of Education for the Province of Cape of Good Hope, South Africa, and from the University of Wisconsin library. For the latter are needed only the following: Vol. III December, Vol. IV July, and Vol. V March. Secretary Cairns will pay cash for these and others as the orders come in. Anyone who has back numbers for sale is requested to send an inventory of the same to the Secretary at once.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Ind.

Dr. F. C. FERRY, dean of Williams College and professor of mathematics, has been elected president of Hamilton College.

Professor R. G. D. RICHARDSON and Assistant Professor C. H. CURRIER, of Brown University, will spend the summer at the University of Chicago in attendance upon the summer session.

Dr. PAULINE SPERRY, of Smith College, will teach mathematics at the University of California summer session.

Dr. HENRY BLUMBERG, University of Nebraska, and Dr. G. A. PFEIFFER, Princeton University, have been elected associate editors of the Annals of Mathematics.

Mr. WALTER C. WRIGHT, consulting actuary and accountant, died at his home in Medford, Mass., on April 23. He was a charter member of the Association.

Professor EDWIN B. WILSON, of the department of mathematics at the Massachusetts Institute of Technology, has been made head of the department of Physics in the same institution.

Professor W. E. EDINGTON, of the University of New Mexico, has been appointed instructor in mathematics at the University of Illinois.

Dr. J. H. WEAVER, head of the department of mathematics in the West Chester, Pennsylvania, High School, has been appointed instructor in mathematics at the United States Naval Academy at Annapolis.

Dr. NATHAN ALTSCHILLER, instructor in mathematics at the University of Oklahoma, has been promoted to an assistant professorship.

Miss MARY C. SUFFA, non-resident fellow of Brown University, resident at the University of Chicago, has been appointed instructor in mathematics at the Chicago Latin School.

Professor FLORIAN CAJORI will spend the latter part of the summer in Chicago, engaged in historical research. He will then proceed to Cleveland to attend the meetings of the Society and the Association.

Dr. H. R. KINGSTON, lecturer in mathematics at the University of Manitoba, spent several weeks at the University of Chicago for reading and research after the close of the spring session at Winnipeg.

Dr. W. L. HART, Benjamin Peirce instructor at Harvard University, has joined the officers' reserve training camp at Fort Sheridan.

Miss FLORA LESTOURGEON, formerly instructor at Beaver College, Pennsylvania, has just received the doctorate at the University of Chicago.

Mr. T. R. HOLLCROFT has been appointed instructor in mathematics at Columbia University.

At Brown University Professor R. C. ARCHIBALD has been promoted to an associate professorship of pure mathematics.

Assistant Professor J. F. REILLY has been promoted to an associate professorship of mathematics at the State University of Iowa.

Professors R. C. ARCHIBALD, Brown University, F. MORLEY, Johns Hopkins University and T. LEVI-CIVITA, University of Padua, Italy, have been elected Fellows of the American Academy of Arts and Sciences.

At the Massachusetts Institute of Technology, Dr. JOSEPH LIPKA and Mr. F. B. HITCHCOCK have been promoted from instructors to assistant professorships of mathematics.

At Washington University Professor C. A. WALDO has retired from active service, and Professor W. H. ROEVER will attend the summer session at the University of Chicago.

Mr. T. McN. SIMPSON, JR., for a number of years in charge of mathematics at Converse College, Spartenburg, S. C., has been appointed instructor in mathematics at the University of Texas. Mr. K. W. LAMSON has been appointed instructor in mathematics at Columbia University. Mr. Simpson and Mr. Lamson have just finished their work for the doctorate at the University of Chicago.

The June issue of the *Annals of Mathematics* has appeared containing the following papers: "Fermat's last theorem and the origin and nature of the theory of algebraic numbers," by L. E. DICKSON; "The modified remainders obtained in finding the highest common factor of two polynomials," by A. J. PELL and R. L. GORDON; "Nomograms of adjustment," by L. I. HEWES; "Closed algebraic correspondences," by A. A. BENNETT; "The intersections of a straight line and hyperquadric," by J. L. COOLIDGE; "The relation between the zeros of a solution of a linear homogeneous differential equation and those of its derivative," by W. B. FITE; "Conjugate planar nets with equal invariants," by L. P. EISENHART. Beginning with the next volume, the *Annals* will be enlarged by 100 pages per volume, and will be in part supported by the Mathematical Association of America. The increase in the size of the volume will be devoted to historical and expository articles in so far as suitable material of this kind is available. The paper by Professor DICKSON in the present issue is an important contribution along this line.

The number of new subscribers to the *Annals of Mathematics* under the terms of cooperation with the Association is now (June 15) 338. The opportunity is still open at the half rate to members of the Association and to applicants for membership.

The National Research Council, formed by the National Academy of Science at the request of the President of the United States, is intended to bring into co-operation governmental, educational, industrial, and other research organizations, with the object of developing national resources and making them available. The chairman of the Council is Dr. George E. Hale, formerly Director of the Yerkes Observatory. Committees have been appointed, eighteen in number, representing all the interests included in the work of the Council. The chairmen

of the committees representing mathematics and physics are Professors E. H. MOORE and R. A. MILLIKAN, respectively, both of the University of Chicago. Professor Millikan has been on duty in Washington for several months.

The *Transactions of the American Mathematical Society*, Vol. 18, No. 2, contains the following papers: "Differential equations and implicit functions in infinitely many variables," by W. L. HART; "On the equivalence of écart and voisinage," by E. W. CHITTENDEN; "On the theory of associative division algebra," by Miss OLIVE C. HAZLETT; "The converse of the theorem concerning the division of a plane by an open curve," by J. R. KLINE; "On the conformal mapping of curvilinear angles," by G. A. PFEIFFER; "Dynamical systems with two degrees of freedom," by G. D. BIRKHOFF.

The Board of Trustees of the University of Chicago has voted to permit, upon recommendation by the head of a department, the attendance of doctors of philosophy of other universities as well as of the University of Chicago as guests of the University with the privilege of attending seminars and of carrying on research in the laboratories and libraries. For these privileges no charge will be made except for laboratory supplies and a nominal laboratory fee when laboratory work is to be done. This plan will be in operation beginning the summer session 1917, and some doctors in the department of mathematics are already taking advantage of the opportunity.

Twenty-seven members of the American Mathematical Society attended the regular meeting held at Columbia University on Saturday, April 28, at which twenty-six papers were presented. Professor L. P. EISENHART was reëlected a member of the editorial committee of the *Transactions*, and a special committee was appointed to make arrangements for the summer meeting of the Society at Cleveland, September 4–5, 1917. This will be the twenty-fourth summer meeting of the Society, and will be held at Adelbert College and Case School of Applied Science. The meeting will be followed by the second summer meeting of the Mathematical Association of America, on September 6–7.

The following mathematical papers were presented at the February and March meetings of the Paris Academy of Sciences: "The theory of convergence of Fourier's series," by W. H. YOUNG; "A simple solution of Mathieu's problem," by M. MESNAGER; "Hyperfuchsian functions and systems of total differentia' equations," by G. GIRAUD; "Characteristic number and radius of curvature,' by E. COTTON; "*Left* algebraic curves," by R. DE BALLORE; "The approximate value of some definite integrals," by M. HARNY; "A new table of divisors of numbers," by E. LEBON; "The reduction of binary forms of any degree," by G. JULIA; "Hyperfuchsian functions" (second paper), by G. GIRAUD; "The Abelian sum of conical volumes," by A. BUHL; "Deformable hyper-surfaces in a real Euclidean space of $n > 3$ dimensions," by E. BOMPIANI; "Summation of ultra spherical sines," by E. KOGBETLIANTZ.

Professor A. R. CRATHORNE, of the University of Illinois, was the leading speaker at the mathematics section of the fourteenth annual conference of high schools of Kansas held at the University of Kansas on March 16, 1917. The paper

by Professor CRATHORNE dealt with the plans of the Committee on Mathematical Requirements of the Mathematical Association of America and of allied organizations, and was therefore of great interest to all members of the Kansas Section of the Association.

At the meeting of the Philosophical Society of Cambridge, February 5, the following mathematical papers were presented: "The direct solution of the quadratic and cubic binomial congruences with prime moduli," by H. C. POCKLINGTON; "The hydrodynamics of relativity," by C. E. WEATHERBURN; "The character of the electric potential in electromagnetics," by R. HARGREAVES; "The fifth book of Euclid's elements," by Dr. M. J. M. HILL.

The *Teachers College Record*, May, 1917, contains a very interesting and important discussion of "Mathematics in the training for citizenship," by Professor DAVID EUGENE SMITH. This is an address that was delivered before the faculty of Teachers College, March 8, 1917. Professor Smith presents six very important reasons for the study of mathematics. These stated briefly are: (1) Mathematics is one of the small group of subjects that are linked up with a large number of branches of human knowledge; (2) it possesses a high value as a mental discipline; (3) it enables every one to experience the "power of thought"; (4) mathematics is one of the eternal verities; (5) it makes one conscious of his position in the universe about him; (6) its study gives humanity a religious sense that cannot be fully developed without it.

One of the charter members of the Association, Mr. H. S. CARD, assistant professor of mathematics and physics in Lombard College, Galesburg, Illinois, is now "doing his bit" as a sergeant in the First Regiment, U. S. Engineers. This regiment is for the present stationed at Washington Barracks, D. C., having only recently returned from service on the Mexican border.

It is desired that all members of the Association who are serving their country in connection with the war should be listed in this column of Notes and News. Special space will be devoted to this purpose in the September issue and it is hoped that all readers of the MONTHLY will take it upon themselves to report such information to the news editor, Professor D. A. ROTHROCK, Indiana University, Bloomington, Indiana.

Professor KARPINSKI writes that his statement in the January MONTHLY with reference to the first appearance of the decimal point: "But all standard authorities on this subject agree that the point or comma was first used by the German Pitiscus in the 1612 edition of his trigonometry," might be replaced with more precision by the statement: "Such standard authorities as Cantor, Tropfke, and D. E. Smith in his monograph on decimal fractions, mention explicitly the appearance in print of the decimal point in the Tables of the 1608 or 1612 edition of the Trigonometry by Pitiscus. Smith cites the 1612 edition which is available in the New York Public Library; and the first of three references made by Cantor to the use of the point by Pitiscus (*Vorlesungen über Geschichte der Mathematik*, Vol. II, second edition, 1900, p. 604; see also pages 619 and 733) cites only the 1612 edition, while the 1608 edition is cited in the other passages.

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There are other journals suited to the Secondary field, and there are still others of technical scientific character in the University field: but the MONTHLY is the only journal of Collegiate grade in America suited to the needs of the non-specialist in mathematics.

THE MATHEMATICAL ASSOCIATION OF AMERICA now has over eleven hundred individual members and over seventy-five institutional members. There are already nine sections formed, representing twelve different states. The Association has held so far two national meetings per year, one in September and one in December. The sections, for the most part, hold two meetings each year. All meetings, both national and sectional, are reported in the Official Journal, and many of the papers presented at these meetings are published in full.

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THE PRESENTATION OF THE NOTION OF FUNCTION.

By JOSEF A. NYBERG, Princeton, N. J.

In a previous article[1] I have shown how the work of the freshman year in colleges can be unified by presenting trigonometry, college algebra, and analytics from the point of view of the theory of functions. I argued further that, because the notion of functional independence is always considered somewhere in every one of the three courses, it would be advisable to treat it once for all at the very beginning of the year. Admitting this, I wish to show in the present article how the notion of function can be developed briefly, but so thoroughly that the student will think of it as something else besides axes, coördinates, and curves, and so comprehensively that the work will be useful regardless of whether the course is trigonometry, algebra, or analytics.

An examination of the textbooks shows that the usual procedure is about the following: (*a*) We have the explanation and illustration of the dependence of one variable upon another; (*b*) Denoting a value of one variable by x and of the other by y, we mark values of x and y on two perpendicular lines; (*c*) Assigning a value to x and computing the value of y, we determine a point in the plane, and the totality of these points defines a curve. Thereafter the student associates a curve with every equation in x, y, but he is most likely to think that the equation is a baptismal name of the curve. Another objection to this presentation is that coördinates are introduced too early, and that consequently the emphasis falls on "the point on the curve" instead of on "the two points on the axes." And further, after coördinates have been introduced, the essential idea of the relation between the two variables, represented now by abscissas and ordinates, is neglected.

The method I advocate aims to bring this idea of relationship between variables into the foreground. I would first present mathematics as a language

[1] The Unification of Freshman Mathematics, in this MONTHLY, April, 1916, page 101.

which facilitates the stating of these relations; second, I would emphasize the correspondence; and lastly, only after the notion of function and correspondence is grasped, would I explain the language of coördinates. Psychologically this can be the only correct order inasmuch as we present problems first and then study the common elements, coördinates, underlying the different problems. Under I, II, and III below, I sketch the process in so far as it differs from the conventional.

I. With the customary illustrations we show that the object of every branch of science is to find relations between variables. The doctor wishes to find the relation between the severeness of a typhoid epidemic and the number of bacteria in drinking water; the psychologist seeks to explain the relations between our variable experiences and emotions; the physicist learns experimentally how the length of an iron bar depends upon its temperature; the botanist relates plant forms to their climate; the astronomer finds that the size or mass of a planet influences the positions of other planets; the economists find that prosperity may depend upon business credit.

In order to state the relations in as concise a way as possible mathematicians have devised symbols (such as $+$, $-$, $\times$, $\div$, $=$, $\int$, ∂, $\surd$, ∇, $\sim$, ∞) which are so unique as to form almost a distinct language. The student must learn to use these symbols and be able to translate any statement in English words into these symbols much as he learns to translate English into French. Thus, the statement "one variable is always equal to twice another" is written mathematically "$y = 2x$," wherein y is a symbol for one variable and x is a symbol for the other. *The equation is a mathematical statement of an idea or law.* Its usefulness is due in part to its brevity and conciseness.

After stating the relation as simply as possible, mathematics aims to picture the relation by lines and curves. These pictures we call scales and graphs.

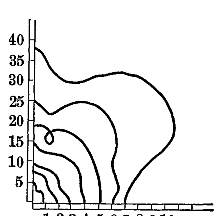

FIG. 1.

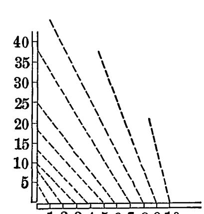

FIG. 2.

II. Assuming that the nature of scales has been explained, we can next place two scales parallel to each other, and then note that opposite numbers on the two scales correspond. Further, on these scales the *numbers* alone are the important things in the figure. In fact we use the figure only to show what number on one scale corresponds to another number on the other scale. Hence, we can do either of two things:

(*a*) Omitting the lines, and writing the numbers vertically, we can show the relation by what we call *data* or a table of corresponding values.

(*b*) We can place the two scales at right angles and show the correspondence, as in Figs. 1 or 2.

The teacher is then ready to explain Figs. 3 and 4. The student should be led to see the advantages of Fig. 4: from it we can approximate new corresponding values that would be impossible from a table of data or from Fig. 2 or 3. A figure like 1 may look ludicrous on the blackboard, and certainly no textbook has yet dared to contain one, but no one will deny that it illustrates a significant step in the evolution of the graph. The teacher must make sure that the student understands that the two lines (axes, as we later call them)

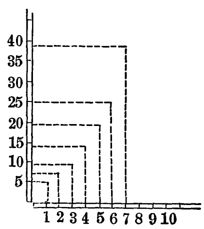

FIG. 3.

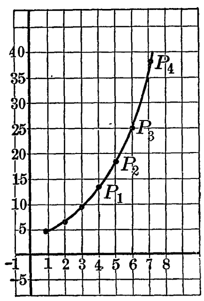

FIG. 4.

need not intersect at the zero point; that with a change of units the graph will look different, that if the problem hinges on the correspondence only over a particular range then the zeros may not even appear on either of the axes. All such matters as tend to emphasize *correspondence* should be considered at this point, and considered fully. It may even be useful to illustrate discontinuous correspondences, multiple valued ones, etc. Further, either here or before beginning II, the student should have some drill in finding functional relations, *i. e.*, stating by an equation a relation given descriptively. Problems involving motion or the measurement of physical magnitudes are the simplest and most useful.

If presenting the notion of function were all that we wish to do we could satisfactorily stop here. And we note that we are able to explain this concept without referring specifically to axes or coördinates. In truth, I believe the value of this mode of presentation lies in the very fact that coördinates and graphing have been kept in the background while the student is studying relations between variables. This is the reverse of what the textbooks do, for they consider coördinates first and then later, preliminary to locus problems, study functional relations. This reversal of order has probably arisen through the teaching of graphing in the high schools. Graphing is a very easy idea for the high school student to grasp; it enlightens the subject of simultaneous linear equations, and the teacher is glad to introduce a geometric idea into algebraic work. As a correlation of geometry and algebra such work is useful, but as an introduction to the notion of function it is misleading.

We can, however, now introduce coördinates in such a fashion that they will appear not at all as a *new* idea, but will appear as *that element which is common to all* the previous exercises. Pedagogically the introduction of coördinates at this stage is proper in that we now focus the student's attention on the underlying processes of his work. And since the student has been unconsciously using them, nothing further is needed except to define the technical terms by which the ideas will be referred to in the future.

III. We have used letters to denote one value from a set. We have axes, units, and points in the plane whose projections are the values which correspond. Conversely, every pair of corresponding values fixes a point in the plane. The numbers which determine the point we now choose to call coördinates. And thus we proceed to develop the rest of the terminology used in the study of relations $f(x, y) = 0$.

The work thus far can be done completely in three days, or even in two, depending upon how much the teacher wishes to drill on the various problems. All that is essential in explaining the notions of function, coördinates, graphing has been covered. If, however, an additional day is available, then some exercises involving the construction of such curves as $xy = 1$, $y = x^3$ are very profitable. While many papers have been published on the use of cross section paper for curve tracing I do not know that any one has pointed out its pedagogic value.

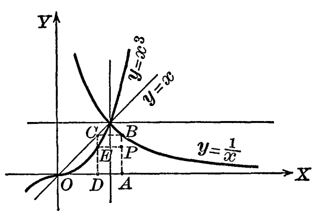

FIG. 5.

As a single illustration let us trace the thoughts of the student when asked for a method for constructing $x^3y = 1$ (see Fig. 5). He writes $y = 1/x^3$ and sees that he is dealing with the reciprocal of a variable and with the cube of a variable. Consequently he must first draw $y = 1/x$ and $y = x^3$. He selects any point A, writes $OA = x$, $AB = 1/x = DC = OD$, and finally $DE = \overline{OD}^3 = 1/x^3$. Consequently P is a point on the desired curve. The reader will see that this problem contains every idea associated with the notion of a function: the length of AB varies with the choice of A but varies according to a definite law; to cube AB it must be laid off horizontally from the origin; P and not E is the desired point on the curve because the ordinate is measured vertically from the end A of the abscissa; and finally, each equation of the problem states like a law the length of one line as compared with another. As the attention shifts from A to B to C, D, E, P the student must trace the fortunes of the variable as it is influenced by the different laws.

Whether the student thinks exactly in this fashion or not, this type of exercise is more valuable than the kind wherein he mechanically computes data for points and draws the curve through them. Even when the student can not himself discover the construction, nevertheless the subsequent reproduction of the teacher's method will be valuable if the student is required to give the reasons for each step. Also, this type of exercise more than any other affords good opportunity for questions by the teacher.

DIRECTED ANGLES AND INVERSION, WITH A PROOF OF SCHOUTE'S THEOREM.

By ROGER A. JOHNSON, Western Reserve University.

In this paper it is proposed to apply the method of directed angles[1] to the theory of inversion, especially to the use of the latter in studying properties of the triangle. None of the theorems below is new, but in most cases the form of presentation is new, and this form seems to be more definite and more easily workable than the usual statements.

We begin with a group of fundamental angle relations connected with the transformation of inversion; these are then applied to the triangle, yielding various interesting theorems. Thirdly, we state some of the best known properties of the so-called circles of Apollonius, and the circles of Schoute which are orthogonal to them, and finally we have a simple proof of a noted theorem of Schoute. This proof resembles to some extent a proof given by W. Gallatly,[2] but avoids some complexities. The original proof given by Schoute[3] was algebraic.

We recall the most useful properties of directed angles from the earlier paper:[4]

(*A*) The addition of directed angles follows the same rules as that of directed segments of a line. A useful identity is

$$\measuredangle\, l_1, l_2 + \measuredangle\, l_3, l_4 = \measuredangle\, l_1, l_4 + \measuredangle\, l_3, l_2.$$

(*B*) Three points A, B, C are collinear if and only if

$$\measuredangle\, ACB = 0$$

or, in other words, if for any other point D

$$\measuredangle\, ACD = \measuredangle\, BCD.$$

(*C*) Four points A, B, C, D lie on a circle or line if and only if

$$\measuredangle\, ABC = \measuredangle\, ADC.$$

(*D*) The Miquel formula for the angles of the pedal triangle of a point P with regard to triangle $A_1A_2A_3$. If PP_1, PP_2, PP_3 are the $\perp$s to the sides, then

$$\measuredangle\, A_2PA_3 = \measuredangle\, P_2P_1P_3 + \measuredangle\, A_2A_1A_3, \text{ etc.}$$

[1] See a paper by R. A. Johnson, *Directed Angles in Elementary Geometry*, this MONTHLY, Vol. XXIV, page 101, 1917.

[2] *The Modern Geometry of the Triangle*, Chap. VIII.

[3] Schoute, *Over een neawer verband tusschen hoek en cirkel van Brocard*, Amsterdam Acad. Proceedings, Series III, vol. 3, 1887, page 22; see Coolidge, *Geometry of the Circle and the Sphere*, page 127.

[4] One who desires to extract the essence of the present note without using the method of directed angles may proceed by drawing in each case a figure, and interpreting any statement of the form $\measuredangle\, x = \measuredangle\, y$ as meaning that the angles x and y of the figure are either equal or supplementary or differ by 180°.

Consider now an inversion with regard to a fixed circle with center O, radius r.

THEOREM 1. *Two pairs of inverse points are either collinear with O, or concyclic on a circle orthogonal to the circle of inversion.*

For

$$\overline{OP} \cdot \overline{OP'} = \overline{OQ} \cdot \overline{OQ'} = r^2.$$

THEOREM 2. *Triangles OPQ, $OQ'P'$ are inversely similar.*

For angle O is common; and by (C) and theorem 1 we have

$$\measuredangle OPQ = \measuredangle P'Q'O, \qquad \measuredangle PQO = \measuredangle OP'Q'.$$

COROLLARY. *The distance between two points, and the distance between their inverses, are connected by*

$$\overline{P'Q'} = \overline{QP}\frac{r^2}{\overline{OP} \cdot \overline{OQ}}.$$

THEOREM 3. *If P, Q, R are any three points, P', Q', R' their inverses,*

$$\measuredangle QPR + \measuredangle Q'P'R' = \measuredangle QOR.$$

Proof.

$$\measuredangle QPO = \measuredangle OQ'P',$$
$$\measuredangle OPR = \measuredangle P'R'O.$$

Adding,

$$\measuredangle QP, PO + \measuredangle PO, PR = \measuredangle OQ', Q'P' + \measuredangle Q'R', R'O,$$

or, applying (A),

$$\measuredangle QPR = \measuredangle R'P'Q' + \measuredangle Q'OR',$$
$$\measuredangle QPR + \measuredangle Q'P'R' = \measuredangle QOR.$$

This formula is exceedingly powerful; in fact, it may be ranked in point of importance with the Miquel formula (D), which it resembles in form. From it many of the important properties of inversion are immediate corollaries.

COROLLARY 1. *For any four points and their inverses,*

$$\measuredangle PQR + \measuredangle RSP = \measuredangle R'Q'P' + \measuredangle P'S'R'.$$

COROLLARY 2. *The inverse of a circle not passing through the center of inversion is also a circle.*

COROLLARY 3. *The inverse of a straight line not through the center of inversion is a circle through that point, and conversely.*

COROLLARY 4. *The angle of intersection of two circles is the same as that of the inverse circles. This is true also for any intersecting curves.*

We turn now to the triangle.

THEOREM 4. *If two points are mutually inverse with regard to the circumcircle of a triangle $A_1A_2A_3$, their pedal triangles are inversely similar.*

For each vertex of the triangle is self-inverse, hence we have from Theorem 3,

$$\measuredangle A_2PA_3 + \measuredangle A_2P'A_3 = \measuredangle A_2OA_3.$$

Applying (D),

$$\measuredangle A_2A_1A_3 + \measuredangle P_2P_1P_3 + \measuredangle A_2A_1A_3 + \measuredangle P_2'P_1'P_3' = \measuredangle A_2OA_3.$$

But

$$2\measuredangle A_2A_1A_3 = \measuredangle A_2OA_3;$$

hence

$$\measuredangle P_2P_1P_3 = \measuredangle P_3'P_1'P_2'.$$

Since all the angles of the pedal triangles satisfy such equations, the triangles are inversely similar.

We may regard as a corollary to this theorem the fact (otherwise much more easily proved) that the pedal triangle of a point on the circumcircle reduces to a line.

THEOREM 5. *If an inversion is performed with any center P, the inverses B_1, B_2, B_3 of any three points A_1, A_2, A_3 form a triangle similar to the pedal triangle of P with regard to $\triangle A_1A_2A_3$.*

For since

$$\measuredangle A_2PA_3 = \measuredangle A_2A_1A_3 + \measuredangle B_2B_1B_3 \qquad \text{(Theorem 3)}$$

$$\measuredangle A_2PA_3 = \measuredangle A_2A_1A_3 + \measuredangle P_2P_1P_3 \qquad (D)$$

it follows that

$$\measuredangle P_2P_1P_3 = \measuredangle B_2B_1B_3, \quad \text{etc.}$$

COROLLARY. *If a triangle is subjected to an inversion with regard to two points which are inverse with regard to the circumcircle, the resulting triangles are inversely similar.*

THEOREM 6. *There is a single point which may be taken as center of inversion, in order that a given triangle may be transformed into a triangle similar in a given sense to a given triangle.*

For the center of inversion P is located uniquely by the equations

$$\measuredangle A_2PA_3 = \measuredangle A_2A_1A_3 + \measuredangle B_2B_1B_3,$$

$$\measuredangle A_3PA_1 = \measuredangle A_3A_2A_1 + \measuredangle B_3B_2B_1.$$

An exceptional case arises when the given triangles are inversely similar.

THEOREM 7. *If four points are subjected to an inversion, the pedal triangle of one of the points with regard to the other three is inversely similar to the corresponding pedal triangle for the inverted points.*

Let an inversion carry A_1, A_2, A_3, P into B_1, B_2, B_3, Q respectively; and let the pedal triangle of P with regard to $A_1A_2A_3$ be $P_1P_2P_3$, and that of Q with regard to $B_1B_2B_3$ be $Q_1Q_2Q_3$. Then

$$\measuredangle A_2PA_3 + \measuredangle B_2QB_3 = \measuredangle A_2OA_3,$$

$$\measuredangle A_2A_1A_3 + \measuredangle P_2P_1P_3 + \measuredangle B_2B_1B_3 + \measuredangle Q_2Q_1Q_3 = \measuredangle A_2OA_3.$$

But

$$\measuredangle A_2A_1A_3 + \measuredangle B_2B_1B_3 = \measuredangle A_2OA_3$$

so that

$$\measuredangle P_2P_1P_3 + \measuredangle Q_2Q_1Q_3 = 0,$$

as was to be proved.

We pass now to the circles of Apollonius and the isodynamic points. The locus of a point whose distances from two vertices of a triangle are proportional to the adjacent sides, is a circle through the third vertex. In any triangle there are three such circles, called the circles of Apollonius. They are well known, and we shall merely mention a few of their properties. It is easily proved that the center of the circle through A_1 lies on A_2A_3; that the circle passes through the feet of the bisectors of angle A_1; and that the tangent to the circumcircle at A_1 passes through its center. The three circles intersect at two points, called isodynamic points; the distances of these from the sides of the triangle are inversely as the lengths of these sides.

THEOREM 8. *If two points are mutually inverse with regard to the circle of*

THEOREM 11. (SCHOUTE'S theorem) *In any triangle, the locus of a point whose pedal triangle has a constant Brocard angle is a circle coaxial with the circumcircle and the Brocard circle.*

The proof is effected by performing the inversion of theorem 9, noting theorem 7, and applying Theorem 10.

COROLLARY. *In particular, the locus of a point whose pedal triangle has the same Brocard angle (in the same sense) as the given triangle is the Brocard circle; and for the same Brocard angle but with the triangle described in the opposite sense, the locus is the Lemoine line.*

For, in the first place, the Brocard circle contains a point O, whose pedal triangle is similar to the given triangle; and further, the points of this circle and of the Lemoine line are mutually inverse with regard to the circumcircle.

All previous proofs by elementary methods, even of the corollary, have made use of a very awkward method of vertical projections from one plane to another (cf. Gallatly, l. c., or Emmerich, Die Brocard'sche Gebilde, §§ 60, 61). It will be noted that most of the steps of the present proof may fairly be regarded as general theorems, very little special proof being required for the theorem itself.

A SIMPLE RELATION BETWEEN ELEMENTARY NUMBER-THEORY AND ELEMENTARY PROJECTIVE GEOMETRY.

By AUBREY J. KEMPNER, Urbana, Ill.

In *L'Enseignement Mathématique,* 1916, p. 332, Mr. A. Reymond has published a note in which, with slight modifications, the following simple graphical method of determining whether a given integer is a prime number, is explained.

Write in a vertical and in a horizontal row the numbers 0, 1, 2, 3, 4, $\cdots$, as

0	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	2								
3	1		3							
4	1	2		4						
5	1				5					
6	1	2	3			6				
7	1						7			
8	1	2		4				8		
9	1		3						9	
10	1	2			5					10

FIG. 1.

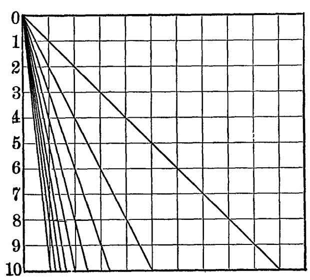

FIG. 2.

far as one likes, and fill out Fig. 1 in the way indicated. Evidently every number in our first vertical row, for example 6, will have for factors just the numbers which are in the same horizontal row, in this case 1, 2, 3, 6. Here the number 1 and the number 6 are both considered factors of 6. The number 7 has only the

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5	1				5					
6	1	2	3			6				
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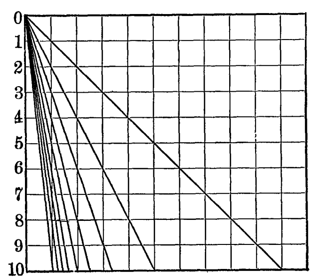
8	1	2		4				8		
9	1		3						9	
10	1	2			5					10

FIG. 1.



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far as one likes, and fill out Fig. 1 in the way indicated. Evidently every number in our first vertical row, for example 6, will have for factors just the numbers which are in the same horizontal row, in this case 1, 2, 3, 6. Here the number 1 and the number 6 are both considered factors of 6. The number 7 has only the

two factors 1 and 7 and is therefore prime. In general, a number is prime when, and only when, in the same horizontal row with it, there are exactly two numbers. (The number 0, which contains every integer $\neq 0$ as a factor, and the number 1, are exceptional, as can be read off from Fig. 1.)

The method by which Fig. 2 is derived from Fig. 1 is obvious and requires no explanation. If we call the points in Fig. 2, in which exactly three straight lines intersect, knot-points, for brevity, then we see that a number is a prime number when and only when the horizontal line belonging to it in the figure has exactly two knot-points (excepting the special numbers 0, 1). In general, the number of distinct divisors of n (including 1 and n among the divisors) is exactly equal to the number of knot-points lying on the corresponding horizontal line.

For practical purposes, the construction is without value (and this quite apart from the fact that its application to any but small numbers becomes impossible on account of the complexity of the figure), because we have no means of deciding, in an empirical drawing, whether three lines meet in a point, or in three points forming the vertices of a small triangle. Also, as Mr. Reymond points out, it seems unlikely that results of theoretical interest may be obtained by a closer study of the construction (which is essentially a graphical interpretation of the method known under the name "Sieve of Eratosthenes." See any book on elementary number-theory). In spite of these objections, the construction is of interest as bringing out clearly the fact that the problem of deciding whether a given integer is a prime number, may be graphically solved by means of ruler and compasses. When one remembers, however, that only rational operations are required in order to determine the prime character of a number n (trial-divisions by integers less than n; more precisely trial-divisions by all prime numbers equal to or less than the square root of n) and even to determine all of its factors, it is clear that a linear construction with straight-edge[1] alone should be possible.

For justification of this statement, in case it should not be evident, see, for example, Weber-Wellstein, referred to below, pp. 210, 211.

To obtain such a construction, we shall only have to construct a figure perspective to Fig. 2. For readers not familiar with the idea of a projective scale, an explanation and references are given.

I. *Projective Scales.*[2] Consider an ordinary metric scale with a set of equidistant points representing for example the positive integers and 0. We wish

[1] Ruler without scale and without marks of any kind; an instrument used solely for drawing straight lines.

[2] See for example: YOUNG, "The theory of sets of points," 1906, p. 9, or WEBER-WELLSTEIN, "Encyklopädie der Elementar-Mathematik," Vol. II, 2d ed., 1907, p. 198. STEINER ("Die geometrischen Constructionen ausgeführt mittelst der geraden Linie und eines festen Kreises," Berlin, 1833. See "Gesammelte Werke," Vol. I, p. 499, or "Ostwald's Klassiker, Nr. 60," Chapter 3) showed that it is possible to carry out all constructions which may be made by use of straight-edge and compasses (equivalent to graded ruler and compass, provided a fixed unit is given) by means of a straight-edge alone, provided one fixed circle and its center are given. D. CAUER (*Mathematische Annalen*, 1912–13, Vol. 73, p. 90) proved that the center of the circle cannot be dispensed with.

to construct an equivalent projective scale. Introducing on our ordinary scale the point at infinity (∞) as representing the ∞ of real numbers (where no distinction is made between $+\infty$ and $-\infty$, the assumption being made in projective geometry that each straight line contains *one* point at infinity), we may interpret our metric scale in the following manner:
Forming the cross-ratio

$$(a_1, a_2, a_3, \infty) = \frac{a_1 - a_2}{a_3 - a_2} : \frac{a_1 - \infty}{a_3 - \infty},$$

we find

$$(a_1 a_2 a_3 \infty) = -1,$$

when a_1, a_2, a_3 are any three equidistant points, that is, when $a_3 - a_2 = a_2 - a_1$, since $(a_1 - \infty)/(a_3 - \infty)$ must be counted equal to unity, as one sees by a simple limiting process. Hence the four points a_1, a_2, a_3, ∞, taken in this order, are harmonic points. We make for the construction of a projective scale the following assumptions. Instead of the point at infinity on our metric scale, we choose on our line an arbitrary point u, which shall play on our projective scale the rôle of the point at infinity on our ordinary scale; we accordingly agree to consider three points a_1, a_2, a_3 as equidistant, that is, $a_3 - a_2 = a_2 - a_1$, when the four points a_1, a_2, a_3, u are harmonic points. By repeatedly using the classical construction for the determination of the fourth harmonic point when three points are given we see the truth of the following theorem:

When we are given on a line three points, a_1, a_2, u, which shall represent, respectively, 0, 1, ∞, then it is possible to construct as far as one likes by means of straight-edge alone[1] a projective scale containing the points representing 0, 1, 2, 3, 4, 5, $\cdots$.[2] The construction of the scale is unique, when 0, 1, ∞ are arbitrarily given; this fact may also be expressed by saying that a projective scale is completely determined by three of its points.

It is also possible to obtain, by straight-edge construction, a projective scale containing as many "rational" points as one likes, but we shall need only the points corresponding to integers. Likewise, it is beyond our purposes to introduce the "negative" numbers of the projective scale. For all this, and for the important question as to what assumptions must be made to ensure continuity in the projective plane, see either Young or Weber-Wellstein, referred to above.

For measurements in projective geometry, these projective scales usually take the place of ordinary scales in metric geometry. The ordinary metric scale is obviously a very special projective scale, obtained by assuming u at infinity and taking the length $a_1 \cdots a_2$ for the unit of length. Note that if 0, 1, 2, 3, 4, $\cdots$, form a projective scale, then also 0, 2, 4, 6, 8, $\cdots$ form a projective scale,

[1] For the construction of the ordinary metric scale, where u is actually the point at infinity on the line, compasses cannot be avoided. Compare, however, the reference to Steiner given above.

[2] We shall generally call these points simply 0, 1, 2, 3, $\cdots$, or $0'$, $1'$, $2'$, $3'$, $\cdots$, or $0''$, $1''$, $2''$, $3''$, $\cdots$, while the point u will be denoted by ∞, or ∞', or ∞''.

the ray $B(1, 1')$. This ray intersects AC in a point which we call $1''$. Construct the projective scale $0''$, $1''$, $2''$, $3''$, $4''$, $\cdots$ on AC, taking $A = 0''$, $C = \infty''$, and draw the rays $B2''$, $B3''$, $B4''$, $\cdots$. We thus have in our triangle a set of rays from each vertex.[1] Mark in the figure all points in which three rays meet. These points we shall call, for convenience, knot-points. By comparing Fig. 3 with Fig. 1 and Fig. 2, we find:

The number r of knot-points on An, $n = 2, 3, 4, 5, \cdots$, gives the number of distinct divisors of n, counting both 1 and n as divisors. Therefore n is prime, when and only when there are two knot-points on An. When $r = 3$, n is the square of a prime number, and, in a similar way, some other elementary results (for example the determination of common prime factors of given integers) may be read off the figure.—Readers familiar with projective geometry will recognize immediately how Fig. 3 may be derived from Fig. 2.

Other projective geometric constructions for determining the prime-character of a given number are easily devised, but the construction given above has the advantage of involving only straight lines.

When one ignores simple relations like the one treated in this note, between projective geometry and number theory, which in reality are only analogies without deeper import, the problem of finding connections between projective geometry and number theory seems to be, according to an interesting remark by Klein (Elliptische Modulfunktionen, Vol. I, p. 242, footnote), a task of great difficulty. (See also Weber-Wellstein, II, 1907, pp. 211–212, footnote.)

CONCERNING PREFERENTIAL VOTING.

By L. L. DINES, University of Saskatchewan.

In a recent number of the MONTHLY, Professor W. V. Lovitt has given an interesting discussion of a problem which may be stated as follows:

In a certain election, each voter expresses his first, second, and third choice. The three candidates A, B, and C receive respectively A_i, B_i, and C_i votes for ith choice. It is required to determine what weights may be assigned to the votes for first, second, and third choice in order that A may win.

While the purely algebraic discussion given in the present note is less vivid than the geometric discussion of Professor Lovitt, it may be of interest as it leads more directly to the necessary conditions for the existence of a solution, and to explicit ranges of possible values for the weights in terms of the data A_i, B_i, and C_i. Furthermore, the algebraic method can be extended to a treatment of the more general problem in which there are n candidates, and each voter

[1] The $\triangle ABC$ now represents, for purposes of projective measurement, the whole first quadrant in an ordinary system of coördinates. It is interesting to see how, by taking on the sides of our triangle (extending each indefinitely in both directions) the "negative" and the "rational" points into account, the whole ordinary plane is covered by a "projective" plane.

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expresses his first, second, $\cdots$, and mth choice. This more general problem is considered briefly in § 2.

§ 1. Let the weights of first, second, and third choice be x_1, x_2, and x_3 respectively, with the condition

$$x_1 > x_2 > x_3 > 0. \tag{1}$$

Then since the number of points received by A is given by $A_1x_1 + A_2x_2 + A_3x_3$, with similar expressions for the number of points received by B and C, it follows that the necessary and sufficient conditions that A shall win are expressed by the inequalities

$$\begin{aligned} &(A_1 - B_1)x_1 + (A_2 - B_2)x_2 + (A_3 - B_3)x_3 > 0, \\ &(A_1 - C_1)x_1 + (A_2 - C_2)x_2 + (A_3 - C_3)x_3 > 0. \end{aligned} \tag{2}$$

These inequalities may be simplified by introducing instead of the weights, the differences between the weights, by the substitutions

$$x_1 - x_2 = \xi_1, \quad x_2 - x_3 = \xi_2, \quad x_3 = \xi_3,$$

or

$$x_1 = \xi_1 + \xi_2 + \xi_3, \quad x_2 = \xi_2 + \xi_3, \quad x_3 = \xi_3,$$

with the condition (1) replaced by the conditions

$$\xi_i > 0 \qquad i = 1, 2, 3. \tag{1'}$$

Expressed in terms of the ξ's, conditions (2) are

$$(A_1 - B_1)(\xi_1 + \xi_2 + \xi_3) + (A_2 - B_2)(\xi_2 + \xi_3) + (A_3 - B_3)\xi_3 > 0,$$

$$(A_1 - C_1)(\xi_1 + \xi_2 + \xi_3) + (A_2 - C_2)(\xi_2 + \xi_3) + (A_3 - C_3)\xi_3 > 0;$$

which upon collecting terms in ξ_1, ξ_2, and ξ_3, and making use of the relations

$$A_1 + A_2 + A_3 = B_1 + B_2 + B_3 = C_1 + C_2 + C_3 = \text{number of voters,} \tag{3}$$

may be written

$$(A_1 - B_1)\xi_1 > (A_3 - B_3)\xi_2 \tag{4}$$

$$(A_1 - C_1)\xi_1 > (A_3 - C_3)\xi_2. \tag{5}$$

Since ξ_3 does not appear in these condition, ξ_3 $(= x_3)$ may have any positive value. Also since (4) and (5) obviously impose a restriction only on the ratio $\xi_1 : \xi_2$, ξ_2 $(= x_2 - x_3)$ may have any positive value. It remains to determine the ranges of values which ξ_1 may have when conditioned by (4) and (5). The nature of these conditions is made more apparent by separating them into cases according to the value of the coefficient of ξ_1, as follows:

(4′) (a) If $A_1 - B_1 > 0$, $\xi_1 > \dfrac{A_3 - B_3}{A_1 - B_1}\xi_2$; (b) If $A_1 - B_1 = 0$, $(A_3 - B_3)\xi_2 < 0$;

(c) If $A_1 - B_1 < 0$, $\xi_1 < \dfrac{A_3 - B_3}{A_1 - B_1}\xi_2$;

(5′) (a) If $A_1 - C_1 > 0$, $\xi_1 > \dfrac{A_3 - C_3}{A_1 - C_1}\xi_2$; (b) If $A_1 - C_1 = 0$, $(A_3 - C_3)\xi_2 < 0$;

(c) If $A_1 - C_1 < 0$, $\xi_1 < \dfrac{A_3 - C_3}{A_1 - C_1}\xi_2$.

In considering the possible order relations which may exist between the three quantities A_1, B_1, C_1, those which differ from each other only in having B_1 and C_1 interchanged may for our purposes be considered equivalent. With this understanding, the following table which can now be filled out by reference to the conditions (4′), (5′), and (1′), gives the complete solution of our problem for all cases.

Case	Necessary Conditions	Range for ξ_1
$A_1 > B_1 \geqq C_1$	....	$\xi_1 > \left\{\dfrac{A_3 - B_3}{A_1 - B_1}\xi_2 \text{ and } \dfrac{A_3 - C_3}{A_1 - C_1}\xi_2\right\}$
$A_1 = B_1 > C_1$	$A_3 < B_3$*	$\xi_1 > \dfrac{A_3 - C_3}{A_1 - C_1}\xi_2$
$A_1 = B_1 = C_1$	$A_3 < B_3$*, $A_3 < C_3$*	ξ_1 any positive value
$C_1 > A_1 > B_1$	$\dfrac{A_3 - B_3}{A_1 - B_1} < \dfrac{A_3 - C_3}{A_1 - C_1}$, $A_3 < C_3$	$\dfrac{A_3 - B_3}{A_1 - B_1}\xi_2 < \xi_1 < \dfrac{A_3 - C_3}{A_1 - C_1}\xi_2$
$C_1 > A_1 = B_1$	$A_3 < B_3$*, $A_3 < C_3$	$\xi_1 < \dfrac{A_3 - C_3}{A_1 - C_1}\xi_2$
$B_1 \geqq C_1 > A_1$	$A_3 < B_3$, $A_3 < C_3$	$\xi_1 < \left\{\dfrac{A_3 - B_3}{A_1 - B_1}\xi_2 \text{ and } \dfrac{A_3 - C_3}{A_1 - C_1}\xi_2\right\}$

It may be noted that on account of the relations (3), the conditions $A_3 < B_3$ and $A_3 < C_3$ when marked with a * in the column of necessary conditions, may be replaced respectively by $A_2 > B_2$ and $A_2 > C_2$.

§ 2. Suppose now there are $n + 1$ candidates: A, and $B^{(i)}$ $(i = 1, 2, \cdots, n)$; and the jth choice is to receive the weight x_j $(j = 1, 2, \cdots, m)$, with the condition $x_k > x_{k+1} > 0$. Then the necessary and sufficient conditions that A be elected are

$$\sum_{j=1}^{m} (A_j - B_j^{(i)})x_j > 0, \qquad i = 1, 2, \cdots, n. \tag{6}$$

If we now introduce as new variables the differences between the weights, by means of the substitutions

$$x_j = \sum_{h=j}^{m}\xi_h, \qquad j = 1, 2, \cdots, m,$$

we obtain in place of (6) the conditions

$$\sum_{j=1}^{m} (A_j - B_j^{(i)}) \sum_{h=j}^{m} \xi_h > 0, \qquad i = 1, 2, \cdots, n,$$

determine certain lower and upper bounds for ξ_2, while those of Type II, together with the conditions that the range for ξ_2 shall contain positive values, will constitute a system of linear, homogeneous inequalities in $\xi_3, \xi_4, \cdots, \xi_m$.

The process indicated can now be repeated, the ranges for $\xi_3, \xi_4, \cdots$ being determined successively, each in terms of the ξ's of higher subscript. After the first $(r - 1)$ ξ*'s* have been thus *eliminated,* we are met by a system of conditions of form

$$(10) \qquad \sum_{h=r}^{m} a_h^{(l)}\xi_h > 0, \qquad l = 1, 2, \cdots, l_r,$$

where the coefficients $a_h^{(l)}$ are rational functions of the differences $A_j - B_j^{(i)}$ This system (10) may be classified into Types I, II, and III, according as the leading coefficients $a_r^{(l)}$ are positive, zero, or negative.

A necessary and sufficient condition that there exist a system of weighting under which A wins, is that at some stage of the process of successive elimination of the variables ξ_i described above, the inequalities of the system (10) *presenting itself shall be all of Type I.*

To show the necessity of the condition, we suppose that the system of conditions presenting itself at each stage of the elimination contains inequalities of Types II or III. Since the elimination of the ξ of lowest subscript from such a system leads to a system of conditions upon the ξ's of higher subscript, it follows that under our assumption, the successive elimination of the ξ*'s* will lead finally to a system of conditions

$$a_m^{(l)}\xi_m > 0, \quad l = 1, 2, \cdots, l_m,$$

which contains only ξ_m. But according to our assumption not all the coefficients $a_m^{(l)}$ are positive. Hence the system of conditions cannot be satisfied by a positive ξ_m.

If, on the contrary, the inequalities (10) at any stage are all of Type I, they determine only *lower* bounds for ξ_r. The variables $\xi_{r+1}, \xi_{r+1}, \cdots, \xi_m$ can then have any positive values, ξ_r can have any positive values greater than these lower bounds, and positive values can be assigned to $\xi_{r-1}, \xi_{r-2}, \cdots, \xi_1$, in order, satisfying the conditions imposed in the elimination process.

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

The Continuum and Other Types of Serial Order. By EDWARD V. HUNTINGTON. Second Edition. Harvard University Press, Cambridge, Mass., 1917.

It is to some extent customary for younger students of our subject to cultivate skill in the use of great mathematical doctrines without perceiving the logical setting, in mathematical and physical science, of the complex and real number systems upon which most of these doctrines depend. /And, indeed, we can all

determine certain lower and upper bounds for ξ_2, while those of Type II, together with the conditions that the range for ξ_2 shall contain positive values, will constitute a system of linear, homogeneous inequalities in $\xi_3, \xi_4, \cdots, \xi_m$.

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$$\sum_{h=r}^{m} a_h^{(l)}\xi_h > 0, \qquad l = 1, 2, \cdots, l_r, \tag{10}$$

where the coefficients $a_h^{(l)}$ are rational functions of the differences $A_j - B_j^{(i)}$ This system (10) may be classified into Types I, II, and III, according as the leading coefficients $a_r^{(l)}$ are positive, zero, or negative.

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the treatment of continuously ordered series, with the emphasis placed on such types as possess an "n-dimensional framework" is noteworthy. It will stimulate better students to reflect on the significance, for any analysis based on a continuously ordered number system, of the presence or absence in that system of segments which are "denumerably framed." The relation of the classical analysis and function theory, which repose ultimately on the system of real numbers, to the distinctive presence in that system of denumerable dense sub-classes will be suggested; a relation responsible for all properties dependent upon the Generalized Heine-Borel Theorem for instance. For a similar situation outside of order theory, the student may consult Hausdorff's fascinating chapters on the "topological" (including the metric) geometries of general classes (Mengen) in his *Grundzüge der Mengenlehre*.

The first edition of the book under review appeared in 1905 as a reprint from two of the author's papers in the *Annals of Mathematics*. In the present edition (1917) the changes made include only a few modifications and additions. The account given of Hartog's paper is of course new. See Discussions, page 345, of this issue.

LESTER S. HILL.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

ALGEBRA.

485. Proposed by J. WALSH, Madison, Wisconsin.

Is it true that to every convergent series of positive terms, $a_1 + a_2 + a_3 + \cdots$, there corresponds a series of the type

$$\frac{M}{1^p} + \frac{M}{2^p} + \frac{M}{3^p} + \cdots, \quad \text{such that} \quad \frac{M}{k^p} > a_k, \ p > 1?$$

486. Proposed by FLORENCE P. LEWIS, Goucher College, Baltimore, Md.

Find the condition which must be satisfied by the coefficients of the quartic,

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$$

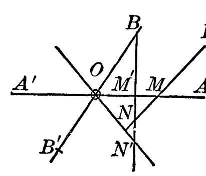
in order that the equation may be solvable by successive applications of the quadratic formula.

GEOMETRY.

518. Proposed by ROGER A. JOHNSON, Cleveland, Ohio.

If one angle of a triangle is 60°, the Euler line (the line through the circumcenter, orthocenter, and median point) is perpendicular to the bisector of that angle; and if one angle is 120°, the Euler line is parallel to the bisector of that angle.

519. Proposed by OTTO DUNKEL, Washington University.



Given the conjugate axes $A'OA$ and $B'OB$ of an ellipse, points of the curve may be constructed as follows: Drop the perpendicular BM' to OA and produce it to N' so that $BN' = AO$. Draw a straight line through O and N'. Upon a straight edge, say that of a slip of paper, the points N, M and P are marked so that $NM = N'M'$ and $MP = M'B$. Place the straight edge so that N falls on ON' and M on OM' and mark the position of P. This gives a point of the ellipse and by sliding the straight edge into new positions other points may be rapidly obtained. If the axes are perpendicular this gives the familiar trammel construction. Prove the correctness of this construction.

CALCULUS.

433. Proposed by LOUIS O'SHAUGHNESSY, University of Pennsylvania.

Solve the differential equation,

$$\frac{d^{1/2}y}{dx^{1/2}} - \frac{y}{x} = 0.$$

434. Proposed by E. W. CHITTENDEN, Champaign, Ill.

Evaluate $\int_0^1 f(x)dx$ where $f(x)$ is defined by the formula $f(x) = \sum_{n=1}^{n=\infty} \frac{1}{n^2}$, signum $(x - x_n)$, and x_n denotes the nth number in the series $\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \cdots$.

Note.—For a real number k, "signum k," denotes $+1, 0, -1$ according as $k >, =, < 0$.

MECHANICS.

350. Proposed by J. B. REYNOLDS, Lehigh University.

If an elastic tube filled with liquid under pressure doubles in length, in what ratio will the radius be increased?

351. Proposed by G. PAASWELL, New York City.

A transition curve is one such that its curvature varies directly with the distance measured along the curve from its point of zero curvature, that is, from the tangent. Its intrinsic equation is given by $da/ds = ks$, the constant being determined from the fact that for a given length of transition the final radius of curvature, *i. e.*, the radius of the circle into which the transition runs, is given together with the length of transition. In making a turnout from the transition curve there is as yet no direct way of computing the functions which would completely locate this turnout. In Figure 1, the point of switch is at B and the frog point at C. The angle F is given, termed the frog angle, and either the location of C or B is given, whence it is required to find either B or C and the radius of the turnout. Note that all these data must be referred finally to the center lines of the tracks and not to the individual rails. In attempting approximate solutions do not replace the transition by the cubic parabola as that is not always a good approximation. (See Crandall, *The Transition Curve* for the discussion of the properties of this curve.)

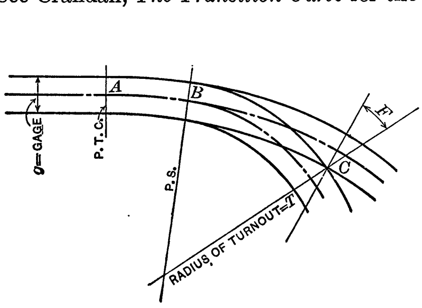


FIG. 1.

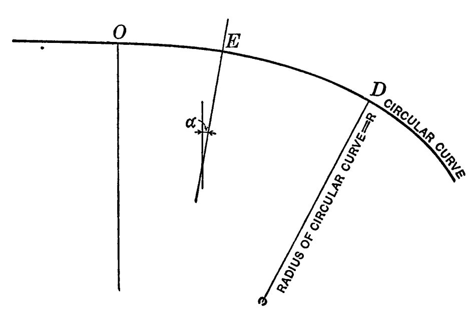


FIG. 2.

NUMBER THEORY.

268. Proposed by FRANK IRWIN, University of California.

Show that in any arithmetical progression, whose first term a_1 and common difference d are positive integers, any required number of consecutive terms may be found, no one of which is a prime number.

269. Proposed by ARTEMAS MARTIN, Washington, D. C.

Find three rectangular parallelepipedons whose edges are rational whole numbers, and their solid diagonals equal, and rational whole numbers.

SOLUTIONS OF PROBLEMS.

GEOMETRY.

504. Proposed by NATHAN ALTSHILLER, University of Oklahoma.

The base of a variable triangle is fixed, the opposite vertex describing a given line. Find the envelope of the side of the pedal triangle opposite the moving vertex.

SOLUTION BY SHELTON P. SANFORD, University of Georgia.

Let the base of the triangle be the line joining the points (0, 0) and $(n, 0)$. Let the given line be (1) $y = mx + b$.

Then one side of the triangle will be (2) $y = ax$.

Solving these equations, we have $x = b/(a - m)$; $y = ab/(a - m)$.

This point of intersection and the point $(n, 0)$ determine the other side of the triangle, (3) $y = abx/H - abn/H$, where $H = b - an + mn$.

The altitude from the point $(n, 0)$ to the line $y = ax$ is (4) $y = -x/a + n/a$.

The altitude from the origin to the line $y = abx/H - abn/H$ is (5) $y = -Hx/ab$.

Solving (2) and (4), we have $x = n/(a^2 + 1)$; $y = an/(a^2 + 1)) \equiv P_1$.

Solving (3) and (5), we have

$$\left.\begin{aligned} x &= a^2b^2\cdot n/(a^2b^2 + H^2) \\ y &= -abnH/(a^2b^2 + H^2) \end{aligned}\right\} \equiv P_2.$$

These two points of intersection (P_1 and P_2) are sufficient to determine the required side of the pedal triangle, which is found to be

$$y = ax(H + b)/(H - a^2b) - abn/(H - a^2b).$$

Substituting for H its value $(b - an + mn)$ and arranging according to powers of the parameter a, we have

$$a^2(nx - by) + a(bn - 2bx - mnx - ny) + (b + mn)y = 0.$$

Differentiating with respect to the parameter and eliminating, we have

$$x^2(4b^2 + 4bmn + m^2n^2) + y^2(4b^2 + 4bmn + n^2) - 2xymn^2 - x(4b^2n + 2mn^2) - 2ybn^2 + b^2n^2 = 0.$$

This equation represents a conic with one of its axes parallel to the given line ($y = mx + b$).

Also solved by PAUL CAPRON and OSCAR ADAMS.

505. Proposed by O. S. ADAMS, Coast and Geodetic Survey, Washington, D. C.

Construct a triangle, having given the sum of two sides, the angle included by these sides, and the altitude from the given angle upon the third side. (See figure on page 330.)

SOLUTION BY MRS. ELIZABETH B. DAVIS, U. S. Naval Observatory.

Let the sum of the two sides be $2k$, the included angle α, and the altitude on the third side R. Construct $\triangle BAC$, making $\angle A = \alpha$ and $AB = AC = k$. Draw AO, the bisector of $\angle A$, and erect perpendiculars to AB and AC at B and C, respectively, meeting AO in E. Then is E the focus of a parabola of which DO is the axis, and BC the tangent at the vertex D. Every tangent to this parabola cuts off on AB and AC intercepts whose sum is $AB + AC = 2k$. Therefore, the third side of the required $\triangle$ is a tangent to the parabola. Since the perpendicular on it from A is equal to R, it is also tangent to the circle whose center is A and radius R.

Therefore, the problem is to construct a common tangent to this parabola and circle.

On CA, lay off $CG = R$. Bisect CG at H, and with H as center and HE as radius describe a semicircle meeting AC, produced if necessary, in K and M. (It is sufficient for this purpose to find the points K and M.)

Then $GK = CM$. Also, E being on the circumference, and EC perpendicular to the diameter KM, we have

$$EC^2 = (R + GK)\cdot CM = (R + CM)\cdot CM.$$

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Solving (3) and (5), we have

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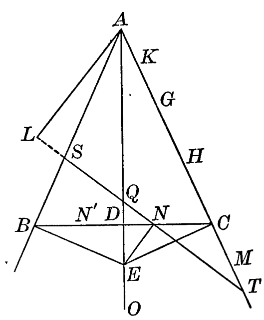
Then $GK = CM$. Also, E being on the circumference, and EC perpendicular to the diameter KM, we have

$$EC^2 = (R + GK) \cdot CM = (R + CM) \cdot CM.$$

From E as center, with radius CM, describe an arc intersecting BC in the points N and N'. Join EN (or EN'). Through N (or N'), draw ST (or $S'T'$), perpendicular, respectively to EN or EN'. Then $\triangle\, AST$ (or $A'S'T'$) is the required triangle.



Proof.—Draw AL (or AL') perpendicular to ST (or $S'T'$), produced if necessary. Since EN, the perpendicular from E to ST, meets ST in its intersection with BC, it follows that ST is tangent to the parabola. Therefore, the sum of its intercepts on AB and AC is $2k$. That is, $AS + AT = 2k$. Also $AL = R$. For, the $\triangle$'s ALQ and ENQ, both being right $\triangle$'s, and the vertical $\angle$'s at Q being equal, the $\triangle$'s are similar, and

$$AL : EN = AQ : EQ.$$

By composition,

$$(AL + EN) : EN = AE : EQ. \tag{1}$$

But in the right $\triangle\, ENQ$, ND is perpendicular to the hypotenuse EQ. Hence, $EQ = \overline{EN}^2/ED$. Substituting this value of EQ in (1), and dividing the second and fourth terms by EN, we have

$$(AL + EN) : 1 = AE : EN/ED,$$

or

$$(AL + EN)\cdot EN = AE\cdot ED.$$

But in the right $\triangle\, ACE$, $AE\cdot ED = \overline{EC}^2$, and it has been shown that $\overline{EC}^2 = (R + CM)\cdot CM$. Therefore, $(AL + EN)\cdot EN = (R + CM)\cdot CM$. But $EN = CM$ by construction. Therefore $AL = R$.

As indicated in the proof, there are two possible solutions.

Also solved by Oscar S. Adams and N. P. Pandya.

506. Proposed by S. A. COREY, Albia, Iowa.

Given a pentagon, plane or gauche, whose sides a, b, c, d, e are represented by the vectors x, y, z, v and $(x + y + z + v)$, respectively; and a second pentagon whose sides a_1, b_1, c_1, d_1, e_1 are represented by the vectors r, s, t, u and $(r + s + t + u)$, respectively, where $r = c_1x - c_5c_2y - c_6c_3z + c_5c_6c_4v$, $s = c_2x + c_1y - c_6c_4z - c_6c_3v$, $t = c_1z + c_3x + c_5c_2v + c_5c_4y$, $u = c_1v - c_4x - c_2z + c_3y$; c_1, c_2, c_3, c_4, c_5, c_6, being ordinary scalars.

Then prove the existence of the following relation between the sides of the two pentagons:

$$(c_1^2 + c_5c_2^2 + c_6c_3^2 + c_5c_6c_4^2)(x^2 + c_5y^2 + c_6z^2 + c_5c_6v^2) = r^2 + c_5s^2 + c_6t^2 + c_5c_6u^2.$$

Solution by the Proposer.

Inasmuch as the relation to be established between the vector sides is identical in form with a well-known algebraic identity, it is sufficient to show that the identity holds when certain of the algebraic (scalar) quantities are replaced by vectors. It is well known that the noncommutative character of vector multiplication does not affect the scalar part of the product of linear vector functions in the particular case where that product is a homogeneous *quadratic* function of the vectors employed. As this condition is satisfied in this problem the identity holds, and the existence of the given relation between the sides of the two pentagons is proved. (In forming the vector sides r, s, t, u the direction of the vectors x, y, z, v must be carefully noted.)

Inasmuch as vector *addition* is commutative it follows that the given pentagons may be replaced by certain other pentagons having the same vector sides placed in some other order of succession. Some of the sides may, of course, be zero, in which case the resulting figure would not be a pentagon. To illustrate, let x, y, z be the coterminous vector edges of any parallelepiped, v in this case being zero, and if $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 1$, then r, s, t and u are the diagonals connecting opposite corners of the parallelepiped, and the formula asserts that the sum of the squares of all its 12 edges equals the sum of the squares of its four diagonals; also that the shape of the parallelepiped may be changed without altering the sum of the squares of its diagonals, provided that no change is made in the length of its sides, or in the sum of their squares.

Similarly, if $c_1 = 0$, $v = 0$, $c_2 = c_3 = c_4 = c_5 = c_6 = 1$, the formula asserts that three times the sum of the squares of any three coterminous edges of any parallelepiped equals the sum of

Hence, $DG = \sqrt{16r^2 - 11r^2} = r\sqrt{5}$. That is, circle LGB' is 5 times the given circle.

Also solved in various ways by J. W. BALDWIN, F. E. CANADAY, C. E. GITHENS, F. E. WOODS, J. E. HATCH, O. S. ADAMS, J. ROSENBAUM, J. M. STETSON, and J. E. MCMAHON, JR.

CALCULUS.

419. Proposed by C. C. YENN, Tangshan, North China.

Find the entire area of the surface $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$.

SOLUTION BY THE PROPOSER.

Using the formula $\int\int \{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2\}^{1/2} dydx$, we have, from the equation of the surface,

$$\partial z/\partial x = -z^{1/3}/x^{1/3}, \qquad \partial z/\partial y = -z^{1/3}/y^{1/3};$$

whence

$$S = 8\int_0^a \int_0^{(a^{2/3}-x^{2/3})^{3/2}} \{(a^{2/3} - x^{2/3})x^{2/3} + (a^{2/3} - x^{2/3})y^{2/3} - y^{4/3}\}^{1/2} x^{-1/3} y^{-1/3} dydx. \tag{1}$$

To integrate with respect to y, let $y^2 = w^3$, so that $y^{-1/3}dy = \frac{3}{2}dw$, also put

$$A = (a^{2/3} - x^{2/3})x^{2/3}, \qquad B = (a^{2/3} - x^{2/3}). \tag{2}$$

Then

$$\int_0^{(a^{2/3}-x^{2/3})^{3/2}} \{(a^{2/3} - x^{2/3})x^{2/3} + (a^{2/3} - x^{2/3})y^{2/3} - y^{4/3}\}^{1/2} y^{-1/3} dy = \frac{3}{2}\int_0^B \{A + Bw - w^2\}^{1/2} dw$$

$$= \frac{3}{2}\left\{\frac{2w - B}{4}(A + Bw - w^2)^{1/2} + \frac{B^2 + 4A}{8}\sin^{-1}\left[\frac{2w - B}{(B^2 + 4A)^{1/2}}\right]\right\}_0^B.$$

Substituting the limits of integration, also the values of A and B from (2), we have from (1),

$$S = 6\int_0^a (a^{2/3} - x^{2/3})^{3/2} dx + 3\int_0^a (a^{2/3} - x^{2/3})(a^{2/3} + 3x^{2/3})x^{-1/3} \sin^{-1}\left\{\frac{a^{2/3} - x^{2/3}}{a^{2/3} + 3x^{2/3}}\right\}^{1/2} dx. \tag{3}$$

To evaluate the first integral, let $x = v^3$, $dx = 3v^2dv$, also put $a^{2/3} = \alpha^2$, then

$$6\int_0^a (a^{2/3} - x^{2/3})^{3/2} dx = 18\int_0^\alpha (\alpha^2 - v^2)^{3/2} v^2 dv$$

$$= -18\frac{v(\alpha^2 - v^2)^{5/2}}{6}\Big|_0^\alpha + 3\alpha^2\int_0^\alpha (\alpha^2 - v^2)^{3/2} dv$$

$$= 3\alpha^2\left\{\frac{v(\alpha^2 - v^2)^{1/2}(5\alpha^2 - 2v^2)}{8} + \frac{3\alpha^4}{8}\sin^{-1}\left(\frac{v}{\alpha}\right)\right\}_0^\alpha$$

$$= \frac{9}{8}\alpha^6\frac{\pi}{2} = \frac{9\pi a^2}{16}.$$

To evaluate the second integral, put $x^{2/3} = \lambda$, $\frac{2}{3}x^{-1/3}dx = d\lambda$, also set $a^{2/3} = k$, then the second term of S in (3) becomes

$$3 \times \frac{3}{2}\int_0^k (k - \lambda)(k + 3\lambda)\sin^{-1}\left\{\frac{k - \lambda}{k + 3\lambda}\right\}^{1/2} d\lambda$$
$$= \frac{9}{2}(k^2\lambda + k\lambda^2 - \lambda^3)\sin^{-1}\left\{\frac{k - \lambda}{k + 3\lambda}\right\}^{1/2}\Big|_0^k + \frac{9}{2}\int_0^k (k^2\lambda + k\lambda^2 - \lambda^3)\frac{kd\lambda}{(k + 3\lambda)\{\lambda(k - \lambda)\}^{1/2}}, \tag{4}$$

where the first term vanishes at both limits of integration. To integrate the second term, let $\sqrt{\lambda(k - \lambda)} = \lambda\xi$, then $\lambda = k/(1 + \xi^2)$, $d\lambda = -2k\xi d\xi/(1 + \xi^2)^2$; and the second term of (4) becomes

$$-9k^3\int_\infty^0 \left(\frac{1}{1 + \xi^2} + \frac{1}{(1 + \xi^2)^2} - \frac{1}{(1 + \xi^2)^3}\right)\frac{d\xi}{\xi^2 + 4} = 9k^3\lim_{c=\infty}\int_0^c \frac{(\xi^4 + 3\xi^2 + 1)d\xi}{(1 + \xi^2)^3(\xi^2 + 4)}.$$

Hence, $DG = \sqrt{16r^2 - 11r^2} = r\sqrt{5}$. That is, circle LGB' is 5 times the given circle.

Also solved in various ways by J. W. BALDWIN, F. E. CANADAY, C. E. GITHENS, F. E. WOODS, J. E. HATCH, O. S. ADAMS, J. ROSENBAUM, J. M. STETSON, and J. E. MCMAHON, JR.

CALCULUS.

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SOLUTION BY THE PROPOSER.

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whence

$$S = 8\int_0^a\int_0^{(a^{2/3}-x^{2/3})^{3/2}}\{(a^{2/3} - x^{2/3})x^{2/3} + (a^{2/3} - x^{2/3})y^{2/3} - y^{4/3}\}^{1/2}x^{-1/3}y^{-1/3}dydx. \tag{1}$$

To integrate with respect to y, let $y^2 = w^3$, so that $y^{-1/3}dy = \frac{3}{2}dw$, also put

$$A = (a^{2/3} - x^{2/3})x^{2/3}, \qquad B = (a^{2/3} - x^{2/3}). \tag{2}$$

Then

$$\int_0^{(a^{2/3}-x^{2/3})^{3/2}}\{(a^{2/3} - x^{2/3})x^{2/3} + (a^{2/3} - x^{2/3})y^{2/3} - y^{4/3}\}^{1/2}y^{-1/3}dy = \frac{3}{2}\int_0^B\{A + Bw - w^2\}^{1/2}dw$$

$$= \frac{3}{2}\left\{\frac{2w - B}{4}(A + Bw - w^2)^{1/2} + \frac{B^2 + 4A}{8}\sin^{-1}\left[\frac{2w - B}{(B^2 + 4A)^{1/2}}\right]\right\}_0^B.$$

Substituting the limits of integration, also the values of A and B from (2), we have from (1),

$$S = 6\int_0^a(a^{2/3} - x^{2/3})^{3/2}dx + 3\int_0^a(a^{2/3} - x^{2/3})(a^{2/3} + 3x^{2/3})x^{-1/3}\sin^{-1}\left\{\frac{a^{2/3} - x^{2/3}}{a^{2/3} + 3x^{2/3}}\right\}^{1/2}dx. \tag{3}$$

To evaluate the first integral, let $x = v^3$, $dx = 3v^2dv$, also put $a^{2/3} = \alpha^2$, then

$$6\int_0^a(a^{2/3} - x^{2/3})^{3/2}dx = 18\int_0^\alpha(\alpha^2 - v^2)^{3/2}v^2dv$$

$$= -18\frac{v(\alpha^2 - v^2)^{5/2}}{6}\Big|_0^\alpha + 3\alpha^2\int_0^\alpha(\alpha^2 - v^2)^{3/2}dv$$

$$= 3\alpha^2\left\{\frac{v(\alpha^2 - v^2)^{1/2}(5\alpha^2 - 2v^2)}{8} + \frac{3\alpha^4}{8}\sin^{-1}\left(\frac{v}{\alpha}\right)\right\}_0^\alpha$$

$$= \frac{9}{8}\alpha^6\frac{\pi}{2} = \frac{9\pi a^2}{16}.$$

To evaluate the second integral, put $x^{2/3} = \lambda$, $\frac{2}{3}x^{-1/3}dx = d\lambda$, also set $a^{2/3} = k$, then the second term of S in (3) becomes

$$\begin{aligned}&3\times\frac{3}{2}\int_0^k(k - \lambda)(k + 3\lambda)\sin^{-1}\left\{\frac{k - \lambda}{k + 3\lambda}\right\}^{1/2}d\lambda\\ &= \frac{9}{2}(k^2\lambda + k\lambda^2 - \lambda^3)\sin^{-1}\left\{\frac{k - \lambda}{k + 3\lambda}\right\}^{1/2}\Big|_0^k + \frac{9}{2}\int_0^k(k^2\lambda + k\lambda^2 - \lambda^3)\frac{kd\lambda}{(k + 3\lambda)\{\lambda(k - \lambda)\}^{1/2}},\end{aligned} \tag{4}$$

where the first term vanishes at both limits of integration. To integrate the second term, let $\sqrt{\lambda(k - \lambda)} = \lambda\xi$, then $\lambda = k/(1 + \xi^2)$, $d\lambda = -2k\xi d\xi/(1 + \xi^2)^2$; and the second term of (4) becomes

$$-9k^3\int_\infty^0\left(\frac{1}{1 + \xi^2} + \frac{1}{(1 + \xi^2)^2} - \frac{1}{(1 + \xi^2)^3}\right)\frac{d\xi}{\xi^2 + 4} = 9k^3\lim_{c=\infty}\int_0^c\frac{(\xi^4 + 3\xi^2 + 1)d\xi}{(1 + \xi^2)^3(\xi^2 + 4)}.$$

Separating into partial fractions, we have

$$\int \frac{(\xi^4 + 3\xi^2 + 1)d\xi}{(1 + \xi^2)^3(\xi^2 + 4)} = \frac{5}{27}\int \frac{d\xi}{\xi^2 + 1} + \frac{4}{9}\int \frac{d\xi}{(\xi^2 + 1)^2} - \frac{1}{3}\int \frac{d\xi}{(\xi^2 + 1)^3} - \frac{5}{27}\int \frac{d\xi}{\xi^2 + 4}.$$

Applying the formula

$$\int \frac{d\xi}{(\xi^2 + a^2)^n} = \frac{1}{2(n-1)a^2}\left\{\frac{\xi}{(\xi^2 + a^2)^{n-1}} + (2n-3)\int \frac{d\xi}{(\xi^2 + a^2)^{n-1}}\right\} \quad [\text{here } a = 1],$$

to the second and third integrals repeatedly, simplifying and integrating, we get

$$\frac{7\xi}{72(\xi^2 + 1)} - \frac{\xi}{12(\xi^2 + 1)^2} + \frac{61}{216}\arctan \xi - \frac{5}{54}\arctan \frac{\xi}{2}.$$

Substituting the limits of integration, letting c approach ∞, and writing $a^{2/3}$ for k, we obtain

$$\frac{41}{48}\pi a^2$$

as the value of the second term of S in (3).

Hence, finally,

$$S = \frac{9\pi a^2}{16} + \frac{41\pi a^2}{48} = \frac{17}{12}\pi a^2,$$

which is the area required.

Note.—The value of S thus obtained is 34/9 [< 4] times the area of the projection of the surface on one of the coördinate planes; the latter area being $\frac{3}{8}\pi a^2$.

Also solved by E. F. CANADAY, PAUL CAPRON, and J. B. REYNOLDS.

421. Proposed by E. H. MOORE, The University of Chicago.

Given n continuous real-valued functions $\phi_g(x)$ $(g = 1, 2, \ldots, n)$ of the real variable x on the interval (01) and set $\exp \int_0^1 \phi_g(x)\phi_h(x) = w_{gh}$ $(g, h = 1, 2, \ldots, n)$. Prove that the determinant $|w_{gh}|$ of the matrix (w_{gh}) is always $\geqq 0$ and that it is > 0 if no two of the functions $\phi_1, \ldots, \phi_n$ are identically equal on (01).

SOLUTION BY C. F. GUMMER, Kingston, Ontario.

[*Note.* Since the appearance in the June issue of my "solution" for this problem. I have learned that it is based on a misinterpretation of the question. Owing probably to the uncommon use of the period in "exp." (as originally printed) I overlooked the meaning of this as a sign for the exponential function, and foolishly read it as an abbreviation for "expression." I wish to make this explanation of the error and to present my apologies to the proposer for doubts cast on the correctness of the problem. C. F. GUMMER].

Let C_{gh} denote

$$\int_0^1 \phi_g(x)\phi_h(x)dx,$$

so that $w_{gh} = \exp c_{gh}$. The theorem will be generalized by putting in the place of the exponential function any function $F(z)$ of the form $\sum_{m=0}^{\infty} a_m z^m$ having a radius of convergence greater than every c_{gh} in absolute value and having the a's, for the first of the theorem positive or zero, and for the second part all positive. Let $W_{gh} = F(c_{gh})$

It was proved in the June number (in mistake for the present theorem) that the determinant $|c_{gh}| \geqq 0$. Since the same is true for the minors of (c_{gh}) coaxial with it, (c_{gh}) is the matrix of a positive or identically vanishing quadratic form, definite or semi-definite according as $|c_{gh}| > 0$ or $= 0$. This form (in variables $y_1, \ldots, y_n$) may be written

$$\sum_{i=1}^{r}\left(\sum_{j=1}^{n} b_{ij}y_j\right)^2,$$

Now let

$$y - z = (1 - z)w, \qquad dy = (1 - z)dw,$$

and

$$1 - y = (1 - z)(1 - w).$$

By substituting these results, we obtain

$$I = \int_0^1 f(z)(1 - z)^{m+n-1}dz \int_0^1 w^{m-1}(1 - w)^{n-1}dw = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m + n)} \int_0^1 f(z)(1 - z)^{m+n-1}dz.$$

This problem is taken from Whittaker & Watson's *Modern Analysis*, p. 250. Given in *Math. Trip.*, 1894.

Note.—A second solution was received but no name signed to it. EDITOR.

423. Proposed by J. B. REYNOLDS, Lehigh University.

Show that the envelope of all circles with their centers on the circle $x^2 + y^2 = a^2$ and tangent to the x-axis is the two-arched epicycloid.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

The general form of equation of circles fulfilling the conditions of the problem is

$$f(x, y, x_1, r) = (x - x_1)^2 + (y - r)^2 - r^2 = 0, \tag{1}$$

with the conditional equation

$$\varphi(x_1, r, a) = x_1^2 + r^2 - a^2 = 0. \tag{2}$$

We are to find the envelope of the system of circles represented by (1), x_1, r being the variable parameters. Using the undetermined multiplier λ and the auxiliary equations

$$\frac{df}{dx_1} = \lambda \frac{d\varphi}{dx_1}, \tag{3}$$

$$\frac{df}{dr} = \lambda \frac{d\varphi}{dr}, \tag{4}$$

we have

$$-(x - x_1) = \lambda x_1, \tag{5}$$

and

$$-(y - r) - r = \lambda r. \tag{6}$$

We must eliminate x_1, r, λ from (1), (2), (5), and (6). Substituting the values of $x - x_1$, and $y - r$ from (5) and (6) in (1),

$$\lambda^2 x_1^2 + r^2(1 + \lambda)^2 = r^2. \tag{7}$$

Also eliminating x_1 from (2) and (7),

$$\lambda(2r^2 + a^2\lambda) = 0. \tag{8}$$

The value of $\lambda = 0$ is irrelevant, but the second factor in (8) gives

$$r^2 = -\tfrac{1}{2}a^2\lambda. \tag{9}$$

Substituting (9) in (2) gives

$$x_1^2 = \frac{a^2}{2}(\lambda + 2) \tag{10}$$

and (5) gives

$$x_1 = \frac{x}{1 - \lambda}, \tag{11}$$

which with (10) gives the cubic in λ

$$a^2\lambda^3 - 3a^2\lambda + 2(a^2 - x^2) = 0. \tag{12}$$

Again, (6) gives

$$r = \frac{y}{\lambda}, \tag{13}$$

and this with (9) gives the second cubic in λ,

$$a^2\lambda^3 + 2y^2 = 0. \tag{14}$$

Eliminating λ from (12) and (14) by Sylvester's method, the required envelope is given by

$$4(x^2 + y^2 - a^2)^3 - 27a^4y^2 = 0. \tag{15}$$

This is a two-cusped epicycloid, as may be shown by eliminating θ from the two parametric equations of the epicycloid, a and b being the radii of the fixed and generating circles,

$$x = (a + b) \cos \theta - b \cos \frac{a + b}{b} \theta, \tag{16}$$

$$y = (a + b) \sin \theta - b \sin \frac{a + b}{b} \theta, \tag{17}$$

first supposing $b = \frac{1}{2}a$.

Note.—Since making this solution, I discovered that this problem was proposed by Sir Arthur Cayley as No. 1812 in the *London Educational Times*, and in the year 1852.

Cayley was one of the most cordial and active contributors to the mathematical section of the *Times*, and it is interesting to notice that his problems and the most of his solutions appearing in the monthly issues of that journal have been included in his *Works*.

Problem 1812 is reproduced there, but no solution is given. Reference, however, is made to problem 1771 and its solution, his statement being that he was led to No. 1771 by his study of 1812.

The statement of 1771 is: "Given a circle and a line, it is required a find a parabola, having the line for directrix, and the circle for its circle of curvature." The solution given is rather intuitional in character, justifying the equation of the parabola required by certain tests.

Employing rectangular axes, such that $x = m$ is the given line, and $x^2 + y^2 = 1$ the given circle, the required parabola is

$$y^2 - 2\left(1 - \frac{4}{9}m^2\right)^{3/2} y + \frac{16}{27}m^3x + 1 - \frac{4}{3}m^2 = 0 \tag{i}$$

and its focus

$$\left\{m - \frac{8}{27}m^3, \quad \left(1 - \frac{4}{9}m^2\right)^{3/2}\right\}. \tag{ii}$$

Taking $(\cos \theta, \sin \theta)$ as coördinates of a variable point on the circumference of $x^2 + y^2 = 1$, Cayley merely states that

$$x = \frac{3}{2} \cos \theta - m \cos 2\theta + \frac{1}{2} \cos 3\theta,$$

$$y = \frac{3}{2} \sin \theta - m \sin 2\theta + \frac{1}{2} \sin 3\theta,$$

are the coördinates of a variable point on the required envelope, adding what is the interesting connection of Nos. 1812 and 1771, viz., the required envelope in 1812 is a curve of the sixth order and has two cusps, which are the *foci* in the result of solution of 1771. It is to be noticed that the unreduced form of Cayley's result shows that there are *two* parabolas.

By a simple transformation of axes it may be easily shown that the polar of (*ii*) with respect to (*i*) is $x = m$, as should be.

The only other place where I have seen our problem is in the American edition of Williamson's *Differential Calculus*, 1884, but there is no certainty about the date when the problem was assigned a place in the manuscript of that text.

Also solved by PAUL CAPRON, S. W. REAVES, H. S. BEERS, I. L. MILLER, WILLIAM WEBER, A. M. HARDING, HORACE OLSON, G. W. HARTWELL, C. P. SOUSLEY, R. A. JOHNSON, F. M. MORGAN, E. W. WORTHINGTON, D. F. BARROW, C. C. YEN, and the PROPOSER.

MECHANICS.

Problem 332 is the same as 490 in Geometry, the solution of which appeared in the February number of the MONTHLY.

333. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

A flywheel 21 feet in diameter makes 100 revolutions per minute. The weight of a cubic foot of its material is 448 pounds. Show that the intensity of stress on a transverse section of rim,

assuming that it is unaffected by the arms, is 1,176 lbs. per sq. in. If the safe stress permissible in the material is 6,000 lbs. per sq. in., show that the greatest speed at which the wheel can be run with safety is about 225 revolutions per minute.

SOLUTION BY WILLIAM W. JOHNSON, Cleveland, Ohio.

The unit stress in the flywheel rim due to centrifugal force is found as follows: In the general equation of force, $F = Ma$, the acceleration a in the present instance is v^2/r, in which v = velocity of rim in feet per second, R = radius of rim in feet, W = weight of rim in pounds, and g = acceleration due to gravity. Hence, centrifugal force $= F = Mv^2/R = Wv^2/gR$.

The resultant of half of this force tends to disrupt one half of the rim from the other half. This rupture is resisted by the two sections of the rim at each end of the diameter.

The total tension T in a cross-section of the rim is to one half the sum of half of the radial forces as the diameter of the flywheel is to half its circumference. Therefore,

$$T = F/2\pi = Wv^2/2\pi Rg.$$

Let $W = 2\pi RAw/144$, in which A = area of cross-section of rim in square inches, and w = weight of rim material in pounds per cubic foot.

Then, $T = Awv^2/144g$. Let $T = AS$, where S = stress per unit area of cross-section of rim. Then, $S = wv^2/144g$.[1]

Put $v = \pi Dn/60$, where D = mean diameter of rim in feet, and n = number of revolutions per minute, we have

$$S = \frac{w(\pi Dn)^2}{60^2 \times 144g}. \tag{A}$$

Solving for n, we have

$$n = \frac{720}{\pi D}\sqrt{\frac{Sg}{w}}. \tag{B}$$

Putting, $D = 21$, $n = 100$, $w = 448$, $S = 6{,}000$, and $g = 32.16$ in formulas (A) and (B) we find

Answer (a): $S = 1{,}170$ lbs. per sq. in., and

Answer (b): $n = 226$ revolutions per min.

Also solved by J. B. REYNOLDS.

334. Proposed by HORACE OLSON, Chicago, Illinois.

A particle of elasticity e is projected with velocity v at an angle φ with a plane inclined to the horizontal at an angle ψ; its plane of motion is perpendicular to the inclined plane. Show that after $2v \sin \varphi/g(1 - e) \cos \psi$ seconds it will cease to rebound and will move along the plane with an initial velocity $v \cos \varphi - 2v \sin \varphi \tan \psi/1 - e$ and a uniform acceleration $g \sin \psi$ down the plane.

SOLUTION BY JOS. B. REYNOLDS, Lehigh University.

The acceleration of the particle may be resolved into a component of $g \cos \psi$ perpendicular to the plane and a component $g \sin \psi$ down the plane. Let v_1 be the initial component of the velocity of the particle perpendicular to the plane, v_2 the component of velocity perpendicular to the plane after the first rebound, v_3 after the second rebound, etc. Also, let t_1 be the time from the instant of projection until the first impact, t_2 the time from the first impact until the second, t_3, from the second until the third, etc. Then

$$v_2 = ev_1, \qquad v_3 = ev_2 = e^2v_1, \quad \text{etc.}$$

$$t_1 = \frac{2v_1}{g \cos \psi}, \qquad t_2 = \frac{2v_2}{g \cos \psi}, \qquad t_3 = \frac{2v_3}{g \cos \psi}, \quad \text{etc.}$$

So that if t is the time before the ball ceases to rebound,

$$t = t_1 + t_2 + t_3 + \cdots = \frac{2v_1}{g \cos \psi}(1 + e + e^2 + \cdots + e^n)_{n=\infty} = \frac{2v_1}{(1 - e)g \cos \psi}$$

$$= \frac{2v \sin \varphi}{(1 - e)g \cos \varphi}, \qquad \text{since} \qquad v_1 = v \sin \varphi.$$

[1] S is independent of the radius and depends only on the rim velocity.

Since the velocity along the plane is not affected by the impacts the velocity v' at the end of time, t, will be upward along the plane end of magnitude

$$\begin{aligned} v' &= v \cos \varphi - g \sin \psi t \\ &= v \cos \varphi - \frac{2v \sin \varphi \tan \psi}{(1-e)}, \qquad \text{since} \qquad t = \frac{2v \sin \varphi}{(1-e) g \cos \psi}. \end{aligned}$$

Also solved by the PROPOSER.

335. Proposed by HAROLD T. DAVIS, Colorado Springs, Colorado.

A heavy particle is projected upwards with a velocity V in a medium resisting as the nth power of the velocity. Prove that the elevation of the particle when the velocity downwards is V is equal to LT, where L is the limiting velocity and T is the time in which the particle falling from rest in the medium will acquire a velocity V^2/L.

SOLUTION BY JOS. B. REYNOLDS, Lehigh University.

As the solution will show, this problem should read as follows: A heavy particle is projected upwards with a velocity V in a medium resisting as the nth power of the velocity. Prove that the whole space (up and down) described when the downward velocity is V is equal to LT, where L is the limiting velocity and T is the time in which the particle falling from rest in the medium will acquire a velocity V^2/L.

The equation of motion for the particle upward is

$$-v\frac{dv}{ds} = g + \mu v^n \tag{1}$$

μ being the proportionality factor, and the equation of motion for the particle downward is

$$v\frac{dv}{ds} = g - \mu v^n = \frac{dv}{dt}. \tag{2}$$

When the particle has acquired the limiting velocity L its acceleration is zero. So by equation (2) we have $g/\mu = L^n$. Also by (2), when falling in the medium,

$$dt = \frac{dv}{g - \mu v^n}, \qquad \text{and} \qquad \therefore \quad T = \int_0^{V^2/L} \frac{dv}{g - \mu v^n} = \frac{1}{\mu}\int_0^{V^2/L} \frac{dv}{L^n - v^n};$$

whence,

$$LT = \frac{L^{n+1}}{g}\int_0^{V^2/L} \frac{dv}{L^n - v^n}. \tag{3}$$

From (1), in the upward motion,

$$ds = -\frac{vdv}{g + \mu v^n};$$

whence, if the particle rises to a height h before coming to rest,

$$h = -\int_V^0 \frac{vdv}{g + \mu v^n} = \int_0^V \frac{vdv}{g + \mu v^n}.$$

When falling, by (2),

$$ds = \int \frac{vdv}{g - \mu v^n},$$

so that if it has fallen a distance h_1 when the velocity is V, we have

$$h_1 = \int_0^V \frac{vdv}{g - \mu v^n}.$$

Then the total distance described (up and down) is

$$h + h_1 = \int_0^V \left(\frac{1}{g - \mu v^n} + \frac{1}{g + \mu v^n}\right) vdv = \int_0^V \frac{2gvdv}{g^2 - \mu^2 v^{2n}},$$

or

$$h + h_1 = \frac{g}{\mu^2}\int_0^V \frac{2vdv}{L^{2n} - V^{2n}} = \frac{g}{\mu^2 L^n}\int_0^V \frac{2vdv}{L^n - (V^2/L)^n}.$$

In this integral, if we let $z = v^2/L$, we shall have

$$h + h_1 = \frac{L^{n+1}}{g}\int_0^{V^2/L}\frac{dz}{L^n - z^n},$$

but this is identical with the integral of (3). Hence $h + h_1 = LT$.

Note.—The incorrectness of the statement of this problem was also pointed out by Professor H. S. Uhler for the special case $n = 2$. However, if in his solution we take the value he finds for h, the height to which the particle rises and the distance z' through which it falls from rest, we have

$$h + z' = \frac{1}{2k}\log\left(1 + \frac{k}{g}V^2\right) + \frac{1}{2k}\log\left(\frac{L^2}{L^2 - V^2}\right) = \frac{1}{2k}\log\left(\frac{L^2 + V^2}{L^2 - V^2}\right),$$

where $\sqrt{g}/\sqrt{k} = V$, the limiting velocity. But he showed that

$$LT = \frac{1}{2k}\log\left(\frac{L^2 + V^2}{L^2 - V^2}\right).$$

Hence, $h + z' = LT$, a result agreeing with that of Professor Reynolds. EDITORS.

NUMBER THEORY.

250. Proposed by JOSEPH E. ROWE, State College, Pa.

Show by comparatively elementary means that the equation $x^{2n} + y^{2n} = z^{2n}$ is impossible of solution in positive integers x, y, z and n, unless at least one of the integers $x, y, z = 0 \pmod 3$. In particular, consider the case $n = 1$.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

Since any number must be of one of the forms $3k$, $3k + 1$, $3k - 1$, its square must be of the form $3l$ or $3l + 1$. Consequently, any perfect square, $x_1^{2n} \equiv 0$ or $1 \pmod 3$. It is clear that the three quantities in the given equation can not all be congruent to $1 \pmod 3$; in fact it is evident that either x^{2n} or y^{2n} must be congruent to $0 \pmod 3$, as otherwise the left-hand member of the equation would be divisible by 3 with remainder 2, and the right-hand member by 3 with remainder 1.

The case $n = 1$ may be handled directly by means of the known solution of the equation $x^2 + y^2 = z^2$, namely, $x = 2mn$, $y = m^2 - n^2$, $z = m^2 + n^2$ (we are supposing that x, y and x are prime to each other) where one of the two quantities n, m is even and the other odd. If either m or n is divisible by 3, x is also; if neither, then as we saw above $m^2 \equiv 1 \pmod 3$, $n^2 \equiv 1 \pmod 3$, and, consequently, $y = m^2 - n^2 \equiv 0 \pmod 3$.

Also solved by C. C. YEN, H. C. FEEMSTER, H. N. CARLETON, and J. W. CLAWSON.

251. Proposed by HERMAN ROLAND KATNICK, Chicago, Ill.

Determine the character of the positive integer n so that the Diophantine system

$$z + n = x^2, \qquad z - n = y^2$$

shall have an integral solution; and exhibit a method for finding all the values of x, y, z for a given n of such character.

SOLUTION BY THE PROPOSER.

[Mr. Katnick died suddenly shortly after this problem was offered for publication. The solution given below is a modified and abridged form of one offered by the proposer at the time the problem was submitted. EDITORS.]

From the given equations we have $x^2 - y^2 = 2n$ or $(x + y)(x - y) = 2n$. If we denote $x + y$ by a and $x - y$ by b we have

$$(1) \qquad x + y = a, \qquad x - y = b, \qquad 2n = ab.$$

Hence it is necessary that $2x = a + b$, $2y = a - b$, while at least one of the numbers a, b is even (since $ab = 2n$). Hence, both a and b are even since x and y are integers. Then we may put

In this integral, if we let $z = v^2/L$, we shall have

$$h + h_1 = \frac{L^{n+1}}{g}\int_0^{V^2/L}\frac{dz}{L^n - z^n},$$

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second), $(p - 1)$ may be filled by 1's in ${}_rC_{p-1}$ ways, and each of these different ways gives rise to $9^{r-(p-1)}$ integers containing the digit 1 p times. Hence, in the first subinterval the number of integers containing 1 at least p times is equal to the sum of expression (1) and ${}_rC_{p-1}\cdot 9^{r-p+1}$. And, therefore, the number of integers in the interval $10^r \leqq x < 10^{r+1}$ which do not contain the digit 1 at least p times, $p \leqq r$, is

$$9\cdot[\text{first } p \text{ terms of expansion of } (9+1)^r] - {}_rC_{p-1}\cdot 9^{r-p+1},$$

or

$$9\cdot\{[\text{first } p \text{ terms of expansion of } (9+1)^r] - {}_rC_{p-1}\cdot 9^{r-p}\}.$$

253. Proposed by HERBERT N. CARLETON, West Newbury, Massachusetts.

Prove that $n^{2k+8} - n^{2k} \equiv 0 \pmod{20}$ for integral values of n and k.

SOLUTION BY R. M. MATHEWS, Riverside, California.

$$n^{2k+8} - n^{2k} = n^{2k}(n^2-1)(n^2+1)(n^4+1)$$

When n is even, $n^{2k} \equiv 0 \pmod 4$. When n is odd, $n^2 - 1 \equiv 0 \pmod 4$.

Next, n being an integer must be of the form $5m$, $5m \pm 1$, or $5m \pm 2$.

For n of the form $5m$, $n^{2k} \equiv 0 \pmod 5$; for n of the form $5m \pm 1$, $n^2 - 1 \equiv 0 \pmod 5$; and for n of the form $5m \pm 2$, $n^2 + 1 \equiv 0 \pmod 5$.

Hence, $n^{2k+8} - n^{2k} \equiv 0 \pmod{20}$, n and k being integers. This is also true of $n^{2k+4} - n^{2k}$.

Also solved by O. S. ADAMS, W. J. THOME, ELIJAH SWIFT, E. B. ESCOTT, C. C. YEN, and the PROPOSER.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence.

REPLIES.

34. Given the mixed integral and functional equation

$$\int_{x=0}^{x=x} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{x}{2}\right) + f(x)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

REMARK BY S. A. COREY, Albia, Iowa.

The prismoidal formula gives the exact value of this integral whenever the fourth derivative of $f(x) = 0$. This was shown in an article entitled "Certain Integration Formulæ Useful in Numerical Computation" in Vol. XIX, Nos. 6 and 7, of this MONTHLY, in which formula $(1r)$ is the prismoidal formula including an expression for the remainder term.

$f(x) = Ax^3 + Bx^2 + Cx + D$ is, therefore, the most general value of the function $f(x)$ for which the prismoidal formula gives the exact value for all values of x.

28. Is it possible to obtain $\int \cos\theta^2 d\theta$ without expanding $\cos\theta^2$? If it is not, can some interesting properties of this integral be determined by treating it as a special function?

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Also solved by O. S. Adams, W. J. Thome, Elijah Swift, E. B. Escott, C. C. Yen, and the Proposer.

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28. Is it possible to obtain $\int \cos\theta^2 d\theta$ without expanding $\cos\theta^2$? If it is not, can some interesting properties of this integral be determined by treating it as a special function?

$$\int \cos\theta^2\,d\theta = \theta\sum_{t=0}\,2^t\prod_{s=1}^{s=t}\frac{1}{2s+1}\,\theta^{2t}\cos\left(t\frac{\pi}{2}-\theta^2\right)$$
$$= \cos\theta^2\left(\theta - \frac{2^2}{1\cdot3\cdot5}\theta^5 + \frac{2^4}{1\cdot3\cdot5\cdot7\cdot9}\theta^9 - + \cdots\right)$$
$$+ \sin\theta^2\left(\frac{2}{1\cdot3}\theta^3 - \frac{2^3}{1\cdot3\cdot5\cdot7}\theta^7 + - \cdots\right).$$

Two interesting special cases deserve mention.

When the exponent of the coefficient is less by unity than that of the argument of the function, the integral assumes a known form; and the value of the other integral becomes known. Hence, expressions of the type $\theta^{rn-1}\sin\theta^n\,d\theta$ are directly integrable in finite form by these formulæ. If in addition n takes the special form $1/r$, the integral of $\sin\theta^{1/r}d\theta$ is found, that is, the integral of sine or cosine of any root of a variable is found in finite form.

Thus, taking the first of the four formulæ, making $m = n - 1$, and transforming a little; and after this, making $n = 1/r$, we obtain the formulæ:

$$\int \theta^{rn-1}\cos\left(r\frac{\pi}{2}-\theta^n\right)d\theta = -\,n^{-1}\prod_{s=r-1}^{s=1} s\cdot\sum_{t=1}^{t=r}\prod_{s=1}^{s=t-1} s^{-1}\cdot\theta^{(t-1)n}\sin\left(t\frac{\pi}{2}-\theta^n\right),$$
$$\int \cos\left(r\frac{\pi}{2}-\theta^{1/r}\right)d\theta = -\,r\prod_{s=1}^{s=r-1} s\cdot\sum_{t=1}^{t=r}\prod_{s=1}^{s=t-1} s^{-1}\cdot\theta^{(t-1)1/r}\sin\left(t\frac{\pi}{2}-\theta^{1/r}\right).$$

As examples, in these formulas making $n = 2$, $r = 4$, we have

$$\therefore\quad \int \theta^7\cos\theta^2\,d\theta \doteq -\,3\left[\cos\theta^2\left(1-\frac{\theta^4}{2}\right)+\sin\theta^2\left(\theta^2-\frac{\theta^6}{3!}\right)\right],$$
$$\int \cos\theta^{1/4} = -\,4!\left[\cos\theta^{1/4}\left(1-\frac{\theta^{1/2}}{2}\right)+\sin\theta^{1/4}\left(\theta^{1/4}-\frac{\theta^{3/4}}{3!}\right)\right].$$

DISCUSSIONS.

I. Relating to a Curve with Unusual Properties.

By Jos. B. Reynolds, Lehigh University.

The curve

$$y = \frac{a^2x}{x^2+a^2}$$

presents some unusual properties. Considering the part of the curve to the right of the y-axis (see the figure) we have for the area between it and the x-axis

$$A = \int_0^\infty y\,dx = a^2\int_0^\infty \frac{x\,dx}{x^2+a^2} = \frac{a^2}{2}\log\left(x^2+a^2\right)\Big]_0^\infty = \infty.$$

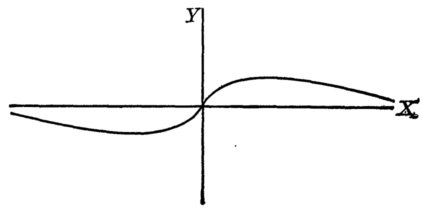
Yet the volume generated by revolving this area about the x-axis is

$$V = \pi\int_0^\infty y^2dx = \pi a^4\int_0^\infty \frac{x^2dx}{(x^2+a^2)^2} = \frac{\pi^2a^3}{4};$$

that is, the revolution of an infinite area gives a finite volume. Again, the ordinate of the center of gravity of this area is

$$\bar{y} = \frac{\int_0^\infty dx \int_0^{a^2x/(x^2+a^2)} ydy}{\int_0^\infty dx \int_0^{a^2x/(x^2+a^2)} dy} = \frac{\frac{\pi a^3}{8}}{\infty} = 0;$$

that is, although the area lies entirely above the x-axis and is indefinitely large,



its center of gravity is indefinitely close to that axis.

By the theorem of Pappus we have

$$2\pi \bar{y} A = V,$$

whence we have an indeterminate form evaluated

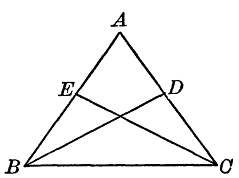
$$2\pi(0)\infty = \frac{\pi^2 a^3}{4}.$$

II. Relating to a Demonstration of a Geometrical Theorem.

By William E. Heal, Washington, D. C.

If the bisectors of two angles of a triangle are equal, the triangle is isosceles.

This is a very celebrated theorem and has been demonstrated in many ways. It was proposed in the Monthly as problem 42 and several demonstrations were published in Vol. II, pp. 157 and 189–191. Of these proofs all but one were by indirect methods; that is, by use of the *reductio ad absurdum.* The following direct method of proof was communicated to the writer some months ago by Dr. Artemas Martin of the U. S. Coast Survey and seems to be eminently worthy of preservation.



If $BD = CE$ we are to prove that $AB = AC$.

We have by a well-known theorem

$$AB \times BC = (AE + BE)BC = BD^2 + AD \times DC, \tag{1}$$

$$AC \times BC = (AD + DC)BC = CE^2 + AE \times EB. \tag{2}$$

If $BD = CE$ these become

$$(AE + BE)BC = BD^2 + AD \times DC, \tag{3}$$

$$(AD + DC)BC = BD^2 + AE \times EB. \tag{4}$$

Subtracting (3) from (4) and transposing,

$$AE \times BC + BE \times BC + AE \times BE = AD \times DC + AD \times BC + DC \times BC. \quad (5)$$

We also have

$$AB : BC = AD : DC,$$

$$AC : BC = AE : EB,$$

$$\therefore \quad BC \times AD = AB \times DC = (AE + BE)DC, \quad (6)$$

$$BC \times AE = AC \times EB = (AD + DC)EB. \quad (7)$$

Substituting these values of $BC \times AD$ and $BC \times AE$ in (5) and dropping $BE \times CD$ from both sides we have

$$AD \times EB + BC \times EB + AE \times EB = AD \times DC + AE \times DC + BC \times DC.$$

Or

$$(AD + BC + AE)EB = (AD + AE + BC)DC.$$

$$\therefore \quad EB = DC.$$

Substituting $EB = DC$ in (5) we get, after dropping $EB \times BC$ from both sides,

$$AE \times BC + AE \times BE = AD \times EB + AD \times BC$$

or

$$(BC + BE)AE = (BC + BE)AD.$$

Hence

$$AE = AD.$$

Therefore

$$AB = AC.$$

III. Concerning Huntington's *Continuum and Other Types of Serial Order*.[1]

By Lester S. Hill, Princeton University.

If A_1 is any subclass of an ordered class A, then all elements a of A which are ordinally less than every element of A_1 ($a < A_1$) constitute a lower segment of A; and all elements $a > A_1$ constitute an end segment. If A_1 and A_2 are two subclasses with $A_1 < A_2$ then all elements a for which $A_1 < a < A_2$ constitute a mid segment. Of course A_1 or A_2, or each of them, may reduce to a single element of A.

The order type of A is continuous if

(D) A is dense: between every two elements of A lie other elements;

(G) A is gap-free: If A' is a lower segment, A'' an end segment, and $A = A' + A''$, then A' has a last element or else A'' has a first. ($A = A' + A''$ implies $A' < A''$.)

The ordered class A of real numbers satisfies (D), (G) and also

(F_0) A is denumerably framed: A contains a *denumerable* subclass B which is dense in A; between every two elements of A lie elements of B.

[1] See Book Review on page 325 of this issue.

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The more general continuous types for which Huntington has space to cite examples satisfy (D), (G) and the following weakened form of (F_0),—

(F) A is denumerably framed by segments: on every segment of A lie segments which are denumerably framed.

It may therefore be of interest to illustrate a simple type of continuous order satisfying neither (F_0) nor (F).

§ I. Let $H = \{h\}$ be the class of all numerical sequences of type ω

$$h = (e_1, e_2, e_3, \cdots),$$

where e_n is any real number for every finite index n, only finite ordinal numbers occurring as indices.

Then, $\quad h' = (e_1', e_2', e_3', \cdots) \quad$ and $\quad h'' = (e_1'', e_2'', e_3'', \cdots)$

being two elements of H we define

$h' < h''$ (h' less than h'') if, from some index on, $e_n' < e_n''$,
$h' > h''$ (h' greater than h'') if, " " " " $e_n' > e_n''$,
$h' = h''$ (h' equal to h'') if, " " " " $e_n' = e_n''$.

If one of these three relations holds we call h' and h'' comparable; otherwise h and h' are incomparable.

Observe that if $h' \lesseqgtr h''$ and $h'' \lesseqgtr h'''$ then $h' \lesseqgtr h'''$.

§ II. We omit the proof that there exist subclasses $K = \{k\}$ of H, every k being an h, with the properties of closure:

(α) K contains no pair of incomparable, and no pair of equal sequences.

(β) The class $H - K$ contains no sequence which is comparable with every k but equal to no k.

§ III. $K = \{k\}$, which is, to to speak, a maximum (closed) subclass of serially ordered sequences in H, has the striking ordinal property:

If K' and K'' are denumerable subclasses of K such that $K' < K''$, then K contains an infinite subclass K''' such that $K' < K''' < K''$.

For, since K' and K'' are denumerable, we can arrange them

$$K' = (k_1'k_2'k_3'\cdots), \qquad K'' = (k_1''k_2''k_3''\cdots),$$

k_1' being the first sequence of K' in this arrangement, etc. We have, more in detail,

$$k_1' = (k_{11}', k_{12}', k_{13}', \cdots); \qquad k_1'' = (k_{11}'', k_{12}'', k_{13}'', \cdots)$$

$$k_2' = (k_{21}', k_{22}', k_{23}', \cdots); \qquad k_2'' = (k_{21}'', k_{22}'', k_{23}'', \cdots)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

and, of course, every k_n' is less than every k_m'' ($K' < K''$).

We can now determine a sequence of indices $n_1 < n_2 < n_3 < \cdots$ such that

$$k_{1n}' < k_{1n}'' \qquad (n \geqq n_1)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$k_{1n}', k_{2n}', \cdots, k_{mn}' < k_{1n}'', k_{2n}'', \cdots, k_{mn}'' \qquad (n \geqq n_m)$$

and, obviously, the sequence $h = (e_1, e_2, e_3, \cdots)$ where the e_n are any real numbers such that

$$k_{1n}' < e_n < k_{1n}'' \qquad (n_1 \leqq n < n_2)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$k_{1n}', k_{2n}', \cdots, k_{mn}' < e_n < k_{1n}'', k_{2n}'', \cdots, k_{mn}'' \quad (n_m \leqq n < n_{m+1})$$

is an element of H greater than K' and less than K''. Therefore, if we suppose the mid segment of K determined by $K' < K''$ to be zero (that is, without elements), it is clear that h is comparable with every element of K, and is itself an element of K; and we have a contradiction. In this argument we use the observation made at the end of § I.

If K' or K'', or each of these classes, reduces to a finite class or to a single element of K, the reader may supply the appropriate argument and obtain the same conclusion as above. Hence K is certainly dense, satisfying (D).

§ IV. Hausdorff defined K and proved that its potency is that of the real number continuum.

We need not trouble to ascertain whether K satisfies condition (G). For we can obtain from K very easily a class J ordered according to (D) and (G), and having the property discussed in § III.

Let $J = \{j\}$ be the class of all those lower segments in K which are without last element; and define $j_1 < j_2$ if the segment j_2 contains elements of K not occurring in j_1. A very hasty reading of Huntington's book will enable the student to see at once that J, thus ordered, satisfies (D) and (G); and that it contains a subclass L which is dense in J and similar (ordinally equivalent) to K: namely, the class of lower segments in K corresponding to partitions $K = K' + K''$ ($K' < K''$), in which K' has no last element, but K'' has a first.

The density of L in J and the ordinal equivalence of L and K furnish an easy proof (based on that property of K discussed in § III) for the statement:

J contains no denumerably framed segment. While continuous it is widely divergent from the continuous types cited in the manual under review. It has a property meaning more than the mere absence of the property (F) which characterizes those types.

The reader may again follow indications by Hausdorff in the construction of number systems "Gröszensysteme" which are serially ordered, but non-Archimedean, and otherwise different from the familiar system of real numbers.

Let $H = \{h\}$ be, as before, the class of all real number sequences; and, if $h' = (e_1', e_2', e_3', \cdots)$, $h'' = (e_1'', e_2'', e_3'', \cdots)$ are two elements of H define:

$$h' \pm h'' = (e_1' \pm e_1'', \quad e_2' \pm e_2'', \cdots)$$

$$h' \gtrless h'' \text{ for } e_m' = e_m'' \ (m < n) \text{ and } e_n' \gtrless e_n''.$$

H is herewith furnished with serial order; and with an associative and commutative addition having the property: $h_1 + h_2 > h_1 + h_3$ if $h_2 > h_3$. The symbols mh, h/n, mh/n where m, n are positive or negative integers have an obvious meaning; and H contains a definite element described by $r_1h_1 + r_2h_2 + \cdots + r_nh_n$ where $h_1 \cdots h_n$ are any elements of H and $r_1r_2 \cdots r_n$ are any rational *real* numbers.

Like the Gröszensystem upon which Veronese bases his geometry of the intuitional continuum, H is not Archimedean; if $h_1 < h_2$ are an arbitrary pair of its elements, we can not assert that for some positive integer n, $nh_1 > h_2$; in fact such an assertion would be false in the obvious instance

$$h_1 = (0, 1, 0, 0, \cdots), \qquad h_2 = (1, 0, 0, 0, \cdots).$$

H contains relative "infinites" and "infinitesimals" of all finite orders; in striking contrast with the real number system, in which no constant number is infinitesimal or infinite with respect to any other constant.

Desiring a continuously ordered Gröszensystem we proceed again to the class $J = \{j\}$ of all lower segments in H without last element; and define $j_1 < j_2$ when the segment j_2 contains elements of H not occurring in j_1. As before, we obtain a continuously ordered J (but of order type very different from that of the former J) containing a subclass L which is dense in J and ordinally equivalent to H. That the extension of H to J is possible appears in the obvious fact that H is dense; that an extension is necessary is shown by the simple partition $H = H' + H''$ $(H' < H'')$ where H' contains all elements of H ordinally less than some $(0, e, 0, 0, \cdots)$, H'' all elements ordinally greater than some $(e, 0, 0, 0, \cdots)$ as e assumes all real values > 0: a partition which explodes condition (G).

The means of defining suitably an operation of addition in J is suggested by Dedekind's procedure in extending the class of rational numbers to the class of real numbers (cf. C. Jordan, *Cours d'Analyse,* Vol. I). Let $j_1 < j_2$ be elements of J representing the lower segments H_1 and H_2 of H. We define $j_1 + j_2$ as that lower segment (clearly without last element) in H which consists of all sums $h_1 + h_2$ of an element in H_1 and an element in H_2.

The properties of addition as defined in H validate this form of definition; and establish the desirable relation: if $l_1 < l_2$ of L correspond to $h_1 < h_2$ of H, then $l_1 + l_2$ is an element of L and corresponds to $h_1 + h_2$ of H. We have in L a class which is equivalent to H with respect to order and also with respect to addition. Since L is thus completely isomorphic with H, and is moreover dense in J, we are more than enabled to argue that J, like H, is non-Archimedean; indeed the Archimedean property involves only a relation between the ordering and the adding in a system with order and addition.

Hausdorff has given a general analysis of a large class of Gröszensysteme with order and addition which are non-Archimedean. He defined the particular case H explicitly but is not responsible for shortcomings which may be found in J.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Ind.

The permanent address of the Secretary, Professor W. D. CAIRNS, is now 27 King Street, Oberlin, Ohio.

At Hamilton College, Associate Professor W. M. CARRUTH has been promoted to a professorship of mathematics.

Professor L. E. DICKSON, of the University of Chicago, is lecturing at the University of California during the first semester.

Professor G. A. MILLER, of the University of Illinois, will contribute the mathematical article to the "American Year Book" for 1917.

Dr. W. H. BESANT, F.R.S., fellow of St. John's College and lecturer on mathematics, died on June 2, 1917, in his eighty-ninth year.

Professor H. HALPERIN, of Vanderbilt University, has been appointed assistant professor of mathematics at the University of Arkansas.

Professor C. A. BARNHART, of Carthage College, will be a member of the mathematical faculty of Colorado College during the coming year.

At the Pennsylvania State College Associate Professors J. H. TUDOR and H. F. STECKER have been promoted to professorships of mathematics.

Dr. W. V. N. GARRETSON, of the University of Michigan, has been appointed associate professor of mathematics at the Pennsylvania Military College.

The death of Dr. G. W. HARTWELL, for eight years professor of mathematics in Hamline University, St. Paul, Minn., occurred July 23 at Columbus, Montana. He was a charter member of the Association.

Dr. H. N. WRIGHT, formerly instructor in the University of California, has been appointed head of the department of mathematics at Whittier College, Whittier, Calif.

Mr. H. C. GOSSARD, assistant professor of mathematics at the University of Oklahoma, has been appointed a member of the mathematical staff in the Naval Academy at Annapolis.

Professor P. P. BOYD, of the University of Kentucky, has been appointed dean of the college of arts and sciences. Professor BOYD has also been made acting president of the university.

Professor G. H. SCOTT, for the fifteen years professor of mathematics and astronomy in Yankton College, Yankton, S. Dakota, has resigned to become principal of Benzonia Academy, Benzonia, Michigan.

Mrs. ELIZABETH BROWN DAVIS, a charter member of the Association, died at her home in Washington, D. C., in April, 1917. She was a computer in the Nautical Almanac division of the U. S. Naval Observatory.

At Washington University, St. Louis, Professor C. A. WALDO has retired from active service and Dr. W. H. ROEVER has been appointed acting head of the department of mathematics and promoted to a full professorship.

Mr. W. W. JOHNSON, 12013 Saywell Ave., Cleveland, Ohio, would like to learn from any reader of the MONTHLY where a copy of Steinhauser's *Berechnung zwanzigstelliger Logarithmen* (published by Carl Gerold's Sohns, Vienna) could be obtained.

The valuable paper by Professor A. R. CRATHORNE, of the University of Illinois, on "Required Mathematics," read before the Fourteenth Annual Conference of Kansas High Schools and Academies, March 16, 1917, appears in full in *School and Society*, July 7, 1917.

At recent meetings of the Academy of Sciences of Petrograd mathematical papers were presented by I. M. VINOGRADOV on "A new method of obtaining asymptotic expressions"; and by V. A. STEKLOV on "The approximation of functions by means of Cebysev's polynomials."

M. EMILE PICARD, professor of mathematics at the University of Paris, has been elected permanent secretary of the Paris Academy of Sciences, succeeding the late M. GASTON DARBOUX. Professor PICARD has also been appointed to represent the French government in the International Geodetic Association.

With a recent number of the *Revista de la Sociedad Matemática Española* the Spanish Mathematical Society has published a Supplement written by the president of the Society, Dr. ZOEL GARCIA DE GALDEANO, upon the development of mathematics under the title "Exposición sumaria de la matemática segũn un nuevo método."

Mr. PHILIP E. B. JOURDAIN is the author of the reports in the quarterly journal, *Science Progress*, on "Recent Advances in Mathematics." In the July, 1917, issue he reports the joint action of the Association and the Society at the New York meeting in respect to possible assistance for the *Revue Semestrielle* and the *Fortschitte*.

When this issue reaches the readers the second summer meeting of the ASSOCIATION will have been ended. The program was practically as announced in the June MONTHLY except that some changes were finally made in respect to those who were to lead the various discussions. The full report of the meeting will appear in the October issue.

The *Rendiconti del Circolo Mathematico di Palermo* has announced that, on account of conditions due to the war, its publication will be suspended until the war is over. The editors of the MONTHLY wish to manifest their sympathy and to state that the exchange copy of the MONTHLY will be sent regularly as heretofore.

Science and Learning in France is the title of a volume of 454 pages recently issued by the Society for American Fellowships in French Universities. The mathematical editors are D. R. CURTISS, T. F. HOLGATE, E. H. MOORE, and E. B. WILSON. The Introduction appears under the two sub-headings "The Mind of France" and "The Intellectual Inspiration of Paris," written by CHARLES W. ELIOT and GEORGE E. HALE respectively, and a list of about one thousand sponsors follows the list of authors.

General WILLIAM H. BIXBY, U. S. Army, retired, a charter member of the Association, was on duty at Kansas City, Missouri, during June in charge of the Kansas City U. S. Engineer Office and its Missouri River improvements. At the end of June he was transferred to St. Louis as president of the Mississippi River commission in addition to his other work.

A communication from the Oxford University Press calls attention to an error in the price of Hill's "Development of the Arabic Numerals" as stated by the reviewer in the December, 1915, issue of the MONTHLY. It should be $2.50 instead of $1.00. Prices of books reviewed are quoted for the benefit of the MONTHLY readers, and it was supposed that this price was correctly given.

The sixth regular meeting of the Association of Mathematics Teachers of New Jersey was held at Rutgers college on May 26, under the presidency of Professor C. O. GUNTHER, of Stevens Institute of Technology. The program consisted of the "Final report of the committee on courses in trigonometry," by C. O. GUNTHER; "Preliminary report of the committee on high school courses," by A. W. BELCHER, of Newark high school; "Rectification of circular arcs," by SYDNEY SEIDLER, of Rutgers college; "Some applications of complex quantities," by C. O. GUNTHER; "Limits in elementary mathematics," by RICHARD MORRIS, of Rutgers College.

North Carolina has effected an organization of the teachers of secondary mathematics for the western portion of the state. The following officers have the work in hand: W. W. RANKIN, of the University of North Carolina, president; J. W. LASLEY, of the University of North Carolina, secretary-treasurer. This organization is the outgrowth of the feeling that the conditions in the secondary schools and colleges of the state would be bettered by a coöperation of the teachers. At the first meeting, held in Greensboro, April 13 and 14 the situation was gone over with care and plans were laid which have as their end better teaching of mathematics in North Carolina.

The special articles of expository or historical character in the September issue of the *Annals of Mathematics* are (1) "Minimal surfaces applicable to surfaces of revolution," by Dr. J. K. WHITTEMORE, of Yale University, and (2) "Ratio, proportion and measurement in the *Elements* of Euclid," by Professor HENRY B. FINE, of Princeton University. This is the second issue of the *Annals* under the new arrangement with the ASSOCIATION, the first issue being in June, 1917, which contained the expository article by Professor L. E. DICKSON on "Fermat's last theorem and the nature and theory of algebraic numbers."

The number of new subscribers to the *Annals* under the terms of coöperation with the ASSOCIATION is now (September 1) 350. The opportunity is still open at the half rate to members of the ASSOCIATION and to applicants for membership.

The great military preparations now engaging the United States have materially reduced the summer session attendance in many of the universities. The attendance during the coming year will, no doubt, be reduced in even a larger proportion. This will entail a reduction in the teaching force already partially discounted by the resignations of members of the faculties. A number of instructors and assistant professors of mathematics have enlisted or entered training camps. The following items, manifestly incomplete, have thus far been reported to the Editors: Mr. P. B. HILL, assistant in mathematics at Wabash College, and Mr. OWENS and Mr. FARRIS, assistants in mathematics at the Alabama Polytechnic Institute, were in the first Reserve Officers' Training Camps. Mr. J. F. CONNER, instructor in mathematics at the Catholic University of America, has been commissioned first lieutenant in the Bureau of Supplies and Accounts, United States Navy. Assistant Professor K. P. WILLIAMS, of Indiana University, has been commissioned captain of a company of field artillery recently mustered into the service of the nation. Professor PETER FIELD, of the University of Michigan, attended the first training camp at Fort Sheridan, Chicago, and is now a captain of ordnance in the coast artillery located at Fortress Monroe. Dr. WALTER L. HART, Benjamin Pierce Instructor at Harvard University, has been commissioned a second lieutenant in the field artillery section of the regular army. Mr. E. P. Hubble, candidate for the doctorate in astronomy and mathematics at the University of Chicago, has been commissioned captain of infantry, officers reserve corps, in the national army. Mr. LESTER S. HILL, instructor in mathematics at Princeton University, has been given a staff commission as ensign in the United States Navy and has been called into active service. Dr. R. W. BURGESS, of Brown University, is in the government service in the statistical department at Washington. Assistant Professor A. L. UNDERHILL, of the University of Minnesota, has a captain's commission in the Coast Artillery, and is now at Fort Constitution, Portsmouth, N. H.

AN APPEAL TO ALL MEMBERS OF THE ASSOCIATION:

Please report to Professor D. A. Rothrock, Chairman of the Committee on Notes and News, Indiana University, Bloomington, Indiana, all items within your knowledge, appropriate for the news column of the MONTHLY such as promotions, new appointments, club meetings, etc., and, particularly at this time, any facts concerning members of the ASSOCIATION (or others in the mathematical field) who have entered, or are about to enter, service for the government in any capacity. It is desired to make this column of genuine interest and value to the members, and to this end universal coöperation is needed. Please do this *at once* and from *time to time*.

Townsend's Functions of a Complex Variable

By E. J. TOWNSEND, Professor and Head of the Department of Mathematics in the University of Illinois. (AMERICAN MATHEMATICAL SERIES.) vii + 384 pp. 8vo. $4.00.

This book is based upon a course of lectures given by the author for a number of years at the University of Illinois. It is intended as an introduction to the subject, suitable for seniors and first year graduate students. It presupposes a knowledge of calculus such as is usually given in the undergraduate course. Such topics from real variables as are not ordinarily included in an elementary course in calculus, and made use of in the development of the theory of functions of a complex variable are discussed in the text.

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SECOND SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

The second summer meeting of the Association was held by invitation of Western Reserve University and Case School of Applied Science at Cleveland, Ohio, on Thursday and Friday, September 6–7, 1917, in conjunction with and following the summer meeting of the American Mathematical Society. There were 90 persons in attendance at the various sessions, including the following 72 members of the Association:

O. P. AKERS, Allegheny College.
FLORENCE E. ALLEN, University of Wisconsin.
R. B. ALLEN, Kenyon College.
L. D. AMES, University of Missouri.
FREDERICK ANDEREGG, Oberlin College.
R. C. ARCHIBALD, Brown University.
G. N. ARMSTRONG, Ohio Wesleyan University.

GRACE M. BAREIS, Ohio State University.
I. A. BARRETT, Chicago, Ill.
MRS. W. E. BECKWITH, College for Women, Western Reserve University.
H. F. BLICHFELDT, Stanford University.
J. W. BRADSHAW, University of Michigan.
R. W. BURGESS, Brown University.

W. D. CAIRNS, Oberlin College.
FLORIAN CAJORI, Colorado College.
W. M. CARRUTH, Hamilton College.
G. E. CARSCALLEN, Hiram College.
E. H. CLARKE, Hiram College.
G. R. CLEMENTS, U. S. Naval Academy.
BYRON COSBY, Kirksville (Mo.) State Normal School.
A. R. CRATHORNE, University of Illinois.
C. H. CURRIER, Brown University.

F. F. DECKER, Syracuse University.
R. M. DEMING, Case School of Applied Science.
L. W. DOWLING, University of Wisconsin.

JOHN EIESLAND, West Virginia University.
L. P. EISENHART, Princeton University.

T. M. FOCKE, Case School of Applied Science.
TOMLINSON FORT, University of Alabama.

M. G. GABA, Cornell University.
D. C. GILLESPIE, Cornell University.
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A. M. HARDING, University of Arkansas.
H. E. HAWKES, Columbia University.
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WILLIAM HOOVER, Ohio University (Retired).
E. V. HUNTINGTON, Harvard University.
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H. L. Olson, Chicago, Ill.

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R. P. Stephens, University of Georgia.
E. B. Stouffer, University of Kansas.
W. T. Stratton, Kansas State Agricultural College.

C. F. Thomas, Case School of Applied Science.
R. P. Thomas, College of Wooster.
M. O. Tripp, Olivet College.

T. O. Walton, William and Vashti College.
D. T. Wilson, Case School of Applied Science.

B. F. Yanney, College of Wooster.
J. W. Young, Dartmouth College.

It is noteworthy that the attendance from the more distant parts of the country included one each from Alabama, California, Colorado, Connecticut, Florida, Georgia, Iowa, Maryland, New Hampshire and New Jersey; two each from Arkansas, Massachusetts, eastern Pennsylvania and Texas; three from Kansas; four each from Canada, eastern New York and Rhode Island; and five from Missouri.

The meetings began with a joint session of the Association with the American Mathematical Society on Thursday morning at nine o'clock. In the absence of Professor L. E. Dickson, president of the Society, President Florian Cajori of the Association called to the chair Professor E. R. Hedrick, ex-vice-president of the Society and ex-president of the Association. Professor L. P. Eisenhart of Princeton University read a paper prepared by invitation of the program committees of the two organizations, the subject being "Darboux's contribution to geometry." This address gave a clear and full analysis of the varied contributions to geometry of this noted mathematician who died last year. In particular Professor Eisenhart reviewed Darboux's work in the field of triply orthogonal systems of surfaces, of the deformation of surfaces and rolling of applicable surfaces, of infinitesimal deformation, of spherical representation of surfaces, and his development of the moving axes of coördinates. Special emphasis was laid upon Darboux's use of imaginary geometric elements, and in particular of isotropic cylinders and developables. The address will be published in the *Bulletin* of the American Mathematical Society.

The joint dinner of the two organizations was held Wednesday evening at the Hotel Statler and proved to be one of the most enjoyable of these periodical dinners, even though the attendance of seventy-seven was not so large as usual. Under the genial toastmastership of Professor Huntington speeches were made by the following: President C. F. Thwing, Western Reserve University; Professor T. M. Focke, Case School of Applied Science; Professor Virgil Snyder, Cornell University; Professor Florian Cajori, Colorado College; Professor W. C. Grau-

stein, Rice Institute; President E. E. Braithwaite, Western University, London, Ontario; Professor O. D. Kellogg, University of Missouri; Professor J. W. Young, Dartmouth College; Professor Tomlinson Fort, University of Alabama; and Professor F. R. Moulton, University of Chicago. After these somewhat formal exercises, the remainder of the evening was spent in social intercourse. Postcard greetings signed by those present were sent to our editor-in-chief, Professor Slaught, who was unable to be present at the meeting.

On Wednesday afternoon between the meetings of the Society and those of the Association an organ recital was given in Amasa Stone Chapel by Professor Clemens, and an opportunity was afforded for a visit to the Cleveland Art Museum. President and Mrs. Charles F. Thwing entertained the members of the Association at a tea given at their home on Thursday afternoon at four o'clock. Particular appreciation was expressed by the ladies because of the arrangements provided for their comfort and entertainment by Mrs. W. E. Beckwith and Dr. Mary F. Curtis of the College for Women of Western Reserve University; this included an automobile ride about the city on Thursday morning. Luncheon was served each day at the Case Club in the immediate vicinity of the campus; this club with the convenient lounging room supplied a means of promoting that good-fellowship which is so important a part of the Association meetings. A formal resolution was adopted at the closing session, expressing the thanks of the Association to the authorities of Western Reserve University and of Case School of Applied Science for their invitation to hold the meetings at Cleveland; to all who assisted in making the stay in Cleveland so pleasurable, in particular to President and Mrs. Thwing for their hospitality, to Professor Clemens for his interesting recital, to those who made the arrangements for the visiting ladies, and to Professors Focke, Pitcher and Wilson who as local members of the committee on arrangements planned so effectively; and finally, to the program committee who under the chairmanship of Professor C. S. Slichter prepared so successful a program.

The sessions of the Association continued through Thursday morning and afternoon, and Friday morning, Professor Cajori presiding except for the last half of the session on Thursday afternoon, when he called to the chair Professor Keppel of the University of Florida. The sessions were all held in the lecture room of the Physics Building of Case School. The program and the abstracts of the prepared papers and discussions follow. It will be seen that the topics prepared by the committee for this meeting are well adapted to the needs of the average college teacher, and, as a natural result, the papers provoked much profitable and lively discussion.

At the close of the session on Friday Professor J. W. Young gave an informal report on the activity of the National Committee on Mathematical Requirements. No account of this will be given here since full and authoritative information from Professor Young will reach our members through a formal report to be printed in the November MONTHLY.

Order of Topics on the Separate Program.

(1) "Undergraduate Mathematical Clubs." Professor H. E. Hawkes, Columbia University.

(2) Discussion led by Professor R. C. Archibald, Brown University, and Professor D. A. Rothrock, Indiana University.

(3) Presidential Retiring Address: "The Significance of Mathematics." Professor E. R. Hedrick, University of Missouri.

(4) "Geometry for Juniors and Seniors." Professor E. B. Stouffer, University of Kansas.

(5) Discussion led by Professor Arnold Emch, University of Illinois, and Professor L. W. Dowling, University of Wisconsin.

(6) "The Treatment of the Applications in College Courses in Mathematics." Professor L. C. Plant, Michigan Agricultural College.

(7) Discussion led by Professor W. A. Hurwitz, Cornell University, and Professor A. M. Kenyon, Purdue University.

Abstracts, numbered to correspond with the numbers on the foregoing program, are printed below, together with reports of some further informal discussions.

Abstracts of Papers.

(1) In speaking of undergraduate mathematical clubs Professor Hawkes first laid emphasis on the importance of such extra-curricula activities of college students as have to do with things of the mind. Although it is difficult for the average undergraduate to understand it, such activities express a deeper and more genuine loyalty to the college than athletics. After pointing out in some detail the benefits that members of a mathematical club might expect to derive from such an organization, he discussed various methods of organizing clubs. He thought that the particular type of organization made little difference provided some member of the faculty was able and willing to devote time and energy to the work of directing the students in the preparation of their papers, the outstanding fact being that the club will not run itself.

The latter part of the paper was devoted to types of topics, some of which are, and some of which are not, well adapted for presentation at the club by undergraduates.

(2) Professor Archibald gave an account of the Brown University mathematics club, whose organization, under the direction of a committee on arrangements and of a program committee, he explained. He then gave a number of facts concerning the organization of 26 undergraduate clubs whose location he indicated. The one at Smith College founded in 1899 was referred to as one of the oldest if not the oldest; and among the youngest were those founded during the past year at Alabama, North Carolina, Oklahoma, Oregon and Texas. In conclusion he spoke as follows:

"I wish to suggest that a special department of the Monthly be devoted to

undergraduate clubs. Probably forty clubs of this kind have been organized already and the increase in this number in the near future is sure to be rapid. There should be some connecting bond, some central source of suggestion, of interchange of ideas, and of inspiration. The MONTHLY has served as such an intermediary in the past, but it appears to me that much more might be done in the future. For instance, in each January issue might be published a list of the clubs with statistics as to membership and meetings. For future issues there would be a wealth of material to draw upon in lists of officers, programs and miscellaneous notes which might be procured from club secretaries. Then, too, such a special department would tend to draw out suggestions, discussions, papers and bibliographies for suitable program topics,—all contributing to the organization and development of these new forces, the undergraduate clubs, which are destined, I believe, in no small way to promote the cause of mathematics in America."

Professor Rothrock gave a report upon the results obtained from a questionnaire sent to 160 colleges and universities of the United States. Only a very brief summary can be given here. There were 110 replies, 31 of which reported the existence of clubs with a total membership of about 900, ranging from 10 to 106 in the different institutions. Membership is, in general, invitational, but the door is open to practically all students above freshmen who are interested in mathematics. Most clubs are conducted by students with the coöperation of the faculty. The programs consist, for the most part, of topics from the history and pedagogy of mathematics, with some attention to the curiosities of mathematics; and, as well, addresses on topics of general interest to all by faculty members and invited guests.

Thirty of the clubs responding to the questionnaire have only the highest praise in favor of the club. According to these it is highly beneficial to the students in giving them opportunity to prepare and present in a public way the results of study, and it gives them insight into many phases of mathematics not otherwise treated in college courses.

In reply to an inquiry, Professor Roe described the mathematical fraternity, Pi Mu Epsilon, at Syracuse University. This club of about 55 members, which takes the place of the usual mathematical club, is a scholarship fraternity, the membership of which is made up from the best members of the higher classes. It was asserted to be a great stimulus both to students and faculty, the students doing better work in their courses in order to qualify themselves for membership. Professor Decker added that in an institution where fraternities abound, the Greek letter mathematical fraternity appeals strongly to the students' interest.

Professors Huntington and Fort described a mathematical club at Harvard University whose membership is confined entirely to advanced students, being dependent solely on their own inspiration and guidance and independent of any embarrassing or dampening effect due to the presence and criticism of faculty members. Messrs. Barnett and Clarke described briefly the mathematical clubs at the University of Chicago, Professor Dowling and Dr. Clements those at the

University of Wisconsin, and Professors Miller and Rietz those at the University of Illinois.

A large number of members took part in this discussion by way of inquiry or suggestion. Some of these related to membership fees (ranging from a simple annual fee of twenty-five cents for postage, etc., to a fee of three dollars to provide for "spreads"), limitations of numbers, voluntary *versus* assigned papers, measures to care for the passive students and to arouse discussion in the clubs, problem solving as an integral part of college work, and still other questions. The eagerness of inquiries and the equal readiness to share experiences made it very evident that the Association, through such conferences or through some still more effective means, can be of great service in bringing to many colleges and universities the benefits to be derived from the development of mathematical clubs. A symposium of these papers and discussions will be published in the MONTHLY.

(3) The presidential address of the retiring president, Professor E. R. Hedrick, dealt with the significance of mathematics in several phases. As an offset to recent criticisms of mathematics, the essential necessity of a widespread knowledge of quantitative relations, as revealed by the existing war, was emphasized. It was urged that the Association might be a center for increasing the appreciation of mathematics of collegiate grade. Finally the recognition of applied mathematics by the Association was mentioned, and the possible function of the Association in stimulating work in applied mathematics was emphasized. The paper will be published in full in an early issue of the MONTHLY.

(4) In his paper on "Geometry for Juniors and Seniors" Professor Stouffer first discussed the need of a course in geometry for juniors and seniors, and then considered some general principles which he believed to be essential to the proper selection and arrangement of the material for such a course. A brief outline of a course in projective geometry was then given which should couple together the earlier courses in geometry and form an introduction to advanced courses which might be taken. This paper will be published at an early date in the MONTHLY.

(5) Professor Emch agreed with Professor Stouffer in the most important points. His principal postulate in his comment was that much more attention should be paid than at present to the constructive side of elementary geometrical instruction with the use of various geometrical instruments. He added that instruction in geometry should run parallel with instruction in algebra. He differs from Professor Stouffer in placing less emphasis on the synthetic as opposed to the analytic side, and in contending that the pupil in a first course in projective geometry should be enabled to solve graphically problems in perspective or central projection rather than to reason successfully on the abstract parts of the subject, the latter being reserved for a second course.

Professor Dowling found himself in substantial agreement with Professor Stouffer's paper. He objected to the vocational *raison d'être* for mathematics in general and for geometry in particular, but wished to emphasize the constructive side of projective geometry on the important ground that prospective

teachers of geometry need training in the power of visualization. He felt that the proposed outline of topics for a course in projective geometry placed the discussion of conics and the consequent construction work too far along in the course. He preferred to follow Reye in this particular, even if the removal of analytic discussion proposed by Professor Stouffer necessitated a postponement of the consideration of continuously projective forms until a later chapter. He emphasized particularly one point of the main paper in saying that the greatest cultural value of projective geometry arises from a study of the generalizing methods and principles characteristic of modern work in geometry.

In the extension of this discussion Professor Hedrick urged that geometry should be studied in this country more than is now the case; Professor Eisenhart called attention to the later volume of Darboux on the principles of analytic geometry which presents the advantages of coördinate systems, a discussion of the geometry of Cayley, and of transformations by inversion, a book which could very properly be used with juniors; Mr. Barnett mentioned a recent Italian text organized somewhat along the lines of Professor Stouffer's course; Professor P. F. Smith pointed out that later college courses in mathematics need not be justified in the way necessary with earlier courses, that the aim in projective geometry might very well be to interest the student and to prepare for his future courses; and Professor Huntington remarked that the treatment of imaginary elements must depend not on visualization but on analytic means.

(6) Professor Plant in his paper on the "Treatment of the Applications in College Courses in Mathematics" pointed out that it would first be necessary to consider why applications, other than a limited number of a geometrical character, should find any place at all in college courses in mathematics. He showed that varied applications are justified if by their use the teacher can (1) arouse an interest in the subject by bringing out its beauty through geometry, or by emphasizing its utility through mechanics; or (2) clarify the fundamental ideas that underlie the subject through geometry, physics, or mechanics; or (3) can more effectively put the student in possession of a powerful instrument for the development of the exact sciences. He also pointed out that the kind of students for whom the applications are written must be kept in mind. Students who elect courses in pure mathematics find pleasure in the logic of the treatment. The majority of students, however, belong to that class who take mathematics because it is a means to an end, and for this class applications will prove of interest only if the student is properly prepared to handle them.

After a discussion of the assumptions made by different teachers and authors concerning the preparation of the student, the conclusion was reached that it is not safe to base the treatment of applications upon ideas and principles which are likely to be outside the student's general experience. Since also instructors are quite prone to think that whatever amount of information the text may contain is sufficient for their classes, it follows that the preparation for the particular applications must be supplied by the author if he wishes to be certain that his readers are prepared to understand them. This, of necessity, reduces

the number of applications he can use and still keep his text within reasonable limits. For this reason, as well as for others, applications should be so selected and treated that they become an integral part of the course, the application bringing out more clearly certain mathematical principles and concepts and at the same time the mathematics appearing to have been created for the application.

The use of a certain kind of problem to supplement, but not to be substituted for, the applications just outlined was emphasized. Such problems may be selected from physics, chemistry, or mechanics, and are stated in the form of assumptions. Since they are so stated the student can work them, even though he does not appreciate their physical significance. While these problems serve to familiarize the student with a type of work actually encountered in the sciences, their greatest value comes from the fact that they furnish assumptions which appeal to the student's interest.

(7) [Both the speakers announced for the discussion of this paper, Professors Carver and Ling, were unable to be present because of serious illness in their families. Professors Hurwitz and Kenyon kindly consented to supply these places on the program, and their discussions, though given on brief notice, were highly appreciated.]

Professor Hurwitz, mentioning two frequently considered reasons for teaching applications in courses in mathematics, namely, the promotion of the student's interest in the subject and the clarification of mathematical principles, emphasized as an important third reason the value to cultural students of such applications as ends in themselves. He deprecated the use of many problems based on concepts neither belonging to the student's normal experience nor admitting of ready explanation to him; for such questions as rest on ideas not in the student's experience, it is advisable to state the general laws involved with brief explanation, clearly marking the distinction between the assumptions and the mathematical deductions. He warned against insistence on pseudo-rigor in the treatment of applications in elementary courses. In concluding, he pointed out the value and the practicability of requiring the students to make problems for themselves, as well as to gather them from the abundant sources of puzzle-supply in the newspapers and magazines.

The primary reason for the use of applications in college courses in mathematics, in Professor Kenyon's opinion, is to give the student an insight into the meaning and implications of the fundamental principles of mathematics. Applications treated under the guidance of this principle are a strong stimulus to the interest of the student. Professor Kenyon has scant regard for courses in which the applications rather than the mathematical principles determine the material, arrangement, and method of presentation. In case of students in scientific and technical courses, a second reason for treating applications is to develop and exercise the ability of the student to make similar applications for himself in subsequent studies and work. To do this independently and confidently requires a certain attitude and maturity of mind which results, if at all, from a somewhat extensive experience and drill in the field of mathematical and scientific ideas,

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H. IVAH THOMSEN, Baltimore, Md.
ROSS B. WILDERMUTH, Assistant professor of mathematics, Capital University, Columbus, Ohio.
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JAY W. WOODROW, Assistant professor of physics, University of Colorado, Boulder, Col.
CHIA-CHEOW YEN, Professor of mathematics, Chinese Government Engineering College, Tangshan, China.

To institutional membership:

CULVER-STOCKTON COLLEGE (formerly Christian University), Canton, Mo.
UNIVERSITY OF PORTO RICO, Mayagüez, P. R.

(2) The Council having voted at the New York meeting that the winter meeting of the Association should be held in Chicago in conjunction with the Chicago Section of the American Mathematical Society, the determination of the exact dates for this meeting was referred by the Council to the joint committee on arrangements, the composition of which, as also of the program committee, is to be announced by President Cajori after consultation with the proper authorities of the Society.

(3) Certain contemplated plans which concern the editorial care of the MONTHLY and a proposal to change the manner of choosing the secretary-treasurer are to be communicated to the whole Council and then laid before the members through the columns of the MONTHLY, in accordance with the provisions of the constitution concerning amendments to the constitution or by-laws.

(4) Measures for taking care of certain expense in the collecting of data for the National Committee on Mathematical Requirements were referred with power to the Committee on Finance of the Council.

(5) A plan for a mathematical dictionary, suggested in the main by Professor G. A. Miller, was brought by him before the Council. It was agreed that the plan, if it can be carried out, is so important and that the questions of the various details, as to size, cost, extent, etc., are so involved, that a committee of five should be appointed by the president, to make an extended study of the plan, and to report to the Council.

W. D. CAIRNS, *Secretary-Treasurer.*

ALGEBRA COURSES FOR COLLEGE JUNIORS AND SENIORS.

Edited by U. G. MITCHELL, University of Kansas.

At the fourth meeting of the Kansas Section of the Mathematical Association of America a large part of the program was devoted to the consideration of

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At the fourth meeting of the Kansas Section of the Mathematical Association of America a large part of the program was devoted to the consideration of

algebra courses for college students. Students taking such courses are, for the most part, preparing to teach in secondary schools, to do research work in mathematics, or to enter applied sciences. The plan of the program committee was to have three men present the subject independently from these three different points of view and by noting the material common to the three selections proposed determine the basis for a course which would answer for all three classes of students. Summaries of the papers in the order of their presentation and of the conclusions brought out in the general discussion are given below.

A. For Students Preparing to Teach, by U. G. Mitchell, University of Kansas.

We take for granted that the college juniors and seniors referred to have taken, during the first two years of their college life, the usual freshman and sophomore courses in college algebra, plane trigonometry, analytical geometry and calculus.

The chief consideration in determining such junior-senior algebra courses is unquestionably the prospective teacher's needs which a study of algebra can supply. Laying aside, for the moment, the question as to what needs can be supplied by algebra, some of the teacher's needs which come immediately to mind are:

1. *Interpreting power.* The teacher needs to grasp quickly and accurately the meaning of a problem, whether abstract or concrete, whether stated orally or in print, and to translate this meaning into mathematical terms. This power depends largely, it is true, upon innate mentality; but it also depends largely upon familiarity with forms of statement and with the concepts and conditions involved. It is a power which may be greatly cultivated, even in persons whose mental processes are not naturally quick.

2. *Analytical Power.* By this term is meant the power to separate conditions or variables from each other and to comprehend readily their interrelations. This implies, of course, synthesizing power as well.

3. *Mathematical perspective.* The teacher needs mathematical knowledge which goes much beyond that which he has to teach in order to estimate properly the relative importance of what he teaches. For example, the teacher who has solved no equations of higher degree than the second and the teacher who is familiar with the solutions of the cubic and biquadratic and knows the essentials of the theory of elimination and the Galois theory of equations will view the quadratic equation in very different perspectives.

4. *Facility in performing algebraic operations.* Facility is here used to imply not only speed but also such mastery as combines ease of performance with accuracy.

5. *An understanding of the number system as a logical development.* The number system is first learned as a practical and not as a logical system. There is no attempt to formulate explicitly the assumptions underlying algebra in any way comparable to the recognition of the definitions, axioms and postulates of elementary geometry. Consequently students have no precise knowledge of

these assumptions or of their limitations. They readily extend to negative and imaginary numbers assumptions applicable to positive numbers only. It will be an unusually bright junior class in which there is unanimity of opinion as to the product of $\sqrt{-3}$ by $\sqrt{-5}$ or in locating the error in such simple fallacies as the "proofs"[1] that $2 = 1$ and $-1 = 1$. The teacher of elementary algebra should certainly know enough concerning the assumptions underlying algebra to be sure of his ground within the scope of his teaching. It is only by such a knowledge that he gains sufficient mastery to know definitely and exactly what operations, processes, and manipulations of symbols are valid and what are not permissible.

6. *An understanding of elementary mathematics as a historical development.* Biologists tell us that, in broad outline, the child in his development repeats the experience of his ancestry—that the theory of recapitulation obtains. Whether or not this be true in general, it is certainly true that the student who has studied successfully the historical development of mathematics gets an objective view of his own mathematical evolution, much as the study of Latin or other foreign language gives the student an objective view of English, which can be obtained in no other way. The teacher needs this view in order to understand and direct most intelligently the mathematical development of those under his instruction. It also proves an excellent source of information from which enriching and stimulating material can be drawn.

Other needs, possibly as important as some of these, might be added; but considering only these, for our present purposes, we turn to the question of what material of an algebraic nature can be used to supply these needs.

For developing *interpreting power*, problems exhibiting functional relations from any other sciences or life-conditions, graphs involving such functional relations, and problems involving pure number relations would seem to be the most suitable material. *Analytical power* can also be developed by such problems as well as by algebraic proofs.

All algebraic material would, of course, contribute to *mathematical perspective;* but the following may be selected as best adapted for the purpose: sequences, limits, theorems on limits, series (including convergence tests, exponential, binomial and logarithmic series and operations with series), permutations and combinations, multinomial theorem, symmetric functions of roots of equations, transformations of equations, solutions of cubic and biquadratic equations, systems of linear equations, determinants and elimination, simultaneous quadratics, methods of approximating roots of numerical equations (Newton's and Horner's), fundamental theorem of algebra and its corollaries.

Facility in performing algebraic operations will be aided by the work in theory of equations and can be obtained by the use of exercises of sufficient difficulty; but the probability is that juniors and seniors will have gained considerable (possibly adequate) facility of this kind from the large amount of algebraic work in the differential and integral calculus.

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[1] Cf. BALL'S *Mathematical Recreations and Essays*, fourth ed., pp. 24–26.

To gain an *understanding of the number system as a logical development* the student may begin with definitions of cardinal and ordinal integers (positive, of course) and formulate exactly the laws of order and procedure for that system. The number system is next enlarged by introducing fractions on an ordinal basis and reconsidering the laws of procedure and order in relation to these new numbers. Similarly, the system is successively enlarged by the introduction of negative, irrational and complex numbers, taking care at the time of each enlargment to make such extensions, restatements and modifications of the previous laws as will apply. Approximately such a development is found, for example, in the first 75 pages of Fine's *College Algebra.*

There is, of course, no strictly algebraic material which can give an *understanding of the number-system as a historical development.* Such an understanding is best obtained from a course in the history of mathematics. If, however, no such course is given to juniors and seniors, it would seem as if some such historical material should be studied in connection with the study of the logical development of the number system.

Sufficient material has been suggested for a three-hour course throughout a year. The first semester's work might well consist of the theory of equations and related topics and the second semester's work of the logical (and somewhat historical) development of the number system, sequences, limits and series.

B. For Students Preparing to Do Research Work, by W. H. Garrett, Baker University.

It is evident that the junior-senior course in algebra will depend as to its character and extent on the course in algebra which has preceded it—the "college algebra" course which the student has taken as a freshman. I shall suppose that the junior or senior has had such a course, in which the usual topics in theory of equations and determinants have been treated in a brief and elementary way. I believe the tendency of the present day is to cut the course in algebra to the smallest dimensions possible, reserving the discussion of the more abstract parts until after the student has had courses in analytic geometry and calculus. In this connection I may say that at Harvard a student may take Mathematics 3 —Introduction to Modern Algebra and Geometry—without having had a course in college algebra and so, in that course, it is necessary to take up the study of determinants from the very beginning. It is my opinion, however, that the average student will do well to have taken the college algebra before entering such a course.

Supposing, then, that the student has had a brief course in college algebra, the question remains as to what topics to present in the junior-senior course. Here, again, I find the tendency is to make the course as brief as possible. In fact, in two of the large universities, there is no mention in their catalogs of such a course being offered to undergraduates. On the other hand, more time and attention is being given to courses along geometrical lines which seem to many to be more interesting and important at that stage of the student's progress. A

three-hour course for one semester should afford ample time to cover all the necessary topics in algebra the undergraduate needs as a preparation for graduate courses and with some classes the ground could be adequately covered in two lectures a week for one semester.

In such a three-hour course there would be time for the presentation of most, if not all, of the topics named below, although not necessarily in the order mentioned. For example, there appears to be no general agreement as to whether the theory of determinants or equations should be taken up first.

Two of the topics would be complex numbers with their geometric representation and the proof of DeMoivre's theorem by mathematical induction. The proof by mathematical induction often given in the college algebra course, would better be postponed, I believe, until this time.

Under the theory of equations I should recommend about such topics and treatment as are given in Dickson's *Elementary Theory of Equations*, including the general properties of polynomials, their continuity and graphical representation, a clear statement of the fundamental theorem of algebra and the necessity for a proof, corollaries following the fundamental theorem, relation of roots to coefficients, imaginary roots, Descartes's rule of signs, elementary transformations of equations, the solution of equations by Newton's method rather than by Horner's, the algebraic solution of the cubic and quartic, multiple roots and Sturm's theorem.

I would not include the proof of the fundamental theorem of algebra. I believe it is time wasted to try to present its proof to the average junior-senior class, since any proof that can be given at this stage will seem long and difficult to the student and will not really teach him anything; whereas, if the proof is postponed to a first course in the theory of functions, it can be made very simple and instructive. In connection with the discussion of Sturm's theorem there may be need for teaching the method of finding the highest common divisor by continued division, a method which was formerly taught in every course in elementary algebra, but which is now generally relegated to the appendix or eliminated altogether.

Under optional topics might be included reciprocal equations, the elementary theory of symmetric functions and its applications to resultants and discriminants of polynomials in a single variable.

In connection with the study of determinants there is a good opportunity to do some constructive work which will be of direct use to the student later on in his graduate work. Following the definition of a determinant and some of the elementary theorems, including Laplace's development and its application to the multiplication theorem, there should be emphasized the topic of linear dependence and the application of determinants to the solution of linear equations, including the solution of systems in which the number of unknowns is not equal to the number of equations, and including systems of homogeneous and non-homogeneous equations. In this connection the first four or five chapters in Bôcher's *Introduction to Higher Algebra* present just what is needed, although the

in series other than power series; partial differentiation, especially of composite functions. [For example, a clear understanding of the expression

$$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x},$$

where $u = f(x, y, z)$ and $z = \varphi(x, y)$.] Line integrals, with Stokes's theorem, vector analysis.

D. Conclusions.

During the general discussion the speakers modified slightly their choices of material. The following was practically agreed upon as the basis for a course suitable for all three classes of students and follows very nearly the material common to the three original selections. The order here given is not to be considered as indicating necessarily the best order of presentation: Systems of linear equations, determinants and elimination, graphical representation of polynomials, transformations of equations, solutions of the cubic and biquadratic, methods of locating and approximating roots of numerical equations—Descartes's rule of signs, Sturm's theorem, multiple roots, Horner's method of approximation, Newton's method of approximation (if only one method of approximation is given, Newton's is to be preferred), fundamental theorem of algebra and its corollaries, symmetric functions of the roots of an equation, permutations and combinations, problems involving pure number-relations.

The subject-matter above suggested would furnish about enough material for a three-hour course during one semester.

A LIST OF MATHEMATICAL BOOKS FOR SCHOOLS AND COLLEGES.[1]

Preliminary Statement. The following list contains the titles of 160 books which it is believed are suitable for purchase by the usual school or college library.[2] The plan has been carried out of forming the books into four divisions of 40 each, corresponding roughly to the mathematical attainments of the freshman, sophomore, junior and senior years, this arrangement also indicating incidentally the relative difficulty which the books present. Preference has been given throughout to books in the English language, and no text-books have been included except a few having a well-recognized value as books of reference. As it might be desirable for a school or college to know in advance the price of a

[1] Reference may be made to a similar list, but intended more especially for high schools and normal schools, which has recently been published under the auspices of the Teachers College, Columbia University, New York. The title is "A brief list of mathematical books suitable for libraries in high schools and normal schools," *Teachers College Bulletin*, Series 8, No. 3 (1916).

[2] The Library Committee was appointed at the Cambridge meeting of the Association in September, 1916, and a preliminary report was made at the New York meeting in December, 1916. (See the February Monthly, page 58.) Of the numerous tentative plans mentioned in this report, the following pages give the results of several months' work of the committee on the one considered of first importance. Editors.

in series other than power series; partial differentiation, especially of composite functions. [For example, a clear understanding of the expression

$$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x},$$

where $u = f(x, y, z)$ and $z = \varphi(x, y)$.] Line integrals, with Stokes's theorem, vector analysis.

D. Conclusions.

During the general discussion the speakers modified slightly their choices of material. The following was practically agreed upon as the basis for a course suitable for all three classes of students and follows very nearly the material common to the three original selections. The order here given is not to be considered as indicating necessarily the best order of presentation: Systems of linear equations, determinants and elimination, graphical representation of polynomials, transformations of equations, solutions of the cubic and biquadratic, methods of locating and approximating roots of numerical equations—Descartes's rule of signs, Sturm's theorem, multiple roots, Horner's method of approximation, Newton's method of approximation (if only one method of approximation is given, Newton's is to be preferred), fundamental theorem of algebra and its corollaries, symmetric functions of the roots of an equation, permutations and combinations, problems involving pure number-relations.

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book which it wished to buy, an effort has been made to ascertain the present selling price and indicate it in each case, but this has not been possible in all cases owing to the uncertainty now existing in this matter, especially as regards books published by foreign houses.

Needless to say, the list contains but a few of the possible books which might appropriately be recommended for the purpose in hand. It is hoped, however, that it indicates what may well constitute a nucleus for the college library.

THE LIBRARY COMMITTEE OF THE ASSOCIATION.

FLORIAN CAJORI, Colorado College.
E. S. CRAWLEY, University of Pennsylvania.
SOLOMON LEFSCHETZ, University of Kansas.
W. R. LONGLEY, Yale University.
R. E. ROOT, U. S. Naval Academy.
W. B. FORD, University of Michigan, *Chairman*.

BOOKS FOR FRESHMEN.

Algebra.

1. FINE (H. B.). The number-system of algebra treated theoretically and historically. Heath. $1.00.
   Admirable introduction to the critical study of algebra.
2. HALL (H. S.) and KNIGHT (S. R.). Higher algebra. Macmillan.
   Gives the technique of a wide variety of topics, with numerous examples.
3. MARSH (H. W.). Technical algebra. Wiley. $2.00.
   Shows the uses of algebra in industry.
4. SMITH (C.). A treatise on algebra. Macmillan.
   Good as a text and as a reference book. Should be in all libraries.

Geometry.

5. CASEY (J.). Sequel to Euclid. Longmans. $1.10.
   Amplifies the ordinary high school course in geometry.
6. GODFREY (C.) and SIDDONS (A. W.). Modern geometry. Cambridge University Press.
   Continues geometry in an interesting way beyond Euclid.
7. HALSTED (G. B.). Rational geometry. Wiley. $1.50.
   Considers the logical foundations of geometry.
8. MARSH (H. W.). Technical geometry. Wiley. $1.25.
   Shows the uses of geometry in modern industry.

Trigonometry and Surveying.

9. HOBSON (E. W.). Treatise on plane trigonometry. Cambridge University Press. $3.00.
   An extensive and thorough treatment of the theoretical side of trigonometry.
10. MARSH (H. W.). Technical trigonometry. Wiley. $1.50.
    Shows uses of trigonometry in industry.
11. TRACEY (J. C.). Plane surveying. Wiley. $3.00.
    Valuable both as text and field manual.

History.

12. BALL (W. W. R.). Primer of the history of mathematics. Macmillan. $0.90.
    Valuable summary of the principal events in the history of mathematics.

13. CAJORI (F.). History of elementary mathematics. Second edition. Macmillan, 1917. $1.75.
    A careful work, interestingly written.
14. HILL (G. F.). The development of Arabic numerals in Europe. Oxford Press. $1.75.
15. SMITH (D. E.) and KARPINSKI (L. C.). The Hindu-Arabic numerals. Ginn. $2.00.
    Scholarly presentation of the theories as to the origin of our numerals.

Miscellaneous.

16. ANDREWS (W. S.). Magic squares and cubes. Open Court Publishing Co. $1.50.
    One of the few treatments of this subject in English.
17. BALL (W. W. R.). Mathematical recreations. Macmillan. $3.50.
    Entertaining and instructive.
18. BRECKENRIDGE (W. E.), MERSEREAU (S. F.) and MOORE (C. F.). Shop problems in mathematics. Ginn. $1.00.
    Excellent collection of applied problems in elementary mathematics.
19. BRINTON (W. C.). Graphic methods for presenting facts. Engineering News Co. $4.00.
    Describes general uses of the graphic method.
20. COMPTON (A. G.). Manual of logarithmic computation. Wiley. $1.50.
21. CRACKNELL (A. G.). Practical mathematics. Longmans. $1.10.
    Gives many well-graded, illustrative problems.
22. DUNCAN (R. H.). Practical curve tracing. Longmans. $1.60.
23. DUNLOP (H. C.) and JACKSON (C. S.). Slide rule notes. Longmans. $0.75.
24. FABER (A. W.). Instructions for calculating rule. Faber, Newark, N. Y.
    A good manual on the slide rule.
25. GEPHART (W. F.). Principles of insurance. Macmillan. 2 vols. $1.60 each.
    Elements of the mathematical aspects of insurance.
26. HENRICI (O. F. M. E.). Congruent figures. Longmans. $0.50.
    Amplifies high-school geometry.
27. HINTON (C. H.). The fourth dimension. Sonnenschein, London.
    Interesting treatment of this popular subject.
28. HORSBURGH (E. M.). Handbook of the exhibition of Napier relics and of books, instruments, and devices for facilitating calculation. The Royal Society of Edinburgh.
29. JOHNSON (J. F.). Practical shop mechanics and mathematics. Wiley. $1.00.
30. KOCH (E. H.). The mathematics of applied electricity. Wiley. $3.00.
31. LAGRANGE (J. L.). Lectures on elementary mathematics. Open Court Publishing Co. $1.00.
    The reprint of a notable work written a century ago by a great mathematician.
32. MANNING (H. P.). Fourth dimension simply explained. Munn and Co., N. Y. $1.50.
    A collection of essays on this popular subject.
33. MARSH (H. W.). Industrial mathematics. Wiley. $2.00.

34. Row (T. S.). Geometric paper folding. Edited by Beman and Smith. Open Court Publishing Co. \$1.00.
    Gives interesting side light on the ordinary construction problems of plane geometry.
35. Sanborne (F. B.). Mechanics problems. Engineering News Co.
    One of the best collections of problems in elementary mechanics.
36. Saxelby (F. M.). Course in practical mathematics. Longmans. \$2.25.
37. Schubert (H.). Mathematical recreations. Open Court Publishing Co. \$0.75.
38. Weld (L. G.). A short course in the theory of determinants. Macmillan. \$1.90.
39. White (W. F.). Scrap-book of elementary mathematics. Open Court Publishing Co. \$1.00.
40. Whitworth (W. W.). Choice and Chance. Deighton, Bell and Co., London.
    Elementary treatment of this interesting subject, with numerous examples.

Books for Sophomores.

Analytical Geometry.

1. Salmon (G.). Conic sections. Longmans. \$3.75.
2. Smith (C.). An elementary treatise on conic sections. Macmillan. \$1.40.
3. Smith (C.). An elementary treatise on solid geometry. Macmillan. \$3.25.

Calculus.

4. Byerly (W. E.) Differential calculus. Ginn. \$2.00.
5. Byerly (W. E.). Integral calculus. Ginn. \$2.00.
6. Lamb (H.). An elementary course of infinitesimal calculus. Cambridge University Press. \$3.00.
7. Leib (D. D.). Problems in the calculus. Ginn. \$1.00.
8. Osgood (W. F.). A first course in the differential and integral calculus. Macmillan. \$2.00.
9. Peirce (B. O.). A short table of integrals. Ginn. \$1.00.
10. Perry (J.). Calculus for engineers. Arnold. London.

Projective Geometry.

11. Cremona (L.). Elements of projective geometry. Translation by Leudesdorf. Oxford Clarendon Press.
12. Emch (A.). Introduction to projective geometry and its applications. Wiley. \$2.50.
13. Hatton (J. L. S.). The principles of projective geometry. Cambridge University Press.
14. Mathews (G. B.). Projective geometry. Longmans.

History.

15. Ball (W. W. R.). A short history of mathematics. Macmillan. \$3.00.
16. Cajori (F.). History of mathematics. Macmillan. Second edition in press.

17. FINK (C.). A brief history of mathematics. Translated by BEMAN and SMITH. Open Court Pub. Co. $1.50.
18. SMITH (D. E.). History of modern mathematics. Wiley. $1.00.

Miscellaneous.

19. BRUHNS. A new manual of logarithms to seven places of decimals. Lemcke and Buechner, New York. (1913.)
Very complete.
20. COBB (H. E.). Elements of applied mathematics. Ginn. $1.00.
21. COOLIDGE (J. L.). The elements of non-euclidian geometry. Oxford. Clarendon Press.
22. DICKSON (L. E.). Elementary theory of equations. Wiley. $1.75.
23. DE MORGAN (A.). Elementary illustrations of the differential and integral calculus. Open Court Pub. Co. $1.00.
34. DE MORGAN (A.). A budget of paradoxes. Revised by D. E. SMITH. Open Court Pub. Co. 2 Vols. $7.00.
35. DUPUIS (N. F.). Elements of synthetic solid geometry. Macmillan. $1.60.
36. FISCHER (A.). The mathematical theory of probabilities and its application to frequency curves and statistical methods. Macmillan. $2.00.
27. KLEIN (F.). Famous problems in elementary geometry. Translated by BEMAN and SMITH. Ginn. $0.50.
28. MACFARLANE (A.). Ten British mathematicians of the nineteenth century. Wiley. $1.25.
29. MANNING (H. P.). Geometry of four dimensions. Macmillan. $2.00.
30. MELLOR (J. W.). Higher mathematics for students of chemistry and physics. Longmans. $4.50.
31. MINCHIN (G. M.) and DALE (J. B.). Mathematical drawing. Arnold. London.
32. MORITZ (R. E.). Memorabilia mathematica. Macmillan. $3.00.
Contains quotations from the writings of prominent mathematicians and anecdotes about them.
33. SCHULTZE (A.). The teaching of mathematics in secondary schools. Macmillan. $1.25.
34. SKINNER (E. B.). Mathematical theory of investment. Ginn. $2.25.
35. SMITH (C.). A treatise on algebra. Macmillan.
36. SMITH (D. E.). The teaching of elementary mathematics. Macmillan. $1.00.
37. SMITH (D. E.). The teaching of geometry. Ginn. $1.25.
38. TURNER (G. C.). Graphical methods in applied mathematics. Macmillan. $1.50.
39. WELD (L. D.). Theory of errors and least squares. Macmillan. $1.25.
40. WOLSTENHOLME (J.). Mathematical problems on subjects for the mathematical tripos examinations. 2,814 problems. Macmillan. 18 shillings.

Books for Juniors.

Geometry.

1. Durell (C. V.). A course of plane geometry for advanced students. 2 Vols. Macmillan. Vol. I, $1.50. Vol. II, $2.00.
2. Heath (T. L.). The thirteen books of Euclid's elements. Cambridge University Press. 3 Vols. $13.50.
3. Hilbert (D.). The foundations of geometry. Translation by Townsend. Open Court Pub. Co. $1.00.
4. Lobachesvki (N. I.). Theory of parallels. Translation by Halsted. Open Court Pub. Co. $1.25.
5. Milne (J. J.). Homogeneous coördinates. Cambridge University Press.
6. Milne (J. J.). Elementary treatise on cross-ratio geometry, with historical notes. Cambridge University Press.
7. Scott (C. A.). Modern analytic geometry. Macmillan.
   Now out of print, but procurable from second-hand book stores.
8. Salmon (G.). Analytic geometry of three dimensions. Revised by R. A. P. Rogers. Longmans. Vol. I (1912) $3.00. Vol. II (1915) $2.25.
9. Snyder (V.) and Sisam (C. H.). Analytic geometry of space. Holt. $2.50.

Algebra.

10. Bôcher (M.). Introduction to higher algebra. Macmillan. $1.90.
11. Burnside (W. S.) and Panton (A. W.). Theory of equations. Longmans. Vol. I $3.00. Vol. II $3.00.
12. Chrystal (G.). Textbook in algebra. 2 vols. Macmillan. $9.00.
13. Fine (H. B.). A college algebra. Ginn. $1.50.

Calculus and Differential Equations.

14. Byerly (W. E.). Problems in differential calculus. Ginn. $0.75.
15. Cohen (A.). Differential equations. Heath. $2.00.
16. Hedrick (E. R.) and Kellogg (O. D.). Applications of the calculus to mechanics. Ginn. $1.25.
17. Johnson (W. W.). A treatise on ordinary and partial differential equations. Wiley. $3.50.
18. Murray (D. A.). Differential equations. Longmans. $1.90.
19. Williamson (B.). A treatise on the differential calculus. Appleton.
20. Williamson (B.). A treatise on the integral calculus. Appleton.

Miscellaneous.

21. Boole (G.). Finite differences. Macmillan.
    Out of print, but procurable from second-hand book stores.
22. Carmichael (R. D.). Theory of numbers. Wiley. $1.00.
23. Coffin (J. G.). Vector analysis. Wiley. $2.50.
24. Dixon (A. C.). Elementary properties of elliptic functions. Macmillan.

25. ENRIQUES (F.). Problems of science. Translation by ROYCE. Open Court Pub. Co. $2.50.
26. FRANKLAND (W. B.). Theories of parallelism. Cambridge University Press. $0.90.
27. GREENHILL (A. G.). The applications of elliptic functions. Macmillan.
28. HANCOCK (H.). Elliptic integrals. Wiley. $1.25. 1917.
29. KELLAND (P.) and TAIT (P. G.). Dynamics of a particle. Macmillan.
30. MERRIMAN (M.) and WOODWARD (R. S.). Higher mathematics. Wiley. Five parts. $0.75 each.
31. MILLER (G. A.). Historical introduction to mathematical literature. Macmillan. $1.60. 1917.
32. MILLER (J. A.) and LILLY (S. B.). Analytic mechanics. Heath. $2.00.
33. PERRY (J.). The teaching of mathematics. Macmillan. $0.75.
34. RUSSEL (B.). Foundations of geometry. Cambridge University Press. $2.00.
35. SOMMERVILLE (D. M. Y.). Elements of non-euclidean geometry. Bell. London.
36. WILLIAMSON (B.). A treatise on dynamics. Appleton.
37. WITHERS (J. W.). Euclid's parallel postulate. Open Court Pub. Co. $1.25.
38. YOUNG (J. W.). Fundamental concepts of algebra and geometry. Macmillan. $1.60.
39. YOUNG (J. W. A.). Monographs on topics of modern mathematics. Longmans. $3.00.
40. ZIWET (A.) and FIELD (P. F.). Introduction to analytical mechanics. Macmillan. $1.60.

BOOKS FOR SENIORS AND FIRST YEAR GRADUATES.

1. BASSET (A. B.). Elementary treatise on cubic and quartic curves. Deighton, Bell and Co. Cambridge.
2. BROMWICH (T. J. I.). Theory of infinite series. Macmillan. $4.75.
3. BURKHARDT (H.). Theory of functions of a complex variable. Translation by RASOR (S. E.). Heath. $4.00.
4. BYERLY (W. E.). Harmonic functions. Wiley. $1.00.
5. BYERLY (W. E.). Fourier Series. Ginn. $3.00.
6. CANTOR (G.). Contributions to the founding of the theory of transfinite numbers. Translation by JOURDAIN. Open Court Pub. Co. $1.25.
7. CARMICHAEL (R. D.). Diophantine analysis. Wiley. $1.25.
8. COHEN (A.). The Lie theory of one parameter groups. Heath.
9. COUTURAT (L.). Algebra of logic. Translation by ROBINSON. Open Court Pub. Co. $1.50.
10. DEDEKIND (R.). On the nature and meaning of numbers. Translation by BEMAN. Open Court Pub. Co. $0.75.
11. DE MORGAN (A.). On the study and difficulties of mathematics. Open Court Pub. Co. $0.75.
12. DICKSON (L. E.). Algebraic invariants. $1.25.

39. ZIZEK (F.). Statistical averages. Translation by PERSONS. Holt. $2.50.
40. ZORETTI (L.). Leçons de mathématiques générales. Paris. Gauthier-Villars.

THE DOUBLE POINTS OF RATIONAL CURVES.

By OSCAR J. PETERSON, University of Wisconsin.

Let P be a point whose homogeneous coördinates are given by the parametric equations:

$$(1)\qquad \begin{aligned} \theta x &= f(\lambda) \equiv a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n, \\ \theta y &= g(\lambda) \equiv b_0\lambda^n + b_1\lambda^{n-1} + \cdots + b_n, \\ \theta z &= h(\lambda) \equiv c_0\lambda^n + c_1\lambda^{n-1} + \cdots + c_n, \end{aligned}$$

where $f(\lambda)$, $g(\lambda)$, $h(\lambda)$ have no common factor. If λ be made to vary continuously from $-\infty$ to $+\infty$, the point P will describe a continuous curve of order n. Such a curve is said to be *rational* or *unicursal*.

If two distinct values s, t of the parameter λ give the same set of coördinates, so that both determine the same point $P_{s,t}$, this point is a *double point*, or singular point of the curve. For these values s, t we have

$$f(s) : g(s) : h(s) = f(t) : g(t) : h(t),$$

or

$$\begin{aligned} g(s)h(t) - h(s)g(t) &= 0, \\ h(s)f(t) - f(s)h(t) &= 0, \\ f(s)g(t) - g(s)f(t) &= 0. \end{aligned}$$

Each of these equations is divisible by $s - t$, since it is zero for $s = t$. After division by $s - t$ let

$$(2)\qquad \begin{aligned} F(s, t) &\equiv \frac{g(s)h(t) - h(s)g(t)}{s - t} = 0, \\ G(s, t) &\equiv \frac{h(s)f(t) - f(s)h(t)}{s - t} = 0, \\ H(s, t) &\equiv \frac{f(s)g(t) - g(s)f(t)}{s - t} = 0 \end{aligned}$$

be the equations. Each pair of values (s, t) which is a solution of the system of equations (2) determines a double point; and, conversely, corresponding to any double point of the curve is a pair of values (s, t) which satisfies equations (2).

(*a*) If $s \neq t$, the curve crosses itself; $P_{s,t}$ is a *node*, or *crunode*.

(*b*) If s and t are conjugate imaginaries, and the point $P_{s,t}$ is real, then $P_{s,t}$

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\theta z &= h(\lambda) \equiv c_0\lambda^n + c_1\lambda^{n-1} + \cdots + c_n,
\end{aligned}
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where $f(\lambda)$, $g(\lambda)$, $h(\lambda)$ have no common factor. If λ be made to vary continuously from $-\infty$ to $+\infty$, the point P will describe a continuous curve of order n. Such a curve is said to be *rational* or *unicursal.*

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is a *conjugate point* (isolated double point, acnode). It is the real intersection of imaginary branches of the curve; there are no other real points in its vicinity.

(*c*) If two consecutive points of the curve coincide, that is, if there is a solution (s, t) of (2) such that $s = t$, the point is a *cusp* (spinode).

In the following discussion we shall assume that all singular points which occur are either nodes, conjugate points, or cusps.

A theorem of fundamental importance in the study of rational curves is: *Every rational curve of the nth order has* $\frac{1}{2}(n-1)(n-2)$ *double points.*

This theorem was first proved by Clebsch (1864).[1] The following proof makes no use of Plücker's formulas. It follows immediately from a consideration of equations (2). Thus,

$$\begin{aligned} g(s)h(t) - h(s)g(t) \equiv\ & (st)^{n-1}[\alpha_{01}(s-t)] \\ & + (st)^{n-2}[\alpha_{02}(s^2-t^2) + \alpha_{12}(s-t)] \\ & + (st)^{n-3}[\alpha_{03}(s^3-t^3) + \alpha_{13}(s^2-t^2) + \alpha_{23}(s-t)] \\ & + \cdots + [\alpha_{0n}(s^n-t^n) + \cdots + \alpha_{n-1,n}(s-t)], \end{aligned}$$

where $\alpha_{ij} = b_i c_j - c_i b_j$.

$$\begin{aligned} F(s,t) \equiv\ & (st)^{n-1}[\alpha_{01}] + (st)^{n-2}[\alpha_{02}(s+t) + \alpha_{12}] \\ & + (st)^{n-3}[\alpha_{03}(s^2+st+t^2) + \alpha_{13}(s+t) + \alpha_{23}] \\ & + \cdots + [\alpha_{0n}(s^{n-1}+s^{n-2}t+\cdots+t^{n-1}) + \cdots + \alpha_{n-1,n}]. \end{aligned}$$

Since $s^k + s^{k-1}t + \cdots + t^k$, where k is a positive integer, can be expressed as a rational integral function of degree k in $s+t$ and st, it is possible to write $F(s,t)$ as a function of degree $n-1$ in $s+t$ and st. Similar expressions are found for $G(s,t)$ and $H(s,t)$.

$$\begin{aligned} F(s,t) &\equiv \bar{F}_{n-1}(s+t, st), \\ G(s,t) &\equiv \bar{G}_{n-1}(s+t, st), \\ H(s,t) &\equiv \bar{H}_{n-1}(s+t, st). \end{aligned}$$

By Bézout's Theorem there are $(n-1)^2$ solutions $(s+t, st)$ of $\bar{F}_{n-1} = 0$ and $\bar{G}_{n-1} = 0$. But each such pair $s+t$, st determines only one distinct pair of values s, t; there are, therefore, precisely $(n-1)^2$ solutions (s, t) of $F = 0$ and $G = 0$. All of these will also satisfy $H = 0$ except those for which both s and t are roots of $h = 0$. The n roots of $h = 0$ can be arranged in pairs in $\frac{1}{2}n(n-1)$ different ways. The number of solutions (s, t) of (2) is

$$(n-1)^2 - \tfrac{1}{2}n(n-1) = \tfrac{1}{2}(n-1)(n-2).$$

[1] A. Clebsch, *Crelle's Journ.*, Vol. 64, pp. 43–65. See also J. C. F. Haase, *Math. Ann.*, Vol. 2 (1870), pp. 515–548; and H. Wieleitner, *Theorie der algebraischen Kurven höherer Ordnung*, pp. 74–75. The proof presented here was worked out in connection with a course in higher geometry conducted by Professor Dowling during the year 1916–17.

in mind, however, that the singularities thus brought out are not the point singularities, but the corresponding line singularities: double tangents, isolated double tangents and inflexional tangents. The equations become

$$\sigma x = \Phi(\lambda) \equiv \chi(\lambda)\psi'(\lambda) - \psi(\lambda)\chi'(\lambda),$$

$$\sigma y = \Psi(\lambda) \equiv \varphi(\lambda)\chi'(\lambda) - \chi(\lambda)\varphi'(\lambda),$$

$$\sigma z = \mathrm{X}(\lambda) \equiv \psi(\lambda)\varphi'(\lambda) - \varphi(\lambda)\psi'(\lambda).$$

If Φ, Ψ, X have no common factor, the order of the curve is $2[2(n-1) - \kappa - 1]$, this being the degree of Φ, Ψ, X; if they have ι factors in common, corresponding to ι inflexions, the order is $2[2(n-1) - \kappa - 1] - \iota$. [*Cf.* the parallel statement, (4) to (4′), above.] The order of the curve is n; hence

$$n = 2[2(n-1) - \kappa - 1] - \iota,$$

(5) $$2\kappa + \iota = 3(n-2).$$

(5) indicates that the number of inflexions is odd or even according as n is odd or even; it also yields an upper limit for the number of inflexions,

(6) $$\iota \leqq 3(n-2),$$

or of cusps,

(7) $$\kappa \leqq \tfrac{3}{2}(n-2).$$

Wieleitner (*loc. cit.*) obtains this result (7) by means of Plücker's formulas.

When n is odd, the right-hand member of (7) is not an integer; but in this case there is at least one inflexion, and $\dfrac{3(n-2)-1}{2}$, which is the integral part of the fraction $\frac{3}{2}(n-2)$, gives the greatest possible number of cusps.

For $n > 4$, the maximum number of cusps is less than the total number of double points.

A PROBLEM IN PERSPECTIVE.

By ARNOLD EMCH, University of Illinois.

1. It is well known that any proper plane quadrangle, *i. e.*, a figure determined by four coplanar points of which no three are collinear, may be considered in an infinite number of ways as the perspective of a square, so that to the vertices of the square correspond in a certain order the vertices of the quadrangle. In such a perspective the points within the surface determined in the ordinary sense by the square and the quadrangle do not necessarily have to correspond to each other. For the sake of clearness of representation the figure shows a case of ordinary pictorial perspective in which also the ordinary surfaces correspond to each other. This does not involve loss of generality.

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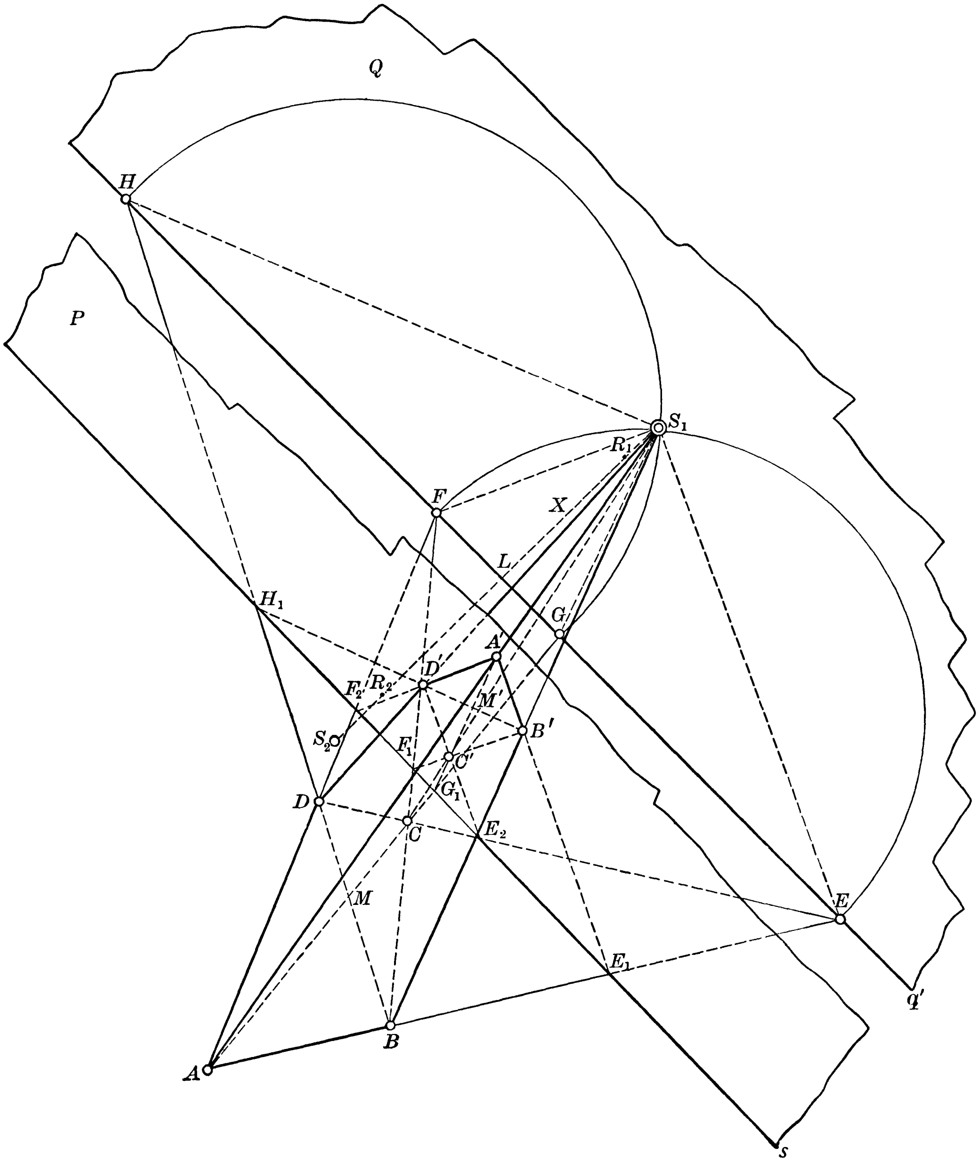
The purpose of the present note is to show how this problem may be treated in an extremely simple manner without going beyond the rudiments of elemen-



tary solid geometry and to emphasize, from a didactic standpoint, the importance of perspective as an introduction to projective geometry.

2. In a plane R assume any proper quadrangle $ABCD$ and let the joins of A and B, and of C and D meet at E, those of B and C, and of A and D at F.

THEOREM. *The locus of the vertices of all pyramids that have a proper coplanar quadrangle with three existing diagonals as a common base and which admit of square plane sections consists of three circles whose centers lie in, and whose planes are orthogonal each for each to, the three diagonals. Each of these circles is the intersection of two spheres with the two segments, like EF and GH, on each diagonal as diameters. If S_1 is a point on any of those circles attached to one of the diagonals, then any plane parallel to the plane determined by S_1 and this diagonal cuts the pyramid $S_1 \cdot ABCD$ in a square.*

3. For the sake of clearness the figure shows the construction for one diagonal, q', only. Any corresponding lines of the quadrangle and the square, like AB and $A'B'$ prolonged, meet in a point of s, the intersection of the planes R and P. The construction is still valid for the intersections R_1 and R_2 of X with the plane R. For these vertices the faces of the pyramid coincide with R. The reason for the validity in these limiting cases can easily be explained but, for the sake of brevity, will be omitted. In the language of central projection or perspective P may be considered as the plane of the object, the square $A'B'C'D'$; R as the picture-plane, S_1 as the center, s as the axis, q' as the vanishing-line of the perspective. E and F are the vanishing points of the two pairs of parallel sides of the square, and the quadrangle $ABCD$ is the perspective of the square.

4. If through S_1 we pass a plane parallel to R, it will cut P in a line t' whose perspective in a projective sense is the infinite line t of R.

By means of this perspective we can easily derive all the polar-properties of a conic from those of a circle. For instance, if we circumscribe a circle K' to the square and construct the pole T' of t' with respect to K', then T' will be projected into the pole T of the infinite line t of R with respect to the perspective K of K', which is a conic circumscribed to the quadrangle $ABCD$. By definition T is now the center of the conic K. According to the disposition in the figure this conic is an ellipse. To avoid the overloading of the figure this part of the construction, which would be simple enough, has been omitted. It follows also without difficulty that two conjugate polars of the circle through T' project into two conjugate diameters of the conic.

Thus we have reached the path by which in the historic development of projective geometry the theory of conics was derived from that of the circle.

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

The Elements of Non-Euclidean Plane Geometry and Trigonometry. By H. S. CARSLAW. Longmans, Green and Co., London, 1916. xii + 179 pages. $1.50.

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interest, ingenuity, and clearness to prove useful to a larger group of readers than that of "teachers of elementary geometry" for whom it was professedly written. The author has imposed on himself various limitations—not to depart from the plane; to avoid imaginary quantities, and, except in one chapter, coördinate systems; to build up a geometry independent of the principle of continuity; to avail himself of dividers (Streckenabträger) instead of compasses, wherever he could. We cannot expect such limitations to be an unmixed good for such a work. For instance, the introduction of imaginaries would have made the essential oneness of elliptic and hyperbolic geometry more apparent. There may be a question, too, as to whether it was worth while, in Chapter III, to avoid Dedekind's postulate so arduously, when Chapter IV had to use it repeatedly. This is, of course, only to say that there is no single all-surpassing way of approach to this subject.

The first chapter begins with a commentary on those axioms of Euclid, expressed or assumed, which are inconsistent with or unnecessary to the geometry to be developed. As Professor Carslaw does not set down his own fundamental assumptions, the reader takes them to be the corrected ones of Euclid. There follow proofs of a few theorems and problems common to non-euclidean and euclidean geometries.

The remainder of the first chapter, and all of the second, are historical. They tell briefly, but with a high degree of discrimination and clearness, of the work of the chief forerunners and founders of the new geometry—Saccheri, Legendre, Gauss, Schweikart, Bolyai, Lobatschevsky, Riemann.

Chapter III gives the theory of hyperbolic plane geometry, basing it on Hilbert's axiom of the existence of two parallels. First come theorems on parallels, angle of parallelism, the Saccheri (bi-rectangular isosceles) quadrilateral. From Lobatschevsky's remarkable correspondence between the five undetermined elements of a right triangle, and those of a tri-rectangular quadrilateral, Carslaw deduces a very elegant series of five associated right triangles. Points at infinity, ideal points (*i. e.*, those on and outside of the absolute) are defined as pencils of lines. The fundamental problems of construction—construction of parallel lines and of the distance corresponding to a given parallel angle—are next solved. Here for the first time compasses must replace dividers; but this does not imply a lapse into continuity, since Hilbert's axiom assures us that the circle and line whose intersection solves the problem do have a common point. The chapter ends with good discussions of the limiting curve (horocycle), the equidistant curve, and areas of triangles and polygons.

The reader who has been so incautious as to omit the preface will not expect continuity to appear unannounced in the fourth chapter (on trigonometry). Yet, on turning to p. 95, he will read: "Since we can find p to satisfy the equation $\Pi(p) = \pi/4$, there is a point Q on the Limiting Curve through P, such that the tangent at Q is parallel to the axis through P, in the opposite sense to that in which the axis is drawn." This means that the given horocycle and the equidistant curve whose distance from the axis satisfies the above equation will

Professor Carslaw has given us, probably, the best balanced elementary book on non-euclidean geometry, both its history and its theory, which has been written; for that he deserves our sincere gratitude. For the future, let us hope that, aided by further coöperation of men so instinct with life and enthusiasm as Liebmann, he can continue to enrich the mathematical world.

EDWARD S. ALLEN.

THE UNIVERSITY OF MICHIGAN.

A History of Elementary Mathematics, with Hints on Methods of Teaching. By FLORIAN CAJORI, Ph.D. Revised and Enlarged Edition. The Macmillan Company, New York and London, 1917. viii + 324 pages. $1.75.

A general demand for information in regard to the historical development of all the great sciences, coupled with the questions that have recently been raised as to the advantage of omitting a considerable part of the present teaching of mathematics, makes Professor Cajori's *History of Elementary Mathematics* a timely and welcome addition to the libraries of both teachers and students. In accordance with the title, the emphasis of the book is upon the foundations of the science of mathematics. Some advanced theories are mentioned, but puzzling details are avoided, much to the relief of the general reader, who knows that many of the upper regions in mathematics can be reached only by the climbing of stairways which are invisible until one has acquired some Aladdin's lamp of knowledge.

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Among racial contributions the book presents with great appreciation the work of the Hindus and the Arabs, and mentions the recently discovered numeral records of the Mayas of Central America as an early attempt on the part of

American races to develop mathematics; but discussion of the possibility of more important discoveries and contributions by the Japanese and Chinese than have so far been credited to them is omitted. The index includes a fairly large number of names of early mathematicians and present-day writers, but those who desire to look up such a connection as that between logic and mathematics find no assistance in the index, though the subject is discussed on p. 289; and there is also no reference to the treatment on p. 300 of the introduction of graphic methods. All these items are of interest to the teacher of the present day and should be included in the index. One wishes also that the question that is quoted so often and in so many kinds of writing as a mathematical problem of the middle ages, "How many angels can stand on the point of a needle?", had been traced to its source, in order to save the time of teachers and students in history, philosophy, literature or mathematics, who are anxious to know its origin.

The tendencies and work of the contributors to mathematics are brought out in a satisfactory way. All the old and famous mathematicians are recognized with appropriate commendations. Where space for details is limited, the lives and qualities and mathematical work of the contributors are presented in a vivid manner by a brief phrase, or even a single adjective, and make the book very readable, even to those who have usually regarded mathematics as a dry subject. Fibonaci is described as "a business man whose leisure hours were given to mathematical study," and Archimedes, "while admired by his fellow citizens for his mechanical inventions," "himself prized more highly his discoveries in pure science." The mathematician who invented a useless new method, more for a pastime than for a practical short cut, is exhibited with a proper reprimand; and the Greek mathematicians, who failed to do their share in geometry because they allowed no construction except by ruler and compasses, receive their due criticism for placing style before progress.

The lack of advance in geometry from the time of the ancient Greeks until Desargues and Pascal in the seventeenth century, and the way their discoveries were "neglected until the close of the eighteenth century" have been carefully noted, as well as the astounding fact that there was a period of "two thousand years when Egyptian mathematics was stationary." These and other points call attention to the curious delays in developments in periods when ideas were ready to flower, but were not cultivated by their owners or observed by more watchful eyes. Over and over again Professor Cajori brings out the lack of advance in theory in time to meet new and necessary demands, because attention was paid so often to arbitrary rules rather than to principles. His presentation of unsatisfactory attitudes toward arithmetic, especially in the 17th century, in the earlier part of the volume, affords considerable material for those readers who wish to take up the question outlined in his quotation from Abraham Flexner, "If, for example, only so much arithmetic is taught as people actually have occasion to use, the subject will sink into modest proportions."

The general tendency of most writers on mathematics who wish to reach modern readers is constantly to use modern notations and methods in the

PROBLEMS AND SOLUTIONS.

Send all communications about problems to B. F. Finkel, Springfield, Mo.

PROBLEMS FOR SOLUTION.

ALGEBRA.

487. Proposed by WILLIAM HOOVER, Columbus, Ohio.

Show in two ways that 0.5623 is not a root of $(1 - m)e^m = 2e^{-1}$, e being the Napierian base. Find the correct value of m, and test the result in two ways.

488. Proposed by NORMAN ANNING, Chilliwack, B. C.

Show that

$$\begin{vmatrix} a_1 & 1 & a_2 \\ 1 & 1/a_4 & 1 \\ a_2 & 1 & a_3 \end{vmatrix} = 0,$$

where

$$a_k = \frac{\sin (k\theta + \alpha)}{\sin k\theta},$$

and θ and α have any values that do not make a denominator zero.

GEOMETRY.

520. Proposed by ALBERT A. BENNETT, University of Texas.

On a given tangent to a circle determine a point such that, if a secant be drawn joining this point to the extremity of the diameter which is perpendicular to the given tangent, the segment of this secant exterior to the circle will be equal in length to a given segment.

521. Proposed by R. M. MATHEWS, Riverside, California.

A variable circle, with center on the line l and passing through a fixed point P, cuts a fixed circle in A and B. Prove that the common chord AB and the perpendicular to l through P intersect in a fixed point.

CALCULUS.

435. Proposed by B. F. FINKEL, Drury College.

Show that

$$\int_0^\infty e^{-x^2-(a^2/x^2)}dx = \frac{\sqrt{\pi}}{2e^{2a}}$$

by a transformation, rather than by the usual method of differentiating under the sign of integration, as, for example, in Byerly's *Integral Calculus*, pages 106–107.

436. Proposed by ARTEMAS MARTIN, LL.D., Washington, D. C.

A circle of radius a is drawn at random on a circular slate of radius r. If another circle of radius a be drawn on the slate, what is the probability that the second circle will intersect the first?

MECHANICS.

352. Proposed by C. N. SCHMALL, New York City.

A glass rod is balanced partly in and partly out of a cylindrical tumbler, with its lower end resting against the vertical wall of the tumbler. If ϕ and ψ are the maximum and minimum angles, respectively, which the rod can make with the vertical plane, and θ is the angle of friction, show that

$$\theta = \tfrac{1}{2} \tan^{-1} \left(\frac{\sin^3 \phi - \sin^3 \psi}{\sin^2 \phi \cos \phi + \sin^2 \psi \cos \psi} \right).$$

353. Proposed by CLIFFORD N. MILLS, Brookings, North Dakota.

A uniform beam of oak, 10 feet long, 15 inches deep and 10 inches wide, sustains, in addition to its own weight, a load of 5,000 lbs. placed at the center. Find the greatest bending moment and the greatest stress in the fibers. Take the specific gravity of oak as .934.

NUMBER THEORY.

270. Proposed by GERSHOM N. CARMICHAEL, Urbana, Ill.

Does there exist a fraction p/q in its lowest terms such that the ratio of the sum of the divisors of p to the sum of the divisors of q is equal to p/q? Give a method of finding such fractions not in their lowest terms.

271. Proposed by HORACE OLSON, Chicago, Ill.

Prove that if x, y, z, u, v, and w are integers such that $x^2 + y^2 = u^2$, $x^2 + z^2 = v^2$, $y^2 + z^2 = w^2$, then the product $xyzuvw$ is divisible by 518400.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

474. Proposed by A. A. BENNETT, University of Texas.

Show that the value of the infinite continued fraction, all of whose coefficients are unity,

$$1 + \frac{1}{1+}\,\frac{1}{1+}\cdots, \quad \text{is} \quad \tfrac{1}{2}(1 + \sqrt{5}).$$

Also find an explicit algebraic formula for the nth convergent.

SOLUTION BY C. C. YEN, Tangshan, North China.

1. Let x denote the value of the continued fraction. Then,

$$x = 1 + 1/x, \quad \text{i. e.,} \quad x^2 - x - 1 = 0,$$

therefore,

$$x = \tfrac{1}{2}(1 + \sqrt{5}),$$

where the sign for the radical is positive, since x is evidently positive.

2. Let p_n/q_n denote the nth convergent. Then

$$p_1 = 1, \qquad q_1 = 1, \qquad p_2 = 2, \qquad q_2 = 1. \tag{I}$$

Also, since all the coefficients are unity,

$$p_n = p_{n-1} + p_{n-2}, \qquad q_n = q_{n-1} + q_{n-2} \qquad (n = 3, 4, 5, \cdots). \tag{II}$$

And, it follows, therefore, from I and II,

$$q_n = p_{n-1} \qquad (n = 2, 3, 4, \cdots). \tag{III}$$

Now consider the series

$$1 + x + 2x^2 + p_3x^3 + p_4x^4 + \cdots + p_nx^n + \cdots,$$

where the coefficients p_n satisfy II. It is a recurring series whose scale of relation is $1 - x - x^2$, and whose generating function is $1/(1 - x - x^2)$. Hence, p_n is the coefficient of x^n of the expansion of this function; and, by III, q_n is the coefficient of x^{n-1} of the same expansion.

If we put $\alpha = -\tfrac{1}{2}(1 + \sqrt{5})$, $\beta = -\tfrac{1}{2}(1 - \sqrt{5})$, we get

$$\frac{1}{1 - x - x^2} = \frac{1}{\alpha - \beta}\left(\frac{1}{\alpha - x} - \frac{1}{\beta - x}\right),$$

whence the general term of the expansion of $1/(1 - x - x^2)$ is

$$\frac{1}{\alpha - \beta}\left\{\left(\frac{1}{\alpha}\right)^{n+1} - \left(\frac{1}{\beta}\right)^{n+1}\right\}x^n = \frac{1}{\alpha - \beta}\left\{\frac{\beta^{n+1} - \alpha^{n+1}}{(\alpha\beta)^{n+1}}\right\}x^n.$$

But $\alpha\beta = -1$, and $q_n = p_{n-1}$. It follows, therefore, that

$$p_n = (-)^n \left\{ \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \right\}, \qquad q_n = (-1)^{n-1} \left\{ \frac{\alpha^n - \beta^n}{\alpha - \beta} \right\}.$$

Hence, finally, the nth convergent is given by

$$-(\alpha^{n+1} - \beta^{n+1})/(\alpha^n - \beta^n),$$

where α, β are the roots of the equation $x^2 - x - 1 = 0$.

Also solved by PAUL CAPRON, N. P. PANDYA, and O. S. ADAMS.

475. Proposed by E. B. ESCOTT, Kansas City, Mo.

A man makes a contract to purchase a house, making a cash payment down and agreeing to make monthly payments of a dollars, interest being charged at 6 per cent., the balance of the monthly payments being credited on the principal. Find a formula for M_n, the balance due after n payments.

SOLUTION BY C. R. DUNCAN, Amherst, Massachusetts.

Let M_0 = balance due after the original cash payment, and r = rate of interest per month $(= .06/12)$, then M_0r = interest due at end of first month, and $a - M_0r$ = amount paid back on the principal M_0.

At the end of the second month the interest would be less, the difference being equal to the interest on the amount paid back the first month, or $(a - M_0r)r$. But as all monthly payments are to be equal this amount would be credited on the principal. Hence, the amount paid back on the principal at the end of the second month is

$$(a - M_0r) + (a - M_0r)r = (a - M_0r)(1 + r).$$

Similarly, the amount paid back at the end of the third month would be equal to the amount paid back the second month plus the difference in interest between the second and third months, or

$$(a - M_0r)(1 + r) + (a - M_0r)(1 + r)r = (a - M_0r)(1 + r)^2.$$

Hence, the amount paid back at the end of the nth month $= (a - M_0r)(1 + r)^{n-1}$. Therefore, the total amount paid back in n months is

$$(a - M_0r) + (a - M_0r)(1 + r) + (a - M_0r)(1 + r)^2 + \cdots + (a - M_0r)(1 + r)^{n-1} = \frac{(a - M_0r)[(1 + r)^n - 1]}{r},$$

and the balance due after n payments is

$$M_n = M_0 - \frac{(a - M_0r)[(1 + r)^n - 1]}{r}.$$

Putting the right-hand member equal to 0 and solving for a, we have a formula for finding the amount of the monthly payments required to pay back the principal M_0 in a given number of months,

$$a = \frac{M_0r(1 + r)^n}{(1 + r)^n - 1}.$$

Also solved by G. W. HARTWELL, E. J. OGLESBY, HORACE OLSON, A. R. NAUER, PAUL CAPRON, G. PAASWELL, H. N. CARLETON, J. B. REYNOLDS, and the PROPOSER.

476. Proposed by W. HAROLD WILSON, University of Illinois.

Prove that if $x_h \neq x_j$ $\qquad (h, j = 1, 2, \cdots, n, h \neq j)$,

then

$$\sum_{i=1}^{n} \frac{x_i^{\,n-1}}{\prod'_{h=1}^{n} (x_i - x_h)} = 1,$$

where the prime indicates the omission of zero factors in the denominator.

II. Solution by the Proposer.

It is convenient to use the notation

$$\Delta = \begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ \cdot & \cdot & \cdot & \cdot \\ x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{vmatrix} = \prod_{h,j=1}^{n} (x_h - x_j), \qquad h < j.^1$$

It is also convenient to designate the minor of Δ with respect to x_i^{n-1} by Δ_i.

$$\Delta_i = \prod_{h,j=1}^{n}{}' (x_h - x_j), \qquad h < j,$$

where the prime indicates that no factor containing x_i enters. It now follows at once that

$$\sum_{i=1}^{n} \frac{x_i^{n-1}}{\prod'_{h=1}^{n} (x_i - x_h)} = \sum_{i=1}^{n} \frac{x_i^{n-1}\Delta_i}{(-1)^{i-1} \prod_{h=1}^{i-1} (x_h - x_i) \prod_{h=i+1}^{n} (x_i - x_h) \prod'_{h,j=1}^{n} (x_h - x_j)}$$

$$= \sum_{i=1}^{n} \frac{(-1)^{i-1} x_i^{n-1}\Delta_i}{\prod_{h,j=1}^{n} (x_h - x_j),\ h < j} = \frac{1}{\Delta} \sum_{i=1}^{n} (-1)^{i-1} x_i^{n-1}\Delta_i = \frac{1}{\Delta}\Delta = 1. \qquad h < j,$$

Also solved by A. A. Bennett, G. W. Hartwell, and E. H. Worthington.

GEOMETRY.

508. Proposed by J. E. ROWE, State College, Pa.

The trilinear coördinates of the Brocard triangle are $(s_3^3, s_1s_2s_3, s_1^3)$, $(s_2^3, s_1^3, s_1s_2s_3)$, and $(s_1s_2s_3, s_0^3, s_2^3)$ where s_i $(i = 1, 2, 3)$ are the sines of the angles of the fundamental triangle. Show that the Brocard triangle and the fundamental triangle are in perspective, and that the trilinear coördinates of the center of perspectivity are s_i^{-3} $(i = 1, 2, 3)$ instead of s_i^3 which are incorrectly given in Clebsch *Vorlesungen über Geometrie*, p. 323.

Solution by William Hoover, Columbus, Ohio.

Use of homogeneous coördinates permits us to replace s_1, s_2, s_3 by a, b, c and we have

$$(0, 2\Delta/b, 0),\ (c^3, abc, a^3); \qquad (0, 0, 2\Delta/c),\ (b^3, a^3, abc); \qquad (2\Delta/a, 0, 0),\ (abc, c^3, b^3)$$

for the pairs of sets of trilinear coördinates of the corresponding vertices of the fundamental and Brocard triangles.

The equation of a straight line passing through two points $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ is

$$\alpha(\beta_1\gamma_2 - \beta_2\gamma_1) + \beta(\gamma_1\alpha_2 - \gamma_2\alpha_1) + \gamma(\alpha_1\beta_2 - \alpha_2\beta_1) = 0. \tag{1}$$

Thus the straight lines joining the pairs of vertices are

$$a^3\alpha - c^3\gamma = 0 \quad (2), \qquad -a^3\alpha + b^3\beta = 0 \quad (3), \qquad b^3\beta - c^3\gamma = 0 \quad (4).$$

The three lines

$$l_1\alpha + m_1\beta + n_1\gamma = 0 \quad (5), \qquad l_2\alpha + m_2\beta + n_2\gamma = 0 \quad (6), \qquad l_3\alpha + m_3\beta + n_3\gamma = 0 \quad (7)$$

are concurrent if

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} = 0. \tag{8}$$

(2), (3), and (4) are, therefore, concurrent and the fundamental and Brocard triangles are in perspective.

[1] L. G. Weld, *The Theory of Determinants*, Art. 89, p. 169.

The proportional values of α, β, γ in (5) and (6) are given by

$$\frac{\alpha'}{m_1n_2 - m_2n_1} = \frac{\beta'}{n_1l_2 - n_2l_1} = \frac{\gamma'}{l_1m_2 - l_2m_1} \tag{9}$$

and those in (2) and (3) by

$$\frac{\alpha'}{b^3c^3} = \frac{\beta'}{a^3c^3} = \frac{\gamma'}{a^3b^3} \quad (10), \qquad \text{or} \qquad \alpha'\Big/\frac{1}{a^3} = \beta'\Big/\frac{1}{b^3} = \gamma'\Big/\frac{1}{c^3} \quad (11),$$

satisfying (4) and proving the theorem.

The fundamental triangle is in perspective with two other Brocard triangles, the centers of perspective being $1/b^2$, $1/c^2$, $1/a^2$; $1/c^2$, $1/a^2$, $1/b^2$.

Also solved by J. W. CLAWSON, C. P. SOUSLEY, and S. W. REAVES.

509. Proposed by NORMAN ANNING, Chilliwack, B. C.

A picture whose coördinates are (0, 0), (50, 0), (50, 50), and (0, 50) is repeated on a smaller scale as part of itself with the coördinates (7, 0), (31, 7), (24, 31), (0, 24). Locate the vanishing point.

SOLUTION BY S. W. REAVES, University of Oklahoma.

Let P, P_1, P_2, $\cdots$ denote the picture and its successive images, and let O_1, O_2, O_3, $\cdots$ be the successive images of the first vertex. We shall find the vanishing point by determining the limiting position of O_n as n increases indefinitely.

It readily follows from the data of the problem that the side of P_1 is one half that of P and makes with it an angle $\theta = \tan^{-1} 7/24$. It is clear that the side of P_n bears the same relations to the side of P_{n-1} for all values of n.

By projecting the successive segments OO_1, O_1O_2, O_2O_3, $\cdots$ on the coördinate axes, it readily follows that the coördinates of the limiting position of O_n are given by the following infinite series:

$$x = 7 + \frac{7}{2}\cos\theta + \frac{7}{2^2}\cos 2\theta + \cdots + \frac{7}{2^n}\cos n\theta + \cdots,$$

$$y = \frac{7}{2}\sin\theta + \frac{7}{2^2}\sin 2\theta + \cdots + \frac{7}{2^n}\sin n\theta + \cdots.$$

Multiplying the second equation through by $i = \sqrt{-1}$, adding the result to the first equation, denoting $\cos\theta + i\sin\theta$ by v, and remembering that $\cos n\theta + i\sin n\theta$ is by De Moivre's formula equal to v^n, we have

$$x + iy = 7\left[1 + \frac{v}{2} + \left(\frac{v}{2}\right)^2 + \cdots + \left(\frac{v}{2}\right)^n + \cdots\right].$$

Using the formula for the sum of a geometric progression and then replacing v by its value

$$\frac{24}{25} + i\frac{7}{25},$$

we have

$$x + iy = \frac{364}{29} + i\frac{98}{29}.$$

Hence, the coördinates of the required vanishing point are (364/29, 98/29).

Also solved by J. B. REYNOLDS and W. R. RANSOM.

CALCULUS.

424. Proposed by OSCAR S. ADAMS, Washington, D. C.

What is the value of

$$\frac{\Gamma'(1)}{\Gamma(1)} - \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})}.$$

SOLUTION BY EDWARD H. WORTHINGTON, University of Pennsylvania.

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx.$$

Hence,

$$\Gamma(1) = \int_0^\infty e^{-x}dx = 1 \qquad \text{and} \qquad \Gamma'(n) = \int_0^\infty x^{n-1}e^{-x} \log x dx,$$

since differentiation may be carried under the sign.

Since

$$\log x = \int_0^\infty \frac{e^{-\alpha} - e^{-\alpha x}}{\alpha} d\alpha, \qquad \Gamma'(n) = \int_0^\infty x^{n-1}e^{-x} \int_0^\infty \frac{e^{-\alpha} - e^{-\alpha x}}{\alpha} d\alpha dx.$$

$$\therefore \quad \Gamma'(1) = \int_0^\infty \int_0^\infty e^{-x}\frac{e^{-\alpha} - e^{-\alpha x}}{\alpha} dx d\alpha = \Gamma(1) \int_0^\infty \left(e^{-\alpha} - \frac{1}{1+\alpha}\right)\frac{d\alpha}{\alpha}.$$

(1) Hence, $\dfrac{\Gamma'(1)}{\Gamma(1)} = \displaystyle\int_0^\infty \left(e^{-\alpha} - \frac{1}{(1+\alpha)}\right)\frac{d\alpha}{\alpha}$; also $\dfrac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} = \displaystyle\int_0^\infty \left(e^{-\alpha} - \frac{1}{(1+\alpha)^{1/2}}\right)\frac{d\alpha}{\alpha}$

(2) and $$\frac{\Gamma'(1)}{\Gamma(1)} - \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} = \int_0^\infty \left(-\frac{1}{1+\alpha} + \frac{1}{(1+\alpha)^{1/2}}\right)\frac{d\alpha}{\alpha}.$$

Changing $1 + \alpha$ to $1/\alpha$, (2) becomes, if the primes are dropped, and convergency considered,

$$\lim_{\epsilon \doteq 0}\left[\int_{1-\epsilon}^0 \frac{1-\alpha^{-1/2}}{1-\alpha} d\alpha = \int_{1-\epsilon}^0 \frac{d\alpha}{-\alpha^{1/2}(\alpha^{1/2}+1)} = \int_{1-\epsilon}^0 -\alpha^{-1/2}[1 - \alpha^{1/2} + \alpha - \alpha^{3/2} + \alpha^2 \cdots]d\alpha\right]$$
$$= +2[1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} \cdots] = +2 \log_e 2.$$

Since $\log_e 2 = 0.69315$

$$\therefore \quad \frac{\Gamma'(1)}{\Gamma(1)} - \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} = +1.38630.$$

Also solved by G. PAASWELL, H. M. TERRILL, and V. M. SPUNAR.

NUMBER THEORY.

254. Proposed by HORACE OLSON, Chicago, Ill.

Find three integers x, y, z, such that $x^2 + y^2$, $x^2 + z^2$, $y^2 + z^2$, and $x^2 + y^2 + z^2$ are all perfect squares.

SOLUTION BY V. M. SPUNAR, Chicago, Ill.

Let

$$x^2 + y^2 = \square = a^2, \quad (1) \qquad y^2 + z^2 = \square = c^2, \quad (3)$$
$$x^2 + z^2 = \square = b^2, \quad (2) \qquad x^2 + y^2 + z^2 = \square = d^2. \quad (4)$$

A complete solution of (4) is as follows:

$$x = m^2 + n^2 - p^2 - q^2, \quad y = 2(mp + nq), \quad z = 2(mq - np), \quad d = m^2 + n^2 + p^2 + q^2. \quad (5)$$

First we remember that one of the two integers A and B satisfying the relation, $A^2 + B^2 = \square$, must be even.

Next, suppose y and z even, then (3) shows that c is even, and after removing the common factor 4 we find again that either $(y/2)^2$, or $(z/2)^2$ is a multiple of 4. But from (5) it is obvious that if x be odd three and only three of the numbers m, n, p, q must be odd or three even. This leads, however, to y and simultaneously z having the same factor 2, which, after suppressing in (3), leads to the conclusion again, that one number must be even, which is impossible.

Hence, the proposition is impossible.

SOLUTION BY EDWARD H. WORTHINGTON, University of Pennsylvania.

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx.$$

Hence,

$$\Gamma(1) = \int_0^\infty e^{-x}dx = 1 \qquad \text{and} \qquad \Gamma'(n) = \int_0^\infty x^{n-1}e^{-x}\log x dx,$$

since differentiation may be carried under the sign.

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$$= +2[1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5}\cdots] = +2\log_e 2.$$

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Hence, the proposition is impossible.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas.

DISCUSSIONS.

RELATING TO REQUIRED MATHEMATICS FOR WOMEN STUDENTS.

By EMILIE N. MARTIN, Mount Holyoke College.

In recent months I have been present at many informal discussions in regard to the required work of a college course. The background of such discussions has been the woman's college, and the aim of all the participants has been to find the best method of training the woman student for her part in the work of the world.

Some argue that with the large amount of required work in college a student has too little time in which to sample new courses or to devote herself to a subject for which she shows an aptitude. Their demand is: "Lessen the required work." Others argue that the required work of a college course is seldom more than sufficient to give the solid foundation of general culture upon which the structure of specialized subjects may be safely and permanently reared.

Some of the arguments of the opponents of required mathematics have a curiously familiar sound. To be sure they are not put as crudely as in the world outside the college. There no one hesitates to ask: "What is the use of mathematics for a girl? She will never need more than the elements of arithmetic." The opponents of the subject in a woman's college, especially if they are themselves women, are somewhat wary of asking that time-honored question. They know that they are supposed to claim the same stiff intellectual training for women as for men, so they avoid stating the question in bald terms of sex. Instead they say: "Every student who enters our college has already had several years of mathematics. She has enough for all practical purposes. [This is nothing but the old query revised.] She has already found out whether or not she has an aptitude for the subject. If she cares for it, by all means let her elect it, but if she has no aptitude for the subject, let her take something that is really worth her while instead of wasting her time on a subject she will never use."

My belief is that in spite of her school training, however thorough, the student needs some college mathematics if she is to have an education that will send her out into life with the best general equipment. Mathematics as taught in college is viewed from an angle different from that used in the school-room. This statement does not apply to solid geometry which is only an extension of the plane geometry of the school, and which we hope some day to see put back in its proper place in connection with plane geometry. Take, however, college algebra and trigonometry. Both of these subjects make use of material in the way of ideas and methods that the student has already worked with in school, but this material is handled in a very different way. In her algebra the school-girl is concerned almost entirely with processes. She needs but little theory.

This little is sometimes explained to her, sometimes she has to remember and reproduce it; but even in the latter case the average student seems to have acquired little grasp of the underlying principles. The consequence is that almost every college freshman has a definite idea of logical reasoning as connected with the subject of geometry, but has little idea of it in connection with any other part of mathematics.

In college the freshman, while once more dealing with symbols and processes that were familiar in school, is now concerned with them from the standpoint of logical combinations. She is applying to them the methods that she had thought confined to geometry, deducing the laws that rule in the subject of her study, and expressing these laws in exact mathematical language. Unless much emphasis is laid upon this side of mathematics in the freshman classes, the claims of its enemies would seem to me to have some foundation. The application of the theory to special problems is necessary as making the subject more vivid and as preparing the student for the application of abstract reasoning to all manner of problems stated in advance; but the true value of the course lies in the training it gives in applying logical processes to the mathematical concepts already familiar from school days.

The average student may not like to find that the mechanical work of her school algebra is replaced in college by this demand upon her reasoning powers, yet she is ordinarily able to master the required amount of freshman mathematics in spite of the extra difficulty. On the other hand there are always a few students to whom such a treatment of mathematics offers an insuperable difficulty,—students who seem to have no idea of logical sequence. In their statement of a geometrical construction no attention is paid to the logical order in which the lines must be drawn, in their statement of a proof the effect precedes the cause, and their minds are so constructed that it seems impossible to convince them of error—one order seems to them as good as another.

This latter class of students is always cited by our opponents as the strongest argument for dispensing with a general requirement of mathematics. "Why torture such a student," they say, "with a subject for which she has no fitness? Why should she not take the subjects for which she shows some aptitude, and let mathematics severely alone?" They usually omit to name the college subjects that do not require at least some modicum of reasoning ability. My feeling, on the contrary, is that no matter what other subjects such a student has to omit, mathematics is one subject absolutely essential to her training. Such a student needs to develop her reasoning powers, and freshman mathematics gives her the best field for practice. Other subjects also require logical ability, but often the logical framework is so obscured by the newness of the material, the unfamiliarity of the nomenclature, and the large number of strange concepts, that the value of the subject from the standpoint of logical training is quite lost.

As for the examples that the opposers of required mathematics quote from time to time of women brilliant in other lines of college work who found themselves totally unable to do even enough work in freshman mathematics to gain a

passing grade, I cannot deny that such women may exist, but I wish to record my deep conviction that the majority of these cases have been diagnosed incorrectly. I have seen so many students of mediocre abilities fight their way through the difficulties of required mathematics by sheer common-sense and will power, that I am sure that most of the cases cited so solemnly are cases of "I will not" and not of "I cannot." Because they did not want to master a difficult subject, they were willing to profess incompetence in order to get their own way.

Here the question of sex enters in. Parents and guardians who would suffer keen mortification if the boy for whose education they are responsible were in danger of being rejected by his college because of his failure in required mathematics will condone any shortcomings of the girl in that line with a deprecating, "You know that one does not think so much of a failure in mathematics for a girl." But if sex must be considered in this matter, why not consider it from this other standpoint, namely, that the woman is prone to look at everything from the personal side? Her own feelings and her background, or lack of background, color the medium through which she views such subjects as history and literature, and affect her judgments of the facts. The personality of her instructor is a factor in inclining her either to believe or disbelieve his interpretations of the theories of economics or philosophy. Of course there are fundamental laws in all these subjects so well established that the personal element cannot enter into their consideration, but there is also a large body of conclusions from these laws, and it is these conclusions from laws that are sometimes only partially understood that are now in question. Herein lies a great advantage of mathematics; it furnishes the woman student with a subject in which the validity of the conclusions drawn from its laws can easily be tested, and in which the personality of the instructor and the bias of the student can play no part.

The foes of the mathematics requirement then say: "Suppose we grant that our students need training in reasoning, and that as women they need especially training in reasoning upon an impersonal subject. Why not then require a course in a science that shall be the equivalent of the mathematics course in these two respects? Any science gives a student the opportunity to acquire the habit of assembling data, rejecting the extraneous elements, and forming the fitting conclusion. No one can bring any personal prejudice into the interpretation of the phenomena that would be discussed in a required course in any established science." Such a suggestion seems quite to overlook the fact that the only sciences that furnish a training at all equivalent to that of mathematics are those that have mathematics as their foundation. Without a preliminary training in mathematics that almost necessitates the inclusion of trigonometry the most rigorous of sciences can only be treated from the more or less popular point of view. The training so acquired is no doubt valuable, but it cannot satisfactorily replace the required work in mathematics.

Another argument that is often brought against this college requirement is that it gives a subject in which the students have already had several years' experience an unfair advantage over others that are not begun until after college

in mathematics possessed by the students. Under such circumstances the science requiring the least amount of mathematical knowledge will naturally be given the preference, and the instructors in even the most mathematical of the sciences will have to confine themselves to a rather popular treatment, if they wish to have a fair proportion of the students elect their subject. In any case much of the value of these sciences as aids to exact thinking will be lost. Of course, for any further work in the selected science some mathematics is necessary, and under the supposed arrangement the student must take in her maturer years in college the fundamental work that she now acquires in her freshman year.

With the two apparently contradictory tendencies at present noticeable—one, to minimize for women even in the science courses in college the necessity of any mathematical training beyond that of the high school course; the other, to encourage these same students to place more and more emphasis upon their work in science, especially in the line of laboratory research,—it is evident that the majority of women workers in science will soon be forced to limit themselves to those fields in science that can be cultivated by means of the very simplest mathematical tools. These fields may be wide and they may be fertile, but by permitting this limitation women are denying to themselves the equality of opportunity with men that has been won for them at such a cost by the pioneers in the struggle for the right of women to share in the higher education.

NOTES AND NEWS.

Edited by D. A. Rothrock, Indiana University, Bloomington, Ind.

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Mr. J. L. Riley, of the State Normal School at Tahlequah, Oklahoma, has been appointed professor of mathematics in the Junior College at Stephenville, Texas.

Mr. J. B. Rosenbach, engineer of maintenance of the Atchison, Topeka and Santa Fe railway, has resigned to become instructor in mathematics at the University of New Mexico.

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Mr. C. A. Epperson, associate professor of mathematics at the First District Normal School, Kirksville, Mo., has been granted leave of absence and is in military training at Fort Sheridan.

Professor L. K. Adkins, head of the department of mathematics at the Wisconsin State Normal School, La Crosse, Wis., has been appointed first lieutenant in the Regular Army, and ordered to France with General Pershing.

Dr. E. A. Kircher, Benjamin Peirce instructor at Harvard University, has been commissioned a captain in the coast artillery and is located at Fort Strong, Boston, Mass.

Mrs. Mqyme Logsdon, of Hastings College, Nebraska, has been appointed to an instructorship in mathematics at Northwestern University.

At the University of Maine, Dr. Norbert Wiener has resigned to enter the military service, and Associate Professor H. R. Willard has been appointed statistician under Mr. Hoover and reported at Washington in September.

Assistant Professor P. L. Thorne, of New York University, is in the military training camp at Plattsburg; and instructor H. H. Pride has completed the training at the Madison Camp and has been commissioned second lieutenant in the new National Army.

Assistant Professor J. A. Nyswander, of the University of Nevada, has been granted leave of absence to serve in the National Army. The vacancy in the department of mathematics was filled by the appointment of Mrs. Nyswander to serve during her husband's absence.

Three instructors of mathematics in the University of Illinois have resigned to assume other positions. Dr. F. W. Reed has entered the aviation service; Dr. Levi Wilson has joined the mathematical staff of the U. S. Naval Academy; and Dr. W. W. Denton has been appointed associate professor of mathematics at the Worcester Polytechnic Institute.

Mr. Volney Wells, instructor in mathematics at the University of Michigan, has accepted a position in the University of Pittsburgh; instructor J. W. Baldwin has been elected to a position in the mathematical department of the Michigan State Normal College, Ypsilanti; and Dr. A. L. Miller, instructor in mathematics, has resigned to enter business.

At Cornell University, Dr. J. V. McKelvey, senior instructor in mathematics, has been granted leave of absence, and after serving in the training camp at Madison Barracks, has been commissioned a second lieutenant in the National Army; Dr. R. E. Gilman, instructor in mathematics, has also been granted leave of absence and has been commissioned captain in the Coast Artillery.

The twenty-fourth summer meeting of the American Mathematical Society was held at Western Reserve University and the Case School of Applied Science, Cleveland, Ohio, on September 4–6, 1917. The attendance at this meeting was unusually large. Nineteen titles of papers appeared upon the printed program and several others not previously announced were presented. The readers represented twelve colleges and universities of the United States. The last session, Thursday forenoon, was in conjunction with the Mathematical Association of America, at which Professor L. P. EISENHART, of Princeton University, gave an address on "Darboux's contribution to geometry." The Society joined with the Association in a dinner at the Hotel Statler on Wednesday evening.

At the University of Chicago the following graduate students have received appointments not previously announced: Mr. G. H. CRESSE, instructor in mathematics at the University of Michigan; Mr. FRANK M. WEIDA, instructor in mathematics at the University of Iowa; Mr. O. W. ALBERT, instructor in mathematics at Grinnell College, Iowa; Dr. FLORA LE STOURGEON, instructor in mathematics at the LIGGETT School, Detroit, Mich., Dr. W. P. OTT, Dr. L. S. SHIVELY, and Dr. H. F. MCATEE, who received their doctorates at the June convocation, return to their former positions at Vanderbilt University, Mt. Morris College, Ill., and William Jewell College, Mo., respectively.

The following promotions of members of the Association have not been previously announced in these columns: LOUIS BRAND and C. N. MOORE to full professorships and J. H. KINDLE and E. S. SMITH to assistant professorships at the University of Cincinnati; E. B. STOUFFER to an associate professorship at the University of Kansas; L. C. MATHEWSON and C. R. DINES to assistant professorships at Dartmouth College; and A. C. LUNN to an associate professorship at the University of Chicago.

The following new appointments have been reported:

Dr. R. B. ROBINS, of Yale University, instructor in mathematics at the University of Michigan; Mr. I. L. MILLER, of Indiana University, professor of mathematics at Carthage College, Carthage, Illinois; Miss MARION E. STARK, graduate fellow in Brown University, professor of mathematics at Meredith College, Raleigh, N. C.; Dr. C. C. CRUMP, of Carleton College, assistant professor of astronomy and director of the Perkins Observatory of Ohio Wesleyan University; Mr. VERN JAMES, of Indiana University, instructor in mathematics at the Carnegie Institute of Technology, Pittsburgh, Pa.

The report of the Library Committee of the Association, which appears in this issue, will, it is hoped, furnish a concrete answer to the numerous inquiries that have been received by the MONTHLY concerning appropriate lists of books for college libraries. It is proposed to provide reprints of this report which may be had at cost by application to the secretary, W. D. CAIRNS, 27 King Street, Oberlin, Ohio. If requests for copies could be made at once it would greatly facilitate the determination of how large an edition to publish.

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THE MATHEMATICAL ASSOCIATION OF AMERICA now has over eleven hundred individual members and over seventy-five institutional members. There are already nine sections formed, representing twelve different states. The Association has held so far two national meetings per year, one in September and one in December. The sections, for the most part, hold two meetings each year. All meetings, both national and sectional, are reported in the Official Journal, and many of the papers presented at these meetings are published in full.

The slogan of the Association is to include in its membership every teacher of collegiate mathematics in America and to make such membership worth while. Application blanks for membership may be obtained from the Secretary at Oberlin, Ohio.

Constitution and By-Laws of the Mathematical Association of America

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ARTICLE I—NAME AND PURPOSE.

1. This organization shall be known as THE MATHEMATICAL ASSOCIATION OF AMERICA.

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field.

ARTICLE II—MEMBERSHIP.

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association; such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

ARTICLE III—OFFICERS.

1. The officers of this Association shall be a President, two Vice-Presidents, a Secretary-Treasurer and twelve additional members of an Executive Council, together with a Committee of three on Publications, who shall be *ex-officio* members of the Council.

2. The President, Vice-Presidents and Secretary-Treasurer shall be elected annually for a term of one year, and four members of the Council shall be elected annually for a term of three years. They shall be eligible for reëlection, but not for more than two consecutive terms, except in the case of the Secretary-Treasurer, whose term may be extended indefinitely. The Committee on Publications, consisting of the Managing Editor and two other members, shall be appointed by the Council.

3. The Council shall transact the official business of the Association and shall report its actions at the annual meeting of the Association and in the official journal. Any proposed action of the Council which makes or alters a question of policy shall be published in the official journal before final action has been taken, so that members of the Association may make known to the Council their individual views.

4. The Council shall have authority to fill vacancies *ad intermim.*

ARTICLE IV—MEETINGS.

1. The annual meeting of the Association shall be held at such time and place as the Council may direct.

2. The Council shall have power to call other meetings of the Association whenever it may be deemed expedient.

ARTICLE V—SECTIONS.

1. Any group of members of this Association may petition the Council for authority to organize a Section of the Association for the purpose of holding local meetings. The Council shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such sections.

Article VI—Official Journal.

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Council shall have power to conduct negotiations with respect to securing an official journal, and shall have full control of its publication and sale.

Article VII—Dues.

1. An individual member of the Association shall pay an initiation fee of two dollars at the time of his election.

2. The annual dues of an individual member shall be three dollars, including a subscription to the official journal.

3. The annual dues of an institutional member shall be five dollars, including two subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list, after due notice.

5. New members entering the Association after April 1, of any year, shall have their dues prorated for the balance of the year, except when they desire to receive the full current volume of the official journal.

Article VIII—Amendments

This Constitution may be amended at any annual meeting of the Association by a two-thirds vote of those present and voting, provided that such amendment shall have been printed in the official journal at least one month before the date of such meeting.

BY-LAWS

1. *Election of Members.* Election to membership shall be by vote of the Council upon written application from the individual or institution seeking admission.

Those who shall be admitted to membership before April 1, 1916, shall constitute the list of charter members.

2. *Nomination and Election of Officers.* Two months before the date of the annual meeting, all members shall be given an opportunity to nominate by mail a candidate for each office for the ensuing year. One month before the annual meeting, the Council shall announce two candidates for each office, one being the person who received the highest vote in the nominations and the other being selected by the Council from among the several nominees next in order.

The election shall be by mail or in person and shall close on the day of the annual meeting.

3. *Committees.* The Committee on Publications shall have charge of the official journal and of all other publications of the Association, under the direction of the Council.

The Council may appoint any other committees and delegate to them such power as may, in its judgment, seem desirable.

4. *Price of Publications.* The Council shall fix the price of the official journal, and of any other publications of the Association to non-members, but in no case shall the journal be sold for less than the annual dues of individual members, as specified in Article VII of the Constitution.

5. *Amendments.* These By-Laws may be amended at any annual meeting under the same conditions as specified in Article VIII of the Constitution.

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THE SIGNIFICANCE OF MATHEMATICS.*

By E. R. HEDRICK, University of Missouri.

Several circumstances combine to render peculiarly fitting a consideration at this time of the significance of mathematics. Of late we have heard much from real or alleged educators, tending to show a lack of appreciation on their part, if not on the part of the public, of the vital role which mathematics plays in the affairs of humanity. These attacks were beginning to receive some hearing in the educational world, on account of their reiteration and their vehemence, if not through intrinsic merit.

A counter influence of tremendous public force, whose import is as yet seen only by those most nearly interested, has now arisen through the existence of war and the necessities of war. To the layman, lately told by pedagogical orators that mathematics lacks useful applications, the evident need of mathematical training on every hand now comes as a distinct surprise.

The attacks on mathematics, and the lay conception of the entire subject, center naturally around elementary and secondary instruction. We ourselves, college teachers of mathematics, have commonly talked of current practice and of reforms largely with respect to secondary education. The third influence which contributes toward the present situation and which may strongly affect its future development is the formation and the existence of this great Association, which affords for the first time in the history of America an adequate forum for the discussion of the problems of collegiate instruction in mathematics.

As retiring president of the Association, I know of no more fitting topic than that which I have chosen. It vitally concerns us; it is bound up with the functions of this Association; and the times in which we live seem to point forcibly toward its consideration. I shall attempt to outline to you my own views on

* Retiring Presidential Address delivered at the second summer meeting of the Mathematical Association of America, at Cleveland, Ohio, September 6, 1917.

the true significance of mathematics, and to sketch what I for one would be glad to see this Association promote.

In speaking of the significance of mathematics, I understand that we mean not at all the baser material advantage to the individual student, not at all a narrow utilitarianism, but rather a comprehensive grasp of the usefulness of mathematics to society as a whole, to science, to engineering, to the nation. Any narrower view would be unworthy of us; any narrower demand by educators means a degraded view of the purposes of education in a democracy.

Especially under the stress of war, public attention may be secured for the real claim of mathematics as a public necessity, not only to be employed by a few specialists, but also to influence and to determine the conduct and the efficiency of thousands.

Thus a knowledge of trigonometry and of the trigonometric theorems of geometry is a prime requisite for the successful and efficient conduct of our armies, not only by a few engineers who are to make maps and to train artillery, but also for all officers to whom the lives of men are entrusted. Any one of these officers, cut off with his force, without a superior engineer at hand, may lose his position and the lives of his men if he is ignorant of the significance of these propositions. Ignorance at such a crisis would be next to treason; it would be incompetence.

Do we, in trigonometry, so bring out the significance of the fundamental ideas on right triangles that the officer who faces such a test will sense the possibility of finding a range, or estimating a distance, without help and without instruments or tables? Frankly I do not believe that we have been doing this, even in such a practical subject as trigonometry. We have been too often content, and too often solely seeking, even here, the knowledge of intricate formalisms, of formulas and rules and theorems, of operations done mechanically. Too often we have omitted, even here, to give insight into the rather obvious significance of these rules and formulas.

On the whole, however, trigonometry is the one subject in which some small measure of insight has usually been secured.

If I now turn to other topics of our curriculum, may I not name scores of equally vital topics usually studied by our students, in which insight is rarely gained? Let me mention some such instances:

In algebra, as taught in colleges, among the topics always considered are fractional exponents, logarithms, and arithmetic and geometric progressions. To many, fractional exponents remain a pure formalism, learned by rote and unappreciated, connected neither with the other topics just mentioned nor with any realities of life. That fractional exponents occur in expressions for air-resistance (as in airplanes), in water resistance (as in measuring stream-flow), in electricity (as in induction), would surprise most students who pass our courses. That these exponents are determinable and are determined by logarithms would surprise students and some teachers, even if the essential equivalence of exponents and logarithms is adequately emphasized. The idea of a compound interest law,

namely, that one quantity may proceed in arithmetic progression as another related quantity proceeds in geometric progression, is ordinarily not brought out, nor is the fact that this same situation leads to a logarithmic law.

The omission of these and similar vital connections, both of mathematics to the exterior world and of one topic in mathematics to another, is directly responsible for the failure of algebra to reach the hearts of our students, and for the failure of the students to gain real insight into the significance of the subjects they so dully learn.

I shall not dwell long on any one topic, for I desire to emphasize the existence of significance for life and society in the entire range of mathematical courses, and I desire to call your attention to the failure—shall I not say *our* failure?—to bring to light that significance.

Let me turn to analytic geometry for another instance of our traditional blindness, if it be that—our sin, if it is not blindness. Here, as before, applications abound. Most of the results of scientific experiment today are known and are recorded not by algebraic formulas of traditional form, but solely by curves traced in our traditional style, showing graphically the functional relations between two or more interdependent variables. Laws of physics, of chemistry, of every quantitative science, expressed by such means abound. The effort of science may well be said to be to deduce from such graphical functions the corresponding laws in algebraic or formalistic form.

Yet to most students of analytic geometry, precisely the reverse view seems to be our aim. The significance of analytic geometry as a piece of scientific machinery is totally lost, and the subject sinks to the level of dubious value in the minds of our students and of half-informed educators. In the present emergency, popular conviction of the real significance of analytic geometry for society is being attained, and may be fostered, through the occurrence of just such graphical laws in the dynamics of airplanes, in artillery performance (ballistics) and in wireless telegraphy. Here, as in general in science, most of our information on functions is now in graphical form, and the desire to express the function in equation form illustrates the fundamental demand of science, and the fundamental significance of analytic geometry.

That the calculus is regarded as dry and uninteresting by many students, and that its value is occasionally doubted, is the strongest proof possible that its significance is not grasped. Here the connection with realities is so easy and so abundant that it is actually a skillful feat to conceal the fact. Yet it is done. I know personally of courses in the calculus (and so may you) in which the pressure to obtain and to enforce memory of formal algebraic rules has resulted in absolute neglect of the idea that a derivative represents a rate of change! I know students whose whole conception of integration is the formalistic solution of integrals of set expressions by devices whose complexity you well know. That an integral is indeed the limit of a summation, and that results of science may be reached through such summation is often nearly ignored and not at all appreciated. That the ideas of the Calculus should fall so low as to consist mainly in

formal differentiation and integration of set expressions must indeed astound anyone to whom the wonderful significance of the subject is at all known. Moreover, it must convince any liberally minded educator who takes our own courses as a true representation of mathematical values that even the calculus is of no importance for real life or for society.

I might proceed to other courses—differential equations as given by Forsyth, the theory of equations as by Burnside and Panton or as by even the most recent writers, the theory of functions (without any hint of its manifold connections with physics), the calculus of variations (denatured, without a hint of its vast importance in mechanics and elsewhere), projective geometry (with no mention of descriptive geometry nor of the representation of space forms).

In all these, tradition has been leading us as far astray as it has in those more elementary courses of the secondary school, which we are wont to criticize. Shall we not search our own house? Shall we not ask if our own collegiate and graduate courses in mathematics demonstrate to students the real significance of the theory they cover? Have we denatured each subject until insight is eliminated and only formalism and logical tricks remain? So long as this blight remains, we must expect and we shall deserve public disdain and sincere doubt of our value to humanity.

It should be unnecessary for me to explain my own deep interest in the logical and cultural side of mathematics. Certainly I would be the last to belittle its great spiritual values. But this is for the specialist rather than for the usual student. Values to the world at large must be stated in terms of more concrete realities. Shall we hide the fact of the immense service of mathematics to society? To emphasize beauty and pleasure to the entire exclusion of the more convincing argument of benefit to mankind is as quixotic and short-sighted as is the corresponding formalization of our courses of instruction. To ignore the significance of our great subject is to spurn our birthright.

Let me then, in retiring from office in the Association, leave with you the sincere hope that a part of the work of this Association may be to impress upon the public the great value of mathematics in its direct effects upon life and upon human society. To accomplish this end, a most effective means, and one ready to hand, is to bring out to our own students, not halfheartedly, but with vigor, not a few but all available facts that shed light on the real meaning of what we teach. Let this Association be a focus from which such doctrine may emanate, a forum in which such views may be emphasized and detailed. Thus I today have mentioned to you a few samples of our neglect, in haste and by name only. Shall we not discuss among ourselves these and other means toward the end, other topics whose significance is commonly lost or neglected, other points of view that will increase insight, even if it be at the expense of a few formulas or theorems that we traditionally treasure.

To the same end, may I now emphasize what seems to me a great, if not the greatest, function of this Association. In America, up to recent years, the beauty and interest centering in pure mathematics has so absorbed all mathematical

that a better insight into the significance of mathematics will prevent or nullify mistaken attacks on the subject as one of little public worth.

Such to my mind should be one function, if not the chief function, of this Association: the regeneration of a significant mathematics, the encouragement of workers in applied mathematics, and the effort to obtain recognition of the true public worth of mathematics in every phase.

THE LINEAR FUNCTION AND THE LINE.

By JOSEF NYBERG, Chicago, Ill.

In a previous paper[1] I showed how the work in college algebra, trigonometry, and analytics may be unified by treating the contents of these subjects under the general theory of functions. In a second paper[2] I then showed how the notion of function should be introduced so that the student may see that coordinates constitute only the language of mathematics, the subject itself dealing with the properties of functions and relations between variables. The present paper begins the study of the line, and closes with some illustrations of how the new point of view eliminates certain difficulties.

I begin the work by stating that the rate of increase of a function is one of its most important properties, and that the simplest function is that which has a known constant rate of increase. For such a function we may define the rate as the ratio of the change in the value of the function to an increase in the value of the variable. We can measure this rate by the change in the function per unit increase in the variable. This idea is impressed by writing some data on the board, as

$$\begin{cases} x & -1 & 0 & 1 & 2 & 3 \\ y & 2 & 5 & 8 & 11 & 14, \end{cases} \qquad \begin{cases} x & -1 & 1 & 3 & 5 \\ y & 4 & 9 & 14 & 19, \end{cases}$$

and asking such progressive questions as: What is the change in x? in y? What is the rate of increase? What is the value of y for $x = 0$? How can the value of y be computed for a value of x not given in the data? When the student has seen that the general relation is $y = y_0 + mx$ the more formal work may be presented in the following order.

I. Assuming that the function has a constant rate of increase m, and that several corresponding values are known, we prove that the graph of the function is a line; and conversely, a line is the graph of a function with a constant rate of increase. The proof naturally hinges on the similarity of certain triangles; and as these triangles will be similar only when m is constant, the student will associate a line with a uniform rate. I prefer the form $y = y_0 + mx$ to the form $y = mx + b$, because the physicists and engineers are accustomed to such relations as $v = v_0 + at$, $s = s_0 + vt$, $p = p_0 + (\rho g)h$ and because $y = y_0 + mx$

[1] "The Unification of Freshman Mathematics," in this MONTHLY, April, 1916, page 101.

[2] "The Presentation of the Notion of Function," in this MONTHLY, September, 1917, p. 309.

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is the only statement of how the student actually does compute any value of y in a numerical case: he adds to y_0, a reference value, the amount mx, which is proportional to the increase in x from 0, another reference value. Even if we are dealing with an abstract relation, b would not be as good a symbol as y_0 because it is entirely lacking in connotation.

II. As a second step we show formally that every linear relation $Ax + By + C = 0$ $(B \neq 0)$ can be written as $y = y_0 + mx$ and that its graph is a line.

III. If the value of y_0 is not known, but the rate of increase and a pair of corresponding values are known, we do not need to compute y_0 explicitly, as the student did in the preliminary exercises, but can show that the relation is $y = y_1 + m(x - x_1)$. Further, if the rate of increase is unknown the student will see that the function is undetermined unless we know several corresponding values from which the rate can be found as

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The work thus emphasizes that a pair of values and a rate determine a function, and does not particularly emphasize such an equation as

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Even the discussion of an abstract line can be based on the form

$$y = y_1 + m(x - x_1)$$

for it can be used to derive the "two intercept" form and the "perpendicular" form, wherein a point $(p \cos \alpha, p \sin \alpha)$ and a slope $(- \cot \alpha)$ are known. I do not consider here the other problems dealing with families of lines or locus problems, as I am concerned only with the introduction to the subject.

There are two points in this method to which I call attention. The first is that all the beginning work is grouped around a central idea—the study of a function determined by a value and a rate of increase. This idea unifies the separate problems. Examining the corresponding treatment of lines in any textbook, we note the great diversity even in a single book in the modes of attack on these problems. Such problems as the line through two points, the point-slope form, the perpendicular form, the determination of the slope from the equation, the condition that lines be parallel or perpendicular, all these appear as *distinct* problems, completed only when a formula is found for writing the answers to analogous problems. Thus, the practice of using an equation like

$$\frac{y - y'}{y' - y''} = \frac{x - x'}{x' - x''},$$

found in several books, is especially pernicious in that it has absolutely nothing in common with other forms, and because the simpler concepts on which the

$P_1P_2P_3$ (irrespective of which is the unknown), determining λ from the data (if it is not given explicitly), and then applying the relation. I have found that the complete elimination of the word "formula," in terms of which the student likes to think, and the substitution of the idea that the equation is a statement of a relation between various quantities is very helpful in all such problems.

A similar situation is met when dealing with the angle between two lines. The student can use the formula

$$\tan \theta = \frac{m_1 - m_2}{1 + m_1m_2}$$

to find $\tan \theta$, but the problem of finding a line making a given angle with a given line is more difficult. This trouble, like the previous one, is avoided by impressing the fact that $(1 + m_1m_2) \tan \theta = m_1 - m_2$ is a relation between three quantities, m_1, m_2, and $\tan \theta$, any two of which being known the relation enables us to determine the third. It is not a mere change of terminology which is used for removing the difficulty; we are giving the student a better conception of the word *equation.*

THE GRAPH OF $F(X)$ FOR COMPLEX NUMBERS.

By A. F. FRUMVELLER, Marquette University.

Teachers of analytic geometry are often asked why the complex roots of $f(x)$ cannot be diagrammed. The answer usually given, that real points of the graph lie in the plane of the blackboard, while complex points are in front of this plane or behind it, is true enough; but it leaves the student unsatisfied. He is not able to carry out the idea for himself; and the teacher may not have thought of working it out, nor can he lay his hand on any systematic treatment of the matter. Yet the topic is important, not to say interesting, since several lines of thought here converge: a rather full development with ample illustrations may consequently be welcomed. The purposes of a mathematical club, or undergraduate seminar, would be beautifully served by a handling of graphs along the following lines. Care has been taken with the notation: x invariably stands for $u + iv$; y stands for $U + iV$; complex constants are designated by Greek letters; thus, $\alpha = a + iA$, $\rho = r + iR$, etc. We restrict $f(x)$ to algebraic types merely for the sake of uniformity in the presentation.

The literature on this subject is so slight that the writer is able to mention only three sources to which the student may go for further information; these are:

1. Schultze's "Graphic Algebra" (The MacMillan Company, 80 pp.): methods are given for the construction of the complex roots of quadratics, cubics, and quartics, the purpose being merely to find these roots, and not to exhibit the complete graph.

2. Hamilton and Kettle's "Graphs and Imaginaries" (Longmans, Green and Company, 40 pp.): the quadratic alone is treated. Complex roots, imaginary

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intersections of two circles, tangent-points of a tangent to a circle from a point within, are constructed most elegantly by means of the polar and an auxiliary conic called the "shadow-curve." The method is not available for general graphing.

3. Phillips and Beebe's "Graphic Algebra" (Henry Holt and Company, 157 pp.): the last 60 pages deal with the complete complex graph of the reduced quadratic, cubic, and quartic, but only when their coefficients are real. Triangular or 60°-paper is used, and the complex branches are plotted in the usual way, point by point, but with many skilful simplifications. The methods of the present paper are not employed, so that the student should by all means consult this book in connection with the following section. See also the article on "Imaginaries in Geometry" by E. W. Davis, in *Nebraska University Studies*, volume 10.

§ 1. X Complex, Y Real.

When x is real, its domain or field is the line OX, and real values y_k of the function $y = f(x)$ are represented by perpendicular ordinates erected at x_k on OX. If now x belongs to the number-system $u + iv$, its field is the plane uov; and just as before, we associate with every point (u, v) of the plane a real perpendicular ordinate y. As x ranges over its plane, the values of $y = f(u + iv)$ will sometimes be real, sometimes complex; the real ordinates will be found standing on certain curves $u = \varphi(v)$, whose equations can at once be found. For example:

(*a*) $y = mx + n$. Since $y = (mu + n) + imv$, real ordinates occur only when $v = 0$, *i. e.*, they stand on the axis of reals, and there are none anywhere else in the plane. The plane uoy is of course identical with the plane YOX of ordinary analytic geometry—a fact to be remembered throughout.

(*b*) $y = kx^2 + mx + n$. When $x = u + iv$, we find

$$y = (ku^2 - kv^2 + mu + n) + iv(2ku + m).$$

If y is to be real, either $u + (m/2k) = 0$, or $v = 0$, so that all real ordinates stand on two straight lines in the x-plane; namely the axis of reals, and a perpendicular to it at $[-(m/2k), 0]$. When $v = 0$, we have $y_1 = ku^2 + mu + n$ as the equation of that branch of the graph lying in the plane uoy; when $u = -(m/2k)$, $y_2 = [n - (m^2/4k)] - kv^2$ is the other branch, lying in a plane parallel to voy. The complete locus consists, therefore, of two parabolic branches of opposite curvature in perpendicular planes with a common vertex-ordinate at $[-(m/2k), 0]$, whose length is $n - (m^2/4k)$. One branch never meets the x-plane; the other pierces it in two points, whose coördinates are precisely the two complex, or the two real, root-values of $f(x)$, according as the discriminant is minus or plus.

To make this clearer, take $y = x^2 - 3x + 5$ (Fig. 1).[1] On $v = 0$, and $u = \frac{3}{2}$, stand the real y's, defining the curves $y_1 = u^2 - 3u + 5$, and $y_2 = \frac{11}{4} - v^2$. This latter pierces the x-plane at $u = \frac{3}{2}$, $v = \pm\sqrt{11}/2$,

[1] The drawings were made by Mr. Andrew Keiding, senior student at Marquette University.

i. e., at $x = \frac{3}{2} \pm (i/2)\sqrt{11}$, or at $x = OQ + iRQ$. In passing, we observe a curious relation between TQ and RQ. The minimum ordinate of $y = ku^2 + mu + n$, standing at $u = -(m/2k)$, is $TQ = (4kn - m^2)/4k = k \cdot RQ^2$; hence we have $x = OQ \pm (i/\sqrt{k})\sqrt{TQ}$.

(*c*) In the cubic $y = x^3$, the real y's stand on the 60° lines, $v = 0$, and $v = \pm u\sqrt{3}$. More generally, let $y = x^3 + mx + n$. When $x = u + iv$, we have $y = (u^3 - 3uv^2 + mu + n) + iv(3u^2 - v^2 + m)$. The real y's now stand on $v = 0$, and $v^2 - 3u^2 = m$, an hyperbola, and its axis, major or conjugate, according to the sign of m. The path-equations are equations of condition.

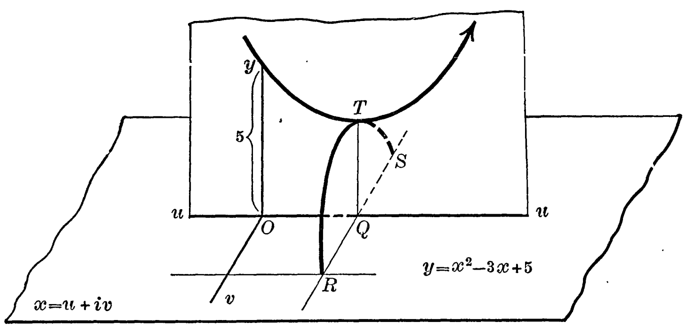

FIG. 1.

If we combine them with the original y-equation so as to eliminate v, we get as length-formulas for y, $y_1 = u^3 + mu + n$, and $y_2 = -8u^3 - 2mu + n$. These are the equations of the graphs that would be obtained by projecting the real ordinates from infinity parallel to ov onto the plane background uoy. (Similarly for the plane voy, by eliminating u.) The points on the hyperbola at which $y = 0$ give the complex roots of $x^3 + mx + n = 0$; they may be found algebraically by putting $y = f_3(u, v) + iF_3(u, v) = 0$, and solving $f_3 = 0$ and $F_3 = 0$ as simultaneous. Note that the plane uov contains only three isolated points; if $y = \sum_0^n a_k x^k = f_n(u, v) + iF_n(u, v)$, there would be n isolated points, since there exist, for every y, exactly n values $x_k = u_k + iv_k$ of x.

Nothing prevents our equations from having *complex coefficients*. Let $\alpha_1 x + \alpha_0 = y$. Then $y = (a_1 u - A_1 v + a_0) + i(a_1 v + A_1 u + A_0)$, where the real y's stand on $a_1 v + A_1 u + A_0 = 0$, which pierces the plane of reals in one point; hence the graph in the plane uoy is merely an isolated point. The quadratic, $(a_2 + iA_2)x^2 + (a_1 + iA_1)x + (a_0 + iA_0) = y$, has real y's on the curve $A_2(u^2 - v^2) + 2a_2 uv + a_1 v + A_1 u + A_0 = 0$; if $v = 0$, the only points on the real axis ou, at which ordinates can stand, are given by $A_2 u^2 + A_1 u + A_0 = 0$ (whose roots are not necessarily real) and the length of these ordinates is furnished by $a_2 u^2 + a_1 u + a_0 = 0$.

In general, if $y = \sum_0^n \alpha_k x^k$, and $v = 0$, we have $\sum_0^n A_k u^k = 0$, which gives, at

most, n points on ou, the real axis; combining this with the conclusion of the previous paragraph, the result may be stated as follows:

Let $y = \sum_0^n \alpha_k x^k$. When x is $\left\{\begin{matrix}\text{complex}\\ \text{real}\end{matrix}\right\}$ and the coefficients $\left\{\begin{matrix}\text{real}\\ \text{complex}\end{matrix}\right\}$, the locus in the $\left\{\begin{matrix}\text{complex plane } uov\\ \text{real plane } uoy\end{matrix}\right\}$ consists of $\left\{\begin{matrix}n \text{ isolated points.}\\ k \text{ isolated points, } (k \leqq n)\end{matrix}\right\}$.

The condition for a real root x_r in $\sum_0^n \alpha_k x^k = 0$ is found by eliminating u from $\sum_0^n A_k u^k = 0$, and $\sum_0^n a_k u^k = 0$ (cf. Hardy, "Pure Mathematics," p. 88). The actual finding of all the real and complex roots of this equation is accomplished by solving $f_n(u, v) = 0$, and $F_n(u, v) = 0$ as simultaneous, either algebraically or graphically. The reader is advised to try the graphic solution in some simple case, such as $ix^2 + 8 = 0$, for instance. (Compare this method with that of Schultze, "Graphic Algebra," pp. 36, 51, 63.)

Implicit functions are dealt with as above, but require a little more care. Take $y^2 = x^3 = (u^3 - 3uv^2) + i(3u^2v - v^3)$; y is real when $v = 0$, and when $3u^2 - v^2 = 0$. This last gives $y^2 = -8u^3$, so that u must be negative for this part of the locus. But above all, the student should carefully construct and visualize the graph of $x^2 + y^2 - r^2 = 0$. The case of $x^2 + y^2 + r^2 = 0$ gives $y^2 = v^2 - (r^2 + u^2)$, with the condition $uv = 0$: the branch $y_1{}^2 = -(r^2 + u^2)$ in the plane uoy is destitute of real ordinates, but in the plane voy we have the real hyperbola $y_2{}^2 = v^2 - r^2$. The locus $x^2 + y^2 = i$ has no real ordinates at all. The student should carefully draw the graph of $y^2 = x^2(x - 2)$, a curve with an isolated point at the origin. Other types suitable for graphic study are furnished by $x, x^2, x^3 = f(y)$; and by $y, y^2, y^3 = \prod_0^n (x - \alpha_k)$, or $1/\sum_0^n \alpha_k x^k$, for $n = 1, 2, 3$, employing in each equation first real, then imaginary, then complex numerical coefficients (cf. Phillips and Beebe, "Graphic Algebra," Chap. V, for a very complete treatment of the quadratic, cubic, and quartic, when the coefficients are real; many graphs are given in illustration).

§ 2. X Real, Y Complex.

At first sight, this section might seem superfluous, since on merely interchanging x and y, we come back to the preceding case. For the purpose we have in view, however, such an inversion is not allowable; we are making a clearcut distinction between the dependent and the independent variable, and while the latter ranges over its domain, the function is to be represented by an ordinate at right angles to this domain. The carrying out of this idea demands that we shall assign to x and to y, throughout this paper, an invariable meaning; the method of graphing to which we are thus led is of the highest theoretical value, since without it a clear understanding of the graph of $f(x)$, when both X and Y are complex, cannot be obtained.

The field of x is therefore a line XOX lying in space; y is of the form $U + iV$, or $y_r + y_i$. On a fixed "base-plane" UOV, perpendicular to OX, let OU, OV, be two directions at right angles for reference. These preparations having been made, let x run over its domain OX. At every point x_k we shall have a perpendicular ordinate y_k: if y_k is real, it stands out from OX parallel to OU—if imaginary, parallel to OV—if complex, at an intermediate angle depending on its amplitude, as given by the equation $y_k = f(x_k)$. The result, in general, will be a spiral curve wound around OX (Fig. 2).

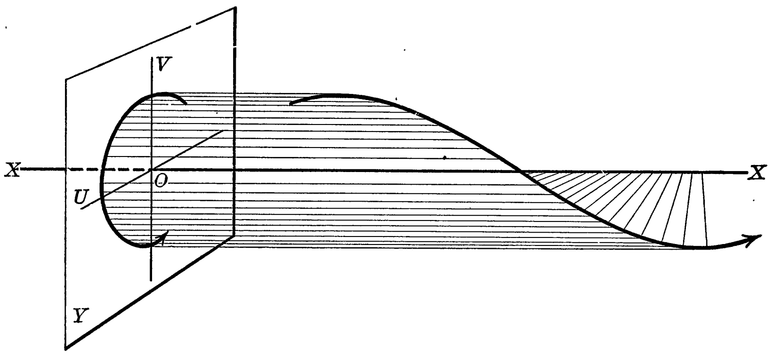

FIG. 2.

If now, we go to infinity and look back along OX, all these ordinates will be projected, unchanged in length and direction, onto the base-plane UOV, and will seem to radiate as vectors from the origin; their endpoints will map out a locus on UOV, say $U = F(V)$. This locus has its points P_k in one-to-one corre-

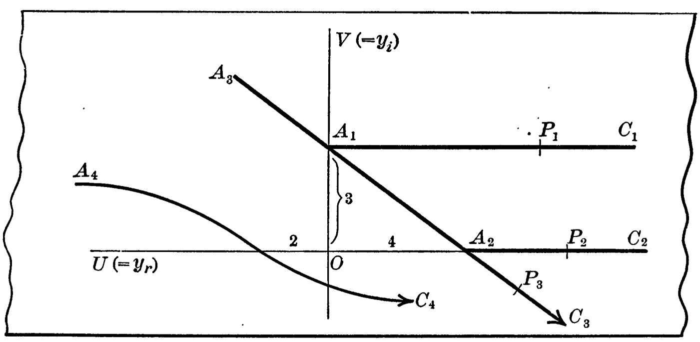

FIG. 3.

spondence with the ordinates y_k, and consequently with the values x_k of the independent variable; it is therefore in a true, though slightly unusual sense, a graph of $y = f(x)$. In reality, it is the projection of the actual locus; but the UV-graph is more compact and easier to handle; and it furnishes all the information that the actual locus could give. It is evident that the function-symbol "f" acts as an operator, transforming the segment $x_m x_n$ of the x-domain into a

segment of the curve $U = F(V)$. This manner of graphing seems to be new, although it is only an extension of the usual process. Since x is thus lost to view, its path must be depicted on a separate diagram.

To have y complex, $f(x)$ must be of the form $\psi(x) + i\phi(x)$. If $\phi(x) = 0$, the graph is altogether in the plane UOX, and its projection is simply a segment of the real axis OU, since the ordinates all pile up on top of one another; similarly for $\psi(x) = 0$. The equation of the UV-curve in parametric form will be $U = \psi(x)$, $V = \phi(x)$. The endpoint of the vector $y_k = OP_k$ is (U_k, V_k), or $(\psi(x_k), \phi(x_k))$, and its angle is $\tan^{-1}(V_k/U_k)$. Let the reader now identify the following graphs (Fig. 3), and try to visualize the space-curves of which they are the projection:

A_1C_1,	Graph of $y = 4x^2 + 3i$,	Equation;	$U = 4x^2, \quad V = 3$,
A_2C_2,	" " $y = 4x^2 + 3$,	"	$U = 4x^2 + 3, \quad V = 0$,
A_3C_3,	" " $y = (4 - 3i)x^2 + 3i$,	"	$3U + 4V - 12 = 0$.
A_4C_4,	" " $y = x^3 - 2 - ix$,	"	$V^3 + U + 2 = 0$.

When the equation is in the form $y = \sum_0^n \alpha_k x^k$, we obtain $U = F(V)$ very readily by Sylvester's method of elimination: thus,

$$y = \sum_0^1 \alpha_k x^k \text{ gives the straight line} \begin{vmatrix} a_1 & a_0 - U \\ A_1 & A_0 - V \end{vmatrix} = 0;$$

$$y = \sum_0^2 \alpha_k x^k \text{ gives the conic,} \begin{vmatrix} a_2 & a_1 & a_0 - U & 0 \\ 0 & a_2 & a_1 & a_0 - U \\ A_2 & A_1 & A_0 - V & 0 \\ 0 & A_2 & A_1 & A_0 - V \end{vmatrix} = 0, \text{ and so on.}$$

Among the simpler types of $f(x)$ we may note the following:

$y = a_0 + iA_0$ is the point $(U, V) = (a_0, A_0)$. The actual locus is a line parallel to OX.

$y = (b + iB) \sum_0^n a_k x^k$ is the line $\dfrac{U}{b} = \dfrac{V}{B}$.

$y = (b + iB) \sum_1^n a_k x^k + (c + iC)$ is the line $\dfrac{U - c}{b} = \dfrac{V - C}{B}$; a special type of which is

$y = \alpha + (\beta - \alpha)x$, or $\dfrac{U - a}{b - a} = \dfrac{V - A}{B - A}$, the line joining $y = \alpha$, and $y = \beta$.

$y = \dfrac{\alpha + \beta x}{\gamma + \delta x}$, a circle. (Write the complex numbers in full, clear of fractions, etc., and finally eliminate x.)

$y = \gamma + \rho \dfrac{1 + ix}{1 - ix}$, a circle in explicit form; on developing this expression as above, we find $(U - c)^2 + (V - C)^2 - (r^2 + R^2) = 0$, so that $y = \gamma$ is the center, and $| \rho |$ the radius. (Hardy, "Pure Mathematics," p. 94.)

The reverse problem—given a graph $U = F(V)$, find $y = f(x)$—can be solved whenever U and V are expressible in parametric form. For example, $U^2 = 4V$ becomes $y = 2x + ix^2$: $U = mV + b$ gives $y = mx + b + ix$, or even $y = mx^n + b + ix^n$; $U^2 + V^2 - r^2 = 0$ becomes $y = r(\cos x + i \sin x)$, and so on. In a similar way, the value of x which makes $y = 0$, *i. e.*, which causes the graph $U = F(V)$ to pass through (0, 0), will be found by putting $U = \psi(x) = 0$, $V = \phi(x) = 0$ simultaneously, and then obtaining the H.C.F. of these expressions.

Implicit functions like $y^2 = 4px$, $y^3 = x(x - 2)$, etc., are handled just as above. They have the peculiarity of furnishing both y_r and y_i when x as well as the coefficients are real. In Fig. 4 we have the complete locus of $x^2 + y^2 = 4$

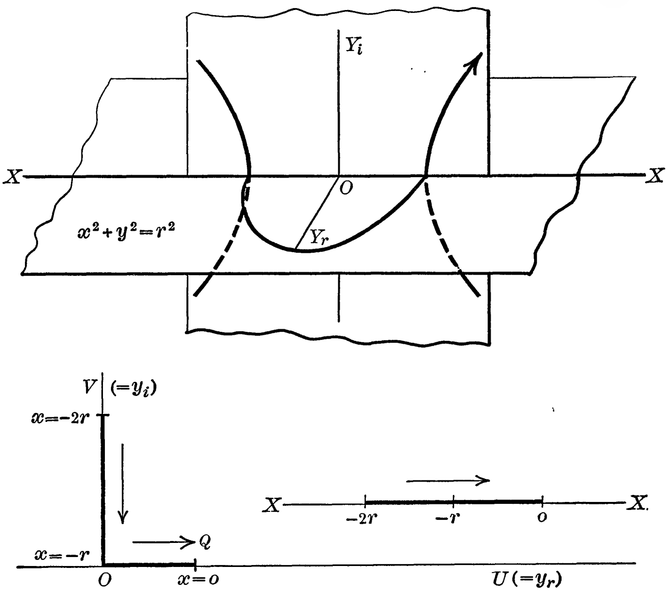

FIG. 4. Graphs of $x^2 + y^2 = r^2$.

where real ordinates lie in UOX, for $-2 < x < 2$. At $x = 2$ there is a sudden turn of 90°, but no discontinuity in the locus; the branch in VOX is $y = i\sqrt{x^2 - 4}$, or $y_i = V = \sqrt{x^2 - 4}$. The revolution of the figure about OX gives a two-sheeted hyperboloid with sphere, from which two mutually perpendicular sections through OX cut out the complete locus. The reader will note that tangents to $x^2 + y^2 = r^2$ may be drawn from points *within the circle*—a thing which at first sight would seem utterly impossible. Such tangents pass out of the two-space UOX into the next higher space. As a further exercise, one might draw the graph of $y^2 = x^2(x - 2)$ for the region $0 < x < 2$.

The general case of $y^n = \psi(x) + i\phi(x)$ gives in a similar manner, at every point x_k, n different ordinates standing out from OX; a simple case for preliminary study would be $y^3 = x$, where two of the roots are complex, and one real. The arrangement of the ordinates can be followed by noting that

$$y = x^{1/3}(+1)^{1/3} = x^{1/3}\left[\cos\left(\frac{0° + n \cdot 360°}{3}\right) + i\sin\left(\frac{0° + n \cdot 360°}{3}\right)\right], \text{ etc.,}$$

by Demoivre's theorem; on UOV, the graph consists of the U-axis, and two straight lines through (0, 0) making with it an angle of 60°. Each of these lines represents a branch of this three-valued function. To indicate this clearly, we may imagine that the plane UOV consists of three transparent, infinitely thin films, not connected with one another at the origin—each film depicting the march of one branch. For the circle, Fig. 4, the situation is different. Let $y = +\sqrt{r^2 - x^2}$ be the positive branch of the function; as x travels from $-\infty$ to $-r$, the point (U, V) travels down the axis OV from $+\infty$ to 0; and thence along OU to Q, while x moves on to zero. As x moves from zero to $+\infty$, (U, V) retraces its path, thereby producing an *overlapping;* to prevent this, conceive UOV as a double film, and cut these films along OU from Q to $+\infty$; then cross-connect the films along this cut, so that the point (U, V), on reaching Q, slides down onto the second film. It must remain on this sheet till x reaches $+\infty$; and in this way, a one-to-one correspondence of the x-path, and of its transformed path in UOV, is effected. A similar cut will go from $(-r, 0)$ to $-\infty$, for the branch $y = -\sqrt{r^2 - x^2}$. (When the plane OUV is thus regarded as composed of several sheets or films cross-connected suitably, it is called a "Riemann surface.") Graphs in UOV, composed of straight-line segments, as above, parallel to the axes, can be defined only by parametric equations.

Our present method of graphing affords a clear explanation of *imaginary intersections* of curves. Take the circles $x^2 + y^2 = 25$, and $(x - 26)^2 + y^2 = 25$; they fail to meet in the plane YOX, and when solved together give $x = 13$, $y = \pm 12i$. Let us state the problem this way: "Where will these two loci meet?" Answer: "On the locus $x = 13$;" which is absolutely correct, according to Fig. 4, since $x = 13$ is a plane perpendicular to the axis OX, on which the two hyperbolic branches cross. The situation is like that in algebra, where the equation is more general than the verbal statement, and applies to conditions not explicitly foreseen. Our imaginary intersections are *actual geometric entities* in a higher space-domain,—they are answers to a wider question in a larger number-field.

In a similar way, the statement that the *polar is the chord of contact of tangents from the pole* is substantially true, even when the pole is within the locus. A circle, for example, sends out, from an inner point P on the X-axis, two actual tangents to the hyperbolic branch, and these have a chord of contact standing *perpendicular* to the axis of reals. A plane at right angles to OX through this chord leaves as its trace on the plane of the circle the line which we call the *polar* of P. The *power-axis* of two non-intersecting circles is accounted for, in the same

way, by the plane through their common chord (*cf.* Hamilton and Kettle, "Graphs and Imaginaries," ch. 5).

As a final application of the above methods, let us graph on the diagram of Fig. 4 the two curves $y^2 + x^2 = r^2$, and $Y^2 + (ix)^2 = -r^2$. On the shelf $y_r ox$, we find the loci $y_r^2 + x^2 = r^2$, and $Y_r^2 + (ix)^2 = -r^2$; on the vertical plane $y_i ox$, we find $Y_i^2 + (ix)^2 = -r^2$, and $y_i^2 + x^2 = r^2$, in the order given. Now, our $y = f(x)$ is essentially a "circular" function, and $Y = f(ix)$ is "hyperbolic." When $r = 1$, it is customary to represent the ordinates of these two classes of functions by the symbols Sin and Sinh, respectively, so that $y = f(x) = \text{Sin}(x)$, and $Y = f(ix) = \text{Sinh}(ix)$. The distances from their feet to the origin along OX are called Cos and Cosh, in like manner. The branches of our two loci are evidently congruent; hence, $\text{Cosh}(ix) = \text{Cos}(x)$, and $\text{Sinh}(ix) = i\,\text{Sin}(x)$, —a new and purely geometrical proof of the fundamental formulas of hyperbolic trigonometry.

The reader will note that if the X-axis of the graphs treated in this § 2 be identified with the real X-axis of the graphs in § 1, the imaginary Y-branches that we have discussed will lie in the fourth dimension.

§ 3. X and Y Both Complex.

The field of x is now the plane uov. When x was linear, its real and imaginary ordinates were at right angles, and both stood perpendicular to OX at the point x_0. The complex intermediate vectors y_c were likewise perpendicular to OX at the same point. To carry out this idea, there must now stand on every point $x_0 = u_0 + iv_0$ a real ordinate y_r and an imaginary ordinate y_i, mutually at right angles, and both perpendicular to uov. We need consequently a 4-dimensional space for our graph, y_i pointing into the fourth dimension. The plane $y_r x_0 y_i$ may be called the "absolutely" perpendicular plane at x_0; in it, radiating from x_0, lie the other complex ordinates y_c, all at right angles to the plane uov at x_0.

The easiest way to deal with this situation graphically is first to ignore one dimension, say y_i, and plot the locus in the 3-space $y_r uv$; then ignore y_r, and plot in the space $y_i uv$. A clear idea can thus be gotten of what is happening in these two directions. Real ordinates will stand on certain paths in uov, and imaginary ones likewise: thus, if $y = x^2 - 2x + 4 = (u^2 - y^2 + 2u + 4) + 2iv(u - 1) = U + iV$, real ordinates whose lengths are, respectively, $y_r = u^2 - 2u + 4$, and $y_r = 3 - v^2$, stand on the paths $v = 0$, $u = 1$. Imaginary ordinates stand on the hyperbola $u^2 - v^2 - 2u + 4 = 0$, and their lengths are given by $y_i = 2(uv - v)$. Combining the path-equation with the length-equation by eliminating u or v, we get the equation of the locus that results from projecting all the y_r's, or y_i's, onto the planes $y_r ou$, $y_r ov$, or $y_i ou$, $y_i ov$.

In this example, U and V were explicit, and the critical paths for x could be read off at first sight; but now, take the circle $x^2 + y^2 - \rho^2 = 0$. Separating the real and imaginary terms, we get (1) $u^2 + U^2 - r^2 - (v^2 + V^2 - R^2) = 0$, (2) $uv + UV - rR = 0$; no obvious x-paths present themselves. If however we assign to v the value $\psi(u)$, we can eliminate u from (1) and (2), and thus

obtain an equation between U and V. To any assigned curve $\psi(u, v) = 0$, there corresponds a definite curve $\phi(U, V) = 0$ in UOV, the "absolutely" perpendicular plane. One locus is the "transformation" of the other. The "f" in $y = f(x)$ is therefore an operator which transforms an arc of the locus $\psi = 0$ in planes 1 and 2 of our 4-space, into a corresponding arc of the locus $\phi = 0$ in planes 3 and 4. Its action is analogous to that of a mirror with a bent surface: the same function ψ, when it is transmitted through the medium of $f_1, f_2, \cdots$, comes out on the plane UOV as $\phi_1, \phi_2, \cdots$. This process is often spoken of as the *transformation* or *depiction* of the plane uov on the plane UOV.

The manner in which this depiction is brought about is as follows: Let there be given a locus $\psi(u, v) = 0$ in the x-plane, with all its ordinates, as computed from $y = f(x)$, standing upon it. Now project the plane from ∞, parallel to itself and to the line ov, onto the axis of reals ou. This leaves the *length* and the *inclination* of the ordinates y_c unchanged, because they are all perpendicular to uov, and their mutual angles are in the plane $y_r o y_i \equiv UOV$, which is perpendicular to the line ov; hence they now stand in proper length and position on the line ou. Now project this line, parallel to itself, onto the plane UOV; all the ordinates pile up against this plane and appear as vectors issuing from the origin, and their end-points map out a locus $\phi(U, V) = 0$, which is the graph of $y = f(x)$ with $x = u + iv$ as invisible parameter.

Since x has been shuffled out of sight by projecting its field uov into the point $(0, 0)$ of the Y-plane, two separate diagrams will be needed in plotting—one, to show the path of x in its own plane—the other, to show the position and length of the vector-ordinates in the plane of Y. We may summarize the relation of these two planes as follows for explicit functions:

Let $y = f(x)$, and $x = u + iv$. To any point (u, v) corresponds a point (U, V), where $U = f_1(u, v)$, $V = f_2(u, v)$, and $y = U + iV$. To the x-path $v = \psi(u)$ corresponds the y-path $U = f_1(u, \psi(u))$, $V = f_2(u, \psi(u))$, or $F(U, V) = 0$; to the y-path $V = \phi(U)$ corresponds the x-path $f_2(u, v) = \phi[f_1(u, v)]$; it being understood that $\psi(u)$, and $\phi(U)$ may be constants. A given ordinate, $y = a + ib$, stands on the points (u_m, v_m) in the x-plane, where u_m, v_m are the common solutions of $U = f_1(u, v) = a$, $V = f_2(u, v) = b$. When $a = b = 0$, we find the complex roots of $f(x) = 0$. For instance, if $x^2 + y^2 = i$, and $U = V = 0$, we get $u = v$, and $uv = 1/2$, which meet at $(\pm (\sqrt{2}/2), \pm (\sqrt{2}/2))$ in the plane uov, where $y = 0$. We have thus found the square roots of i. From $x^3 + y = i$, we get in like manner $(0, -1)$, $(\pm (\sqrt{3}/2), 1/2)$; hence $-i$, and $(i/2) \pm (\sqrt{3}/2)$ are cube roots of i.

For implicit functions $\chi(x, y) = 0$, the separation of real and imaginary terms will yield $\chi_1(u, v, U, V) = 0$, $\chi_2(u, v, U, V) = 0$. To find the locus in UOV corresponding to $v = \psi(u)$, we must regard u as a parameter, and eliminate it, as above. If this be impossible, we can plot by points, though this is hardly ever useful in practice. Now let the reader try all this out on some simple function like $y = x^2$, $y^2 = \alpha x$, or $y = \alpha x^2 + \beta x + \gamma$—allowing x (or y) to run along lines parallel to the axes, or through the origin; or along rectangular hyperbolas, etc. The following examples afford useful practice:

bridged the gap between analytic geometry and function-theory, by way of the fourth dimension. There are many other ways of graphing imaginaries, no doubt; but the above is to be preferred, because it is a logical and systematic scheme resting at both ends on the approved processes.

AN INTRINSIC EQUATION SOLUTION OF A PROBLEM OF EULER.

By PAUL R. RIDER, Washington University.

In his *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes*, Lausanne (1744), p. 64, Euler proposed and solved the following problem: Given two points P_0 and P_1, and directed lines P_0Q and QP_1 through them, to determine an arc tangent to these two lines at P_0 and P_1, which with its evolute and its normals at P_0 and P_1 will enclose the minimum area. This area is of course expressed by the definite integral $\int_{s_0}^{s_1} \frac{1}{2}\rho ds$, in which ρ is the radius of curvature. The ordinary method of solution is to express the integral in the form

$$\int_{x_0}^{x_1} \frac{(1+y'^2)^2}{2y''} dx, \tag{1}$$

and to find, by methods of the calculus of variations, the form of the curves that will give it a minimum value. (It is to be observed that y'', the second derivative of y with respect to x, occurs in the integrand. The problem is of particular interest on this account.) It is necessary to make some substitutions and transformations that are not altogether simple, before it is evident that the minimizing curves are cycloids.[1]

I wish to give here a shorter solution by obtaining the intrinsic equation of the extremals (*i. e.*, minimizing curves). The intrinsic equation of a curve is defined as the relation between s, the length of arc measured from a fixed point on it, and ϕ, the angle of deviation of the tangent at any point from the tangent at the fixed point taken as origin. This is in the sense of Whewell.[2]

If the area integral (1) is written in the form

$$\int_{s_0}^{s_1} \frac{ds}{2\kappa}. \tag{2}$$

where κ is the curvature, we have a special case of an integral studied by Radon,[3]

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[2] See *Cambridge Philosophical Transactions*, Vols. 8 and 9; or Williamson, *Differential Calculus*, 9th edition, revised (1899), p. 304.

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$$s - s_0 = a \sin (\theta - \theta_0).$$

But this is the intrinsic equation of the cycloid.[1]

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

An Elementary Course in Synthetic Projective Geometry. By DERRICK NORMAN LEHMER. Ginn and Company, Boston, 1917. xiii + 123 pages.

Lehmer's *Projective Geometry* "is intended to give in as simple a way as possible the essentials of synthetic projective geometry." The author has done this in the 74 pages of Chapters I—VI, VIII by the methods of pure geometry without the use of anharmonic ratios, circles or any other metric notions. It is his idea that "a purely projective notion ought not to be based on metrical foundations." "The course is not intended to furnish an illustration of how a subject may be developed from the smallest possible number of fundamental assumptions. The author is aware of the importance of work of this sort, but he does not believe it is possible at the present time to write a book along such lines which shall be of much use for elementary students." For the purpose of this course the student should have a thorough knowledge of high-school plane geometry and enough solid geometry to understand the proof of Desargues's theorem about perspective triangles: "*If two triangles ABC and $A'B'C'$ are so situated that the lines AA', BB', CC' all meet in a point, then the pairs of sides AB and $A'B'$, BC and $B'C'$, CA and $C'A'$ all meet on a straight line, and conversely.*"

The first chapter of the book is devoted to the important notion of one-to-one correspondence. In it is explained the need of the fiction about points and lines at infinity. On page 7, after some remarks on one-to-one correspondences being continuous, the author says: "In the case of point-rows this continuity is subject to exception in the neighborhood of the point 'at infinity.'" The reviewer doubts that the student will understand what the "neighborhood of the point at infinity" is.

The theorem of Desargues is very easily proved when the two triangles are not in one plane. When they are in one plane the usual method of proof consists in showing that a third triangle $A''B''C''$ can be found which is perspective to both ABC and $A'B'C'$ from two different centers of projection, and in twice making use of the theorem for two perspective triangles not in one plane. The only difficulty which the student is likely to have with this proof is in the making and the visualizing of the diagram. Lehmer proves the theorem for two triangles not in one plane in the usual way, and draws an illustrative figure;

[1] See Williamson, *loc. cit.*, p. 338.

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[1] See Williamson, *loc. cit.*, p. 338.

then he disposes of the more difficult part of the theorem as follows: "If now we consider the figure a plane figure, the points P, Q, and R still all lie on a straight line, which proves the theorem." This appeal to the student's knowledge of perspective drawing really assumes that if ABC and $A'B'C'$ are two perspective triangles in a plane it is always possible to find two other perspective triangles not in one plane of which ABC and $A'B'C'$ are the perspective picture. When this is argued in detail, as it is in the eleventh edition of the "Encyclopedia Britannica," volume 22, page 428, in the article on "Projection," and a diagram is drawn to illustrate, the proof is just as difficult as the usual proof mentioned above. The author should have given a little more space to this important theorem. The proof is hardly convincing in its brief form. In this same chapter are the definitions of four harmonic points, lines and planes and also this definition of projectivity: "Two fundamental forms are projectively related to each other when a one-to-one correspondence exists between the elements of the two and when four harmonic elements of the one correspond to four harmonic elements of the other." One likes to think of the adjective *projective* in the phrase *two projective point-rows* as meaning that one point-row can be obtained from the other by a series of projections and sections; in fact in paragraph 9 in the first chapter our author has used the phrase *projectively related* in just this sense. But the definition of projectivity quoted above has its advantages. Reye uses it in his *Geometrie der Lage;* but later, to use his own words, he "justifies the use of the term *projective*" by proving "that two projective one-dimensional forms may always be considered as the first and last of a series of forms of which each is perspective both to the one preceding it and to the one following it." Lehmer's book does not contain this proof. Reye's proof is based on this fundamental theorem of projective geometry: "*If two projective point-rows, superposed upon the same straight line, have more than two self corresponding points, they must have an infinite number, and every point corresponds to itself; that is, the two point-rows are not essentially distinct.*" Lehmer proves this theorem in Chapter III by making use of the postulate of continuity in this form: "We now assume, explicitly, the fundamental postulate that the correspondence is *continuous*, that is, that *the distance between two points in one point-row may be made arbitrarily small by sufficiently diminishing the distance between the corresponding points in the other.*" In this same Chapter III a point-row of the second order is defined as the locus of the points of intersection of corresponding rays of two projective pencils; and there is a similar definition of a pencil of rays of the second order. Point-rows of the second order are studied in Chapter IV. It is there proved that such a curve is uniquely determined by any five of its points. Pascal's theorem is proved in its original form and also in the cases in which the inscribed hexagon has degenerated into a pentagon, a quadrangle, and a triangle. Enough construction problems are given to show the great power of the theorem. It is pointed out that circles and conic sections are point-rows of the second order and the statement is made that "it will appear later that a point-row of the second order is a conic section. In the future, therefore, we shall refer to the point-row of the second order as a conic."

Chapter V is a similar chapter on pencils of rays of the second order; the development is about the same as in Chapter IV with Brianchon's theorem taking the place of Pascal's. At the end of this chapter the author calls attention to the principle of duality as evidenced by a comparison of Chapters IV and V. In some books the principle of duality is only a working principle; its validity is not rigorously established but the method of translating from a theorem to its correlative is explained and the validity of the latter is established by translating the proof of the former. In such books it is quite the custom to exhibit the theorem and its correlative in parallel columns on the same page. In other books, for example in "Projective Geometry" by Veblen and Young, it is shown that the duality exists in the axioms on which projective geometry is based, and it is argued that the duality therefore exists in the theorems derivable from those axioms. Poncelet, the discoverer of the principle, established its validity by means of the theory of poles and polars. This is the method used by Lehmer in Chapter VI. The starting point of this chapter is the following pair of theorems: "If a quadrangle be inscribed in a conic, two pairs of opposite sides and the tangents at opposite vertices intersect in four points, all of which lie on a straight line." "If a quadrilateral be circumscribed about a conic, the lines joining two opposite points of contact and the lines joining two pairs of opposite vertices are four lines which meet in a point." Lehmer is not always careful to distinguish between quadrangle and quadrilateral. For instance, in the first of these two theorems he says quadrilateral when he means quadrangle.

An involution of points on a line is often defined as the special case of two superposed point-rows in which every point on the line has the same correspondent whether it is thought of as belonging to the one point-row or the other. In Chapter VIII Lehmer introduces the student to the study of involution in a simpler way by first defining, in terms of a quadrangle and a transversal, what is meant by saying that three pairs of points on a line are in involution and by using this definition to show what is meant by saying that all the points of a line are in involution. Later he shows that points in involution on a line form two superposed projective point-rows.

The seven chapters of 74 pages that have been mentioned constitute Lehmer's course in the essentials of synthetic projective geometry. But this is not all that the book contains. Chapter VII is entitled "Metrical Properties of the Conic Sections." It is here that the author proves that point-rows of the second order are really conic sections. He gets hold of the metrical properties by means of the infinitely distant elements of the plane. The polar line of an infinitely distant point is a diameter, the pole of the infinitely distant line is the center, etc. Point-rows of the second order are classified as hyperbola, parabola and ellipse according as the curve has two points, one point or no point at infinity; and then their analytic geometry equations are found to be the well-known forms $xy = constant$, $y^2 = 2px$, and $x^2/a^2 + y^2/b^2 = 1$. This identifies the point-rows of the second order with the hyperbola, parabola and ellipse as they are known to the student of analytic geometry. If that student had in his analytic

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

ALGEBRA.

489. Proposed by S. A. COREY, Albia, Iowa.

Prove or disprove the following:

$$\begin{vmatrix} -x & -ay & -bu & abv \\ y & x & -bv & -bu \\ u & av & x & ay \\ -v & -u & y & -x \end{vmatrix}^2 + a\begin{vmatrix} x & -x & -bu & abv \\ y & y & -bv & -bu \\ u & u & x & ay \\ v & -v & y & -x \end{vmatrix}^2$$

$$+ b\begin{vmatrix} x & -ay & -x & abv \\ y & x & y & -bu \\ u & av & u & ay \\ v & -u & -v & -x \end{vmatrix}^2 + ab\begin{vmatrix} x & -ay & -bu & -x \\ y & x & -bv & y \\ u & av & x & u \\ v & -u & y & -v \end{vmatrix}^2 = \begin{vmatrix} x & -ay & -bu & abv \\ y & x & -bv & -bu \\ u & av & x & ay \\ v & -u & y & -x \end{vmatrix}^2.$$

490. Proposed by HENRY HEATON, Atlantic, Iowa.

Show that $\sin 3° = \frac{1}{16}(\sqrt{30} + \sqrt{10} - \sqrt{6} - \sqrt{2}) + \frac{1}{8}(\sqrt{5 + \sqrt{5}} - \sqrt{15 + 3\sqrt{5}})$.

491. Proposed by J. W. LASLEY, University of North Carolina.

Solve the equations $xy = x^2 - y^2$ and $x^2 + y^2 = x^3 - y^3$ for x and y.

GEOMETRY.

522. Proposed by GEORGE Y. SOSNOW, Newark, N. J.

Prove that the sum of the squares of the edges of a tetrahedron is equal to four times the sum of the squares of the lines joining the middle points of the opposite edges.

523. Proposed by H. CAMPBELL, St. Johnsbury, Vt.

Given the difference of the segments of the base made by the perpendiculars let fall from the vertical angle, the difference of the base angles, and the sum of the sides of a triangle, to construct the triangle.

CALCULUS.

437. Proposed by LEIGH PAGE, Yale University.

Integrate

$$\int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2}\, dx$$

without the use of Gamma Functions.

438. Proposed by PAUL CAPRON, U. S. Naval Academy.

Find the locus of the equation

$$y^6 - 3(a^2 - x^2)y^4 - 2a^2x^2y^3 + 3(a^2 - x^2)y^2 - 6ax^2(a^2 - x^2)y + a^2x^4 - (a^2 - x^2)^3 = 0,$$

first showing that it can be reduced to the form $y = kx^n \pm (a^2 - x^2)^m$, and finding the points of maximum abscissas, of maximum ordinates, and of inflection.

439. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

MECHANICS.

354. Proposed by G. PAASWELL, New York City.

The acceleration of an electric train is constant and equal to a feet per sec. per sec. Its braking or deceleration is variable and equal to the square root of the velocity. If the distance between stations is 5,000 feet, show that the acceleration must cease and braking ensue when the train is about 960 feet from the stopping point; also that the maximum velocity attained for a minimum time run is 88 m.p.h. and the time of run 54 seconds.

355. Proposed by HORACE OLSON, Chicago, Illinois.

A solid spheroid, axes a, a, b, is placed with its axis of revolution vertical. From its highest point, a particle is projected horizontally with a speed s. Where will it leave the spheroid, assuming that it slides on the surface without friction?

NUMBER THEORY.

272. Proposed by C. C. YEN, Tangshan, North China.

How many integers prime to n are there in each of the sets:

(a) $$1\cdot 2,\quad 2\cdot 3,\quad 3\cdot 4,\quad \cdots,\quad n(n+1);$$

(b) $$1\cdot 2\cdot 3,\quad 2\cdot 3\cdot 4,\quad 3\cdot 4\cdot 5,\quad \cdots,\quad n(n+1)(n+2);$$

(c) $$\frac{1\cdot 2}{2},\quad \frac{2\cdot 3}{2},\quad \frac{3\cdot 4}{2},\quad \cdots,\quad \frac{n(n+1)}{2};$$

(d) $$\frac{1\cdot 2\cdot 3}{6},\quad \frac{2\cdot 3\cdot 4}{6},\quad \frac{3\cdot 4\cdot 5}{6},\quad \cdots,\quad \frac{n(n+1)(n+2)}{6}?$$

273. Proposed by V. M. SPUNAR, Chicago, Illinois.

The ratio of the chances that all numbers ending in 1 or 9 and those ending 3 or 7 are composite is 3 : 2.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

477. Proposed by J. L. RILEY, Junior College, Stephenville, Texas.

Evaluate the product $(1+r+r^2+r^3)(1+r^2+r^4+r^6)\cdots(1+r^{2^{n-1}}+r^{2^n}+r^{3.2^{n-1}})$.

SOLUTION BY LOUIS O'SHAUGHNESSY, University of Pennsylvania.

Since

$$1+r+r^2+r^3=\frac{1-r^4}{1-r},\quad 1+r^2+r^4+r^6=\frac{1-r^8}{1-r^2},\quad 1+r^4+r^8+r^{12}=\frac{1-r^{16}}{1-r^4},\quad\cdots,$$

$$1+r^{2^{n-1}}+r^{2^n}+r^{3.2^{n-1}}=\frac{1-r^{2^{n+1}}}{1-r^{2^{n-1}}},$$

the required product is equal to the product of the following fractions:

$$\frac{1-r^4}{1-r}\cdot\frac{1-r^8}{1-r^2}\cdot\frac{1-r^{16}}{1-r^4}\cdot\frac{1-r^{32}}{1-r^8}\cdots\frac{1-r^{2^{n-1}}}{1-r^{2^{n-3}}}\cdot\frac{1-r^{2^n}}{1-r^{2^{n-2}}}\cdot\frac{1-r^{2^{n+1}}}{1-r^{2^{n-1}}}.$$

The result, manifestly, is

$$\frac{(1-r^{2^n})(1-r^{2^{n+1}})}{(1-r)(1-r^2)}.$$

Also solved by H. C. FEEMSTER, PAUL CAPRON, ELIJAH SWIFT, and HORACE OLSON.

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(b) $$1\cdot 2\cdot 3,\quad 2\cdot 3\cdot 4,\quad 3\cdot 4\cdot 5,\quad \cdots,\quad n(n+1)(n+2);$$

(c) $$\frac{1\cdot 2}{2},\quad \frac{2\cdot 3}{2},\quad \frac{3\cdot 4}{2},\quad \cdots,\quad \frac{n(n+1)}{2};$$

(d) $$\frac{1\cdot 2\cdot 3}{6},\quad \frac{2\cdot 3\cdot 4}{6},\quad \frac{3\cdot 4\cdot 5}{6},\quad \cdots,\quad \frac{n(n+1)(n+2)}{6}?$$

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Since

$$1+r+r^2+r^3=\frac{1-r^4}{1-r},\quad 1+r^2+r^4+r^6=\frac{1-r^8}{1-r^2},\quad 1+r^4+r^8+r^{12}=\frac{1-r^{16}}{1-r^4},\quad \cdots,$$

$$1+r^{2^{n-1}}+r^{2^n}+r^{3.2^{n-1}}=\frac{1-r^{2^{n+1}}}{1-r^{2^{n-1}}},$$

the required product is equal to the product of the following fractions:

$$\frac{1-r^4}{1-r}\cdot\frac{1-r^8}{1-r^2}\cdot\frac{1-r^{16}}{1-r^4}\cdot\frac{1-r^{32}}{1-r^8}\cdots\frac{1-r^{2^{n-1}}}{1-r^{2^{n-3}}}\cdot\frac{1-r^{2^n}}{1-r^{2^{n-2}}}\cdot\frac{1-r^{2^{n+1}}}{1-r^{2^{n-1}}}.$$

The result, manifestly, is

$$\frac{(1-r^{2^n})(1-r^{2^{n+1}})}{(1-r)(1-r^2)}.$$

Also solved by H. C. FEEMSTER, PAUL CAPRON, ELIJAH SWIFT, and HORACE OLSON.

478. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

Solve the equations, $l/x + y/m + z/n = 1$, $x/l + m/y + z/n = 1$, $x/l + y/m + n/z = 1$.

Solution by Henry D. Thompson, Princeton N. J.

Take X, Y, Z as symbols for $x/l, y/m, z/n$, respectively. Then the equations to be solved are:

$$(1)\ 1/X + Y + Z = 1, \qquad (2)\ X + 1/Y + Z = 1, \qquad (3)\ X + Y + 1/Z = 1.$$

The difference of (2) and (1) gives (4) $(X - Y) + (X - Y)/XY = 0$, one solution of which is (5) $X = Y$. This substituted in (1), (2), (3) leaves to be solved (6) $1/X + X + Z = 1$, (7) $2X + 1/Z = 1$, and (7) gives $Z = 1/(1 - 2X)$.
This in (6) gives the cubic

$$1/X + 1/(1 - 2X) = 1 - X, \qquad \text{or} \qquad (1 - X)/X(1 - 2X) = (1 - X),$$

the solutions of which are $X = 1$, and $X = 1/4 \pm \sqrt{-7}/4$. These with (5) and (7) give for $[x, y, z]$ the three solutions:

$$\text{(I, II)}\ [(1 \pm \sqrt{-7})l/4,\ (1 \pm \sqrt{-7})m/4,\ (1 \pm \sqrt{-7})n/4], \qquad \text{(III)}\ [l, m, -n].$$

The other solution of (4) viz.: (8) $1 + 1/XY = 0$ in (1) or (2) gives $Z = 1$, whence (3) becomes $X + Y = 0$, which with (8) gives $X^2 - 1 = 0$, and $X = 1$ with $Y = -1$, or $X = -1$ with $Y = 1$. Two other solutions are:

$$\text{(IV)}\ [l, -m, n], \qquad \text{(V)}\ [-l, m, n].$$

To find the infinite solutions, set X/W, Y/W, Z/W, respectively, for X, Y, Z in (1), (2), (3) and these equations rid of denominators, and with $W = 0$ are:

$$(9)\ X(Y + Z) = 0, \qquad (10)\ Y(Z + X) = 0, \qquad (11)\ Z(X + Y) = 0.$$

The equation (9) is satisfied when $X = 0$, which set in (10) or (11) gives either $Y = 0$ or $Z = 0$. The other solution of (9) viz.: (12) $Y = -Z$ set in (10) and (11) gives $Y(X - Y) = 0$ and $-Y(X + Y) = 0$; whence $Y = 0$, and from (12), $Z = 0$. Thus the infinite solutions (VI), (VII), (VIII) are the points at infinity on the axes when x, y, z are taken as coördinates in a rectilineal system.

Note that each of the three hyperbolic cylinders (1), (2), (3) has for its asymptotic planes a coördinate plane and a plane parallel to the like named axis.

Also solved by B. J. Dining, Gertrude I. McCain, J. L. Riley, E. F. Canady, S. W. Reaves, J. Q. McNatt, G. W. Hartwell, R. A. Johnson, C. H. Worthington, Paul Capron, Horace Olson, and G. Y. Sosnow.

479. Proposed by S. A. COREY, Albia, Iowa.

Prove or disprove

$$\left\{ \begin{vmatrix} x & -v & -z \\ -y & -z & v \\ -z & y & -x \end{vmatrix}^2 + \begin{vmatrix} y & -v & -z \\ x & -z & v \\ v & y & x \end{vmatrix}^2 + \begin{vmatrix} x & y & -z \\ -y & x & v \\ -z & v & -x \end{vmatrix}^2 + \begin{vmatrix} x & -v & y \\ -y & -z & x \\ -z & y & v \end{vmatrix}^2 \right\}$$

$$\div \begin{vmatrix} x & -y & -z & v \\ y & x & -v & -z \\ z & v & x & y \\ v & -z & y & -x \end{vmatrix}^2 = (x^2 + y^2 + z^2 + v^2)^{-1}.$$

Solution by Henry D. Thompson, Princeton, N. J.

Let Q be a symbol for $(x^2 + y^2 + z^2 + v^2)$, and represent the determinants in the order in which they appear in the equation by Z, V, X, Y, H, so that the equation is

$$\{Z^2 + V^2 + X^2 + Y^2\} \div H^2 = Q^{-1}.$$

But by (1), c_4 is the intersection of a_3m_{12}, a_1m_{23}, or the incenter of $a_1a_2a_3$. Therefore c_i as defined above is identical with the incenter of the triangle formed by omitting a_i, and the problem is proved.

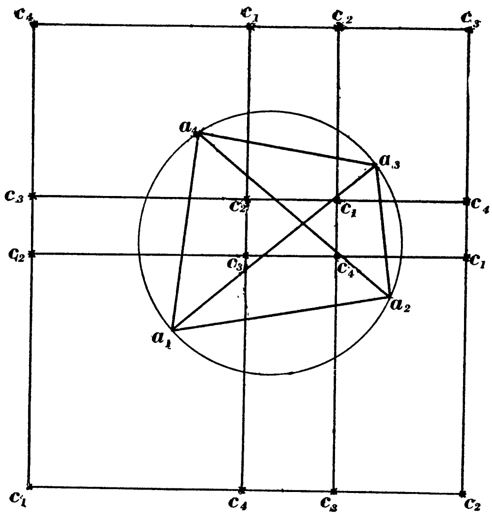


FIG. 2.

The extension suggested to be proved by analogous methods is shown in Fig. 2, where the four incenters and twelve excenters, of the four triangles formed by omitting a_i in turn, are the intersections of two sets of four perpendicular lines.

Also solved by O. J. RAMLER, R. A. JOHNSON, J. E. ROWE, J. W. CLAWSON, and PAUL CAPRON.

CALCULUS.

409. Proposed by B. J. BROWN, Victor, Colorado.

Integrate the differential equation,

$$\frac{\partial^2 z}{\partial x \partial y} + \frac{1}{x+y}\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) - \frac{2}{(x+y)^2} z = 0.$$

II. SOLUTION BY W. W. BEMAN, Ann Arbor, Michigan.

This problem is found in Gregory's *Examples* and Forsyth's *Treatise on Differential Equations.* Gregory puts

$$y + x = u, \qquad y - x = v$$

and makes use of the solution of a previous problem.

Maser in the German translation of Forsyth's *Treatise* puts

$$z = \frac{u}{(x+y)^2} \text{ and writes the new equation } (x+y)\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.$$

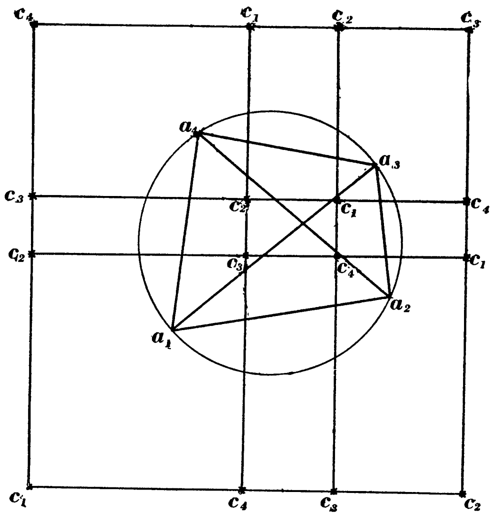
Differentiating with respect to x, he gets

$$(x+y)\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^2 u}{\partial x^2}, \text{ and from this } \frac{\partial^2 u}{\partial x^2} = (x+y)\phi'''(x).$$

Integrating twice with respect to x, he obtains u and finally z as in Mr. Adams's solution on p. 129 of the March, 1917, MONTHLY. It may be observed that in Mr. Adams's solution

$$w = \frac{\partial^2 u}{\partial x^2}.$$

But by (1), c_4 is the intersection of a_3m_{12}, a_1m_{23}, or the incenter of $a_1a_2a_3$. Therefore c_i as defined above is identical with the incenter of the triangle formed by omitting a_i, and the problem is proved.



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$$w = \frac{\partial^2 u}{\partial x^2}.$$

418. Proposed by B. F. FINKEL, Drury College.

A rectangular tract of land is to be bought for the purpose of laying out a quarter-mile track with straightaway sides and semicircular ends. In addition a strip 35 yards wide along each straightaway is to be bought for grandstands, training quarters, etc. If the land costs \$200 an acre, what will be the least possible cost of the land required?

Granville's *Differential and Integral Calculus*, p. 116.

Is there anything wrong with this problem? Explain the contradiction involved in the solution.

SOLUTION BY CLARIBEL KENDALL, University of Colorado.

For the cost of the land in this problem to be a minimum the area must be a minimum.

Let x = radius of ends in yards, y = length of straightaway sides in yards, and v = value of land.

Then $(2x + y)2x$ = area of the rectangular tract, and $70y$ = area of grandstands.

(1) $A = 70y + (2x + y)2x$, total area to be bought.

(2) $2\pi x + 2y = 440$ yds., length of the track, or $y = 220 - \pi x$, $y' = -\pi$, where $y' = dy/dx$.

(3) $A' = 70y' + 8x + 2xy' + 2y = 0$; or, substituting for y and y',

$$(4 - 2\pi)x = 35\pi - 220, \quad \text{and} \quad x = \frac{220 - 35\pi}{2\pi - 4} = \frac{385}{8} \text{ yds.} \left(\pi = \frac{22}{7}\right).$$

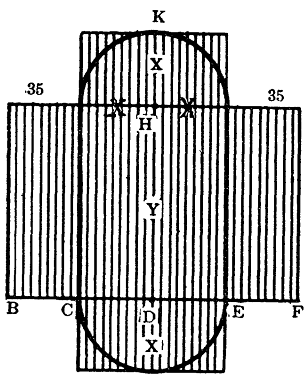

FIG. 1.

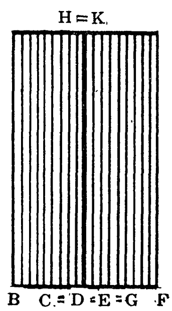

FIG. 2.

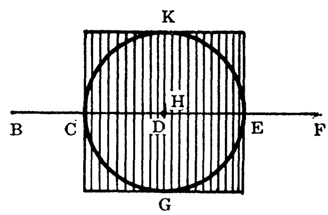

FIG. 3.

Whence

$$y = \frac{275}{4} \text{ yds.,} \quad \text{and} \quad A = 70y + 2x(2x + y) = \frac{824775}{4} \text{ sq. yds.} = 4.28 \text{ acres.}$$

At \$200 an acre, the cost v is \$856 as given by Granville.

From (3) above

$$A'' = 8 - 4\pi,$$

showing that A'' is a negative constant. This proves that A is a maximum, and that no minimum is even possible, under the customary definitions of maxima and minima.

We next inquire whether there may not be, after all, some sort of practical minimum, based on the exclusion of negative values for the end-radius x and for the straightaway side y.

From (2) above $\pi x + y = 220$. Plainly, x must lie between 0 and 70, while y must lie between 220 and 0.

When $x = 0$, $y = 220$; $A = 15400$, $v = \$636.36$. The land bought is now a rectangle. The semi-circular ends have degenerated into points and the track is a strip of zero width between the front edges of the grandstands.

When $x = 70$, $y = 0$; $A = 19600$, $v = \$809.92$, and the ground bought is now a square, the track is the inscribed circle, while the stands reduce to the 35-foot extensions of one of the two diameters of contact.

As x increases from 0 to 70, A and v increase up to the maximum and then decrease. The values for A and v are greater, however, for $x = 70$ than for $x = 0$. (See Fig. 4.) A practical

minimum for v may therefore be secured by taking x as small as possible to allow for an easy-running curve at the ends.

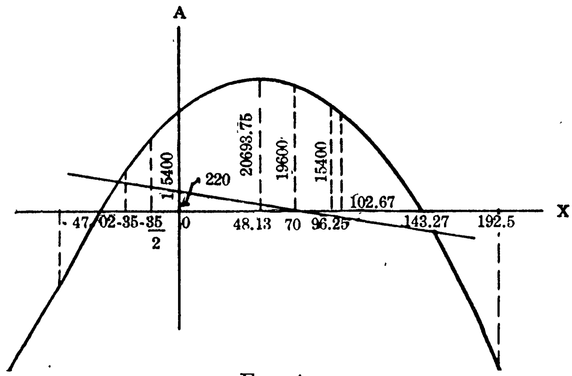

FIG. 4.

As far as has been discovered, there is no contradiction involved in the solution, but rather a misconception in the statement of the problem.

Aside from the question of determining a maximum or a minimum, this problem is curious when we try to represent the area for negative values of x and y. We will trace x from $-\infty$ to $+\infty$.

The following conventions will be adopted: Positive areas will be designated by vertical shadings; negative areas by horizontal shadings, positive areas counted twice by diagonal cross-bar shadings; positive and negative areas cancelling one another by vertical and horizontal shadings (negative areas counted twice do not occur). The positive part of the race track will be designated by a heavy solid line, the negative part by a heavy dotted line. In the following figures, as in Figs. 1, 2, 3, $BC(=EF=35)$ the width of the grandstands will be considered as essentially positive and these grandstands always face in opposite directions; $x = CD = DE$, with the usual conventions as to directed line segments, positive when measured to the right and negative when measured to the left; $y = DH$, positive when measured upward, negative when measured downward; $70 + 2x = BF$, $2x + y = GK$, with the same sign-conventions as above. One of the semi-circular ends of the track always goes from C to E and correspondingly for the other end.

The area of the tract of land may be represented in two ways:

$$A = (70 + 2x)y + 4x^2, \tag{1a}$$

$$A = 70y + (2x + y)2x. \tag{1b}$$

$4x^2$ represents the area of the two rectangles, including the semi-circular ends of the race-track; $(70 + 2x)y$ is the area of the rectangle having BF and DH as dimensions; $70y$ is the area of the two grandstands; $(2x + y)2x$ is the area of the rectangle having CE and GK as dimensions.

For $-\infty < x < 0$, $y > 0$ it is preferable to consider A in the form (1a). $4x^2$ is always positive and since $y > 0$,

$$(70 + 2x)y \gtreqless 0 \text{ according as } 70 + 2x \gtreqless 0, \text{ i. e., for } x \gtreqless -35. \tag{4}$$

For $0 \leqq x \leqq 70$, $y \geqq 0$ either form is applicable since all the above areas are then positive or zero.

For $70 < x < +\infty$, $y < 0$ the form (1b) is preferable. $70y$ is always negative, and since $x > 0$,

$$(2x + y)2x \gtreqless 0 \text{ according as } 2x + y \gtreqless 0, \text{ i. e., for } y \gtreqless -2x(= -385) \text{ or } x \gtreqless \tfrac{385}{2}(= 192.5), \tag{5}$$

obtained by substituting for y in equation (2).

In the following constant reference should be made to the graph in Fig. 4.

I. When $x < -47.02$ then $y > 0$ and $A < 0$. The tract of land is represented by Fig. **5**, $(70 + 2x)y < 0$ by (4) since $x < -35$. And since $A < 0$, $|(70 + 2x)y| > 4x^2$, *i. e.*, the area of the rectangle $BF \times DH$ is greater than the area of the rectangles including the semi-circular ends of the race track.

VI. When $x = 143.27$ then $y < 0$ and $A = 0$. Since $x < 192.5$, $(2x + y)2x > 0$ by (5). And since $A = 0$, $(2x + y)2x = |70y|$. Fig. 12 is again representative and the positive and negative areas just balance.

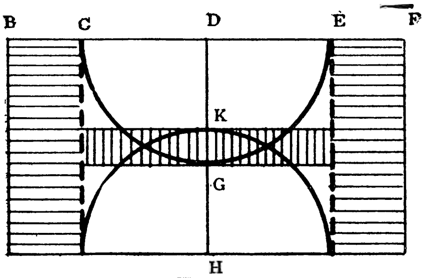

FIG. 12.
$102.67 < x < 192.5$, $y < -x$.

VII. When $x > 143.27$ then $y < 0$ and $A < 0$. On account of (5) we must consider three cases.

(*a*) When $143.27 < x < 192.5$, $(2x + y)2x > 0$ by (5). And since $A < 0$, $(2x + y)2x < |70y|$, *i. e.*, the area of the rectangle $GK \times CE$ is less than the area of the grandstands. Fig. 12 is again representative.

(*b*) When $x = 192.5$, $(2x + y)2x = 0$ by (5) and $A = 70y$, a negative area. Fig. 13 represents the tract.

(*c*) When $x > 192.5$, $(2x + y)2x < 0$ by (5). Now all of the areas included in A are negative and Fig. 14 is representative.

In V, VI, VII, the race-track is partly positive and partly negative. In equation (2) the $2\pi x$ is positive and the $2y$ is negative as is indicated in the figures by the solid and dotted parts of the track.

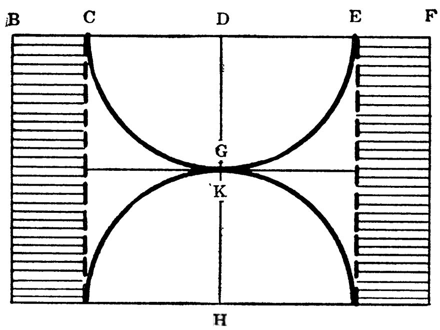

FIG. 13.
$x = 192.5$, $y = -2x$.

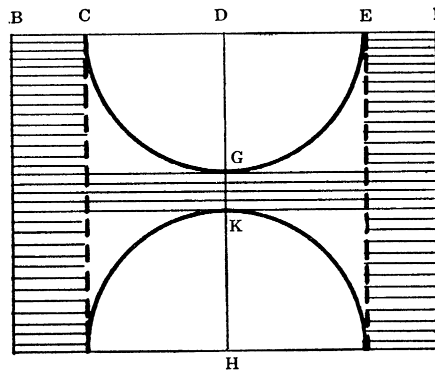

FIG. 14.
$192.5 < x + \infty$, $y < -2x$.

Also solved by HORACE OLSON, S. E. URNER, PAUL CAPRON, and J. W. BALDWIN.

MECHANICS.

336. Proposed by C. N. SCHMALL, New York City.

An inclined plane, length l, makes an angle $\phi < (\frac{1}{4}\pi)$ with the horizontal plane through its foot. From its foot, a body is projected upward along the plane, with a velocity equal to that of a falling body at the height h, so as to pass over the top and strike the horizontal plane at the maximum distance, x, from the foot of the inclined plane. Show by methods of the calculus that $x = h/(\sin\phi\cos\phi)$, and that the corresponding value of l is $(2h\cot 2\phi)/\cos\phi$.

SOLUTION BY THE PROPOSER.

Let v_1 = velocity at time of projection; v = velocity on reaching top of plane; a = height of inclined plane; x_1 = horizontal distance traversed by body after leaving plane.

Then, by the given conditions, we have

$$v_1^2 = 2gh, \tag{1}$$

$$a = l \sin \phi, \tag{2}$$

$$v^2 = 2g(h - a), \tag{3}$$

$$x = x_1 + a \cot \phi, \tag{4}$$

or,

$$x_1 = x - a \cot \phi. \tag{5}$$

Now taking the top of the inclined plane as the origin, the equation of the path of the body is

$$\left.\begin{aligned} -a &= x_1 \tan \phi - \frac{g}{2}\frac{x_1^2}{v^2 \cos^2 \phi} \\ &= x_1 \tan \phi - \frac{g}{2}\frac{x_1^2}{2g(h-a)\cos^2 \phi} \end{aligned}\right\} \tag{6}$$

by equation (3). Whence,

$$(h - a)(a + x_1 \tan \phi) = \frac{x_1^2}{4 \cos^2 \phi}$$

which, putting $x - a \cot \phi$ for x_1 from (5), becomes

$$(h - a)x \tan \phi = \frac{(x - a \cot \phi)^2}{4 \cos^2 \phi}.$$

Hence,

$$x^2 - 2x(a \cot \phi \cos 2\phi + h \sin 2\phi) + a^2 \cot^2 \phi = 0, \tag{7}$$

where x is to be made a maximum.

Differentiating with respect to a, we get

$$\{x - (a \cot \phi \cos 2\phi + h \sin 2\phi)\}\frac{dx}{da} - x \cot \phi \cos 2\phi + a \cot^2 \phi = 0. \tag{8}$$

Hence,

$$\frac{dx}{da} = 0, \qquad \text{if} \qquad x = a\frac{\cot \phi}{\cos 2\phi}.$$

Putting this value of x in (7) and reducing, we get

$$a = h\frac{\cos 2\phi}{\cos^2 \phi}. \tag{9}$$

Again, substituting this value of x in the coefficient of dx/da in (8), we get

$$a\frac{\cot \phi}{\cos 2\phi} - a \cot \phi \cos 2\phi - h \sin 2\phi,$$

which is nearly equal to $h \sin 2\phi$.

It is evident, therefore, that as a increases through the value $h(\cos 2\phi)/(\cos^2 \phi)$, the coefficient of dx/da remains about equal to $h \sin 2\phi$, a positive quantity, while the remainder of equation (8), namely, $-x \cot \phi \cos 2\phi + a \cot^2 \phi$, changes from $-$ to $+$, for x is constant and a is increasing. Hence, dx/da changes from $+$ to $-$. Hence x is a maximum when

$$x = a\frac{\cot \phi}{\cos 2\phi} = \frac{h}{\sin \phi \cos \phi}. \qquad \text{by (9)}$$

Also, from (2), we get

$$l = \frac{a}{\sin \phi} = 2h\frac{\cot 2\phi}{\cos \phi}. \qquad \text{by (9)}$$

Also solved by O. S. ADAMS and HORACE OLSON.

NUMBER THEORY.

255. Proposed by FRANK IRWIN, University of California.

Given any arithmetical progression whose first term a and common difference d are relatively prime integers, and any finite set of positive integers $m_1, m_2, \cdots$ also relatively prime to d, it is required to determine an integer n such that the multiples of $m_1, m_2, \cdots$ may occupy the same

positions in the series of natural numbers beginning with n as they do in the arithmetical progression. This is to say that if the kth, the $(m_1 + k)$th, the $(2m_1 + k)$th, $\cdots$ terms of the progression are divisible by m_1, so also will be the kth, the $(m_1 + k)$th, the $(2m_1 + k)$th $\cdots$ terms of the series $n, n + 1, n + 2, \cdots$, etc. Show that n may be determined as the solution of a congruence $An + B \equiv 0 \pmod{C}$ whose coefficients, A, B, are constants independent of the number and value of the m's.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

By the conditions of the problem the following equations hold: (1) $a + (k - 1)d = m_1\alpha$, (2) $n + k - 1 = m_1\beta$, where α and β are integers. Eliminating k, we obtain,

$$dn - a = m_1(\beta d - \alpha). \tag{3}$$

Consequently we see that n must be a solution of the congruence $dn - a \equiv 0$ (mod L.C.M. of $m_1, m_2, \cdots$). Conversely any solution of this congruence satisfies the given condition. For let n be such a solution. Then (3) is true. We are to show that if (1) holds, (2) will also: and conversely. But adding the equations (1) and (3) we see that $d(n + k - 1)$ is divisible by m_1, and since d is prime to m_1, $n + k - 1$ is divisible by m_1. Similarly (2) implies (1).

Also solved by C. F. GUMMER and HORACE OLSON.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence.

DISCUSSION.

RELATING TO A CONSTRUCTION FOR THE GRAPH OF A CUBIC.

BY PAUL CAPRON, U. S. Naval Academy.

If a cubic is given by the equation

$$\alpha y = x^3 + \beta x^2 + \gamma x + \delta, \qquad (\alpha > 0)$$

and the origin is shifted to (x_0, y_0), where

$$x_0 = -\frac{\beta}{3}, \qquad y_0 = \frac{1}{27\alpha}(2\beta^3 - 9\beta\gamma + 27\delta),$$

the new equation is

$$\alpha y = x^3 + (\gamma - \beta^2/3)x.$$

It is convenient to distinguish three cases, according to the value of $(\gamma - \beta^2/3)$.

Case (1). $\gamma - \beta^2/3 < 0$. Write $\alpha = a^2$, $\gamma - \beta^2/3 = -c^2$; then the equation is

$$a^2y = x^3 - c^2x.$$

$dy/dx = 1/a^2(3x^2 - c^2)$. At the inflection, I, $(0, 0)$, we have $dy/dx = -c^2/a^2 = 1/3\alpha(3\gamma - \beta^2)$, and $dy/dx = 0$ at $[\pm c/\sqrt{3}, \mp (2/3\sqrt{3})(c^3/a^2)]$.

At $(\pm c, 0)$, $dy/dx = 2(c^2/a^2)$, numerically twice the slope at I.

$x = a^2/k$, $y = (a^4/k^3) - (c^2/k)$ are parametric equations for the graph.

Hence the following construction for the graph in Case (1):

positions in the series of natural numbers beginning with n as they do in the arithmetical progression. This is to say that if the kth, the $(m_1 + k)$th, the $(2m_1 + k)$th, $\cdots$ terms of the progression are divisible by m_1, so also will be the kth, the $(m_1 + k)$th, the $(2m_1 + k)$th $\cdots$ terms of the series n, $n + 1$, $n + 2$, $\cdots$, etc. Show that n may be determined as the solution of a congruence $An + B \equiv 0 \pmod{C}$ whose coefficients, A, B, are constants independent of the number and value of the m's.

Solution by Elijah Swift, University of Vermont.

By the conditions of the problem the following equations hold: (1) $a + (k - 1)d = m_1\alpha$, (2) $n + k - 1 = m_1\beta$, where α and β are integers. Eliminating k, we obtain,

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Also solved by C. F. Gummer and Horace Olson.

QUESTIONS AND DISCUSSIONS.

Send all communications to U. G. Mitchell, University of Kansas, Lawrence.

DISCUSSION.

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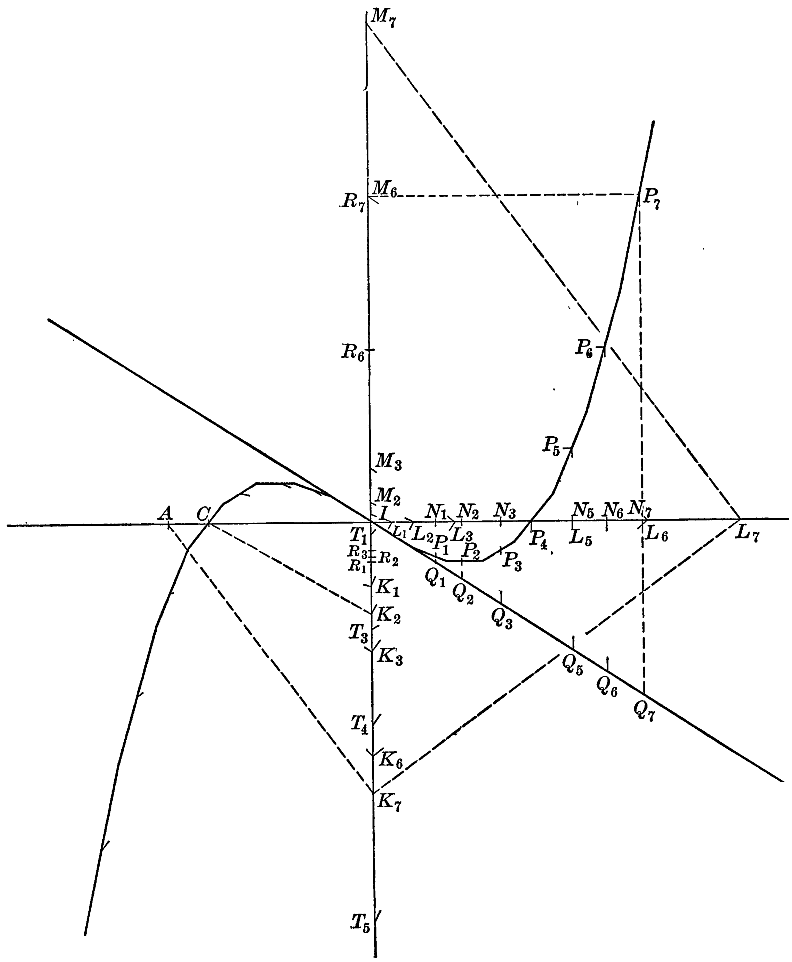
$x = a^2/k$, $y = (a^4/k^3) - (c^2/k)$ are parametric equations for the graph.

Hence the following construction for the graph in Case (1):

Locate the inflection I; through it draw AIL and KIM parallel to the axes of abscissas and ordinates respectively, and in the positive direction in each case. Through I draw the inflectional tangent IQ, giving it the slope $-(c^2/a^2) = (1/3\alpha)(3\gamma - \beta^2)$. Make $AI = a$. From A draw AK to any point K of KM; thence $KL \perp AK$; thence $LM \perp KL$. On IL lay off $IN = KI$ (by rotating IK contraclockwise through $\pi/2$); through N draw $PQ \perp IL$. On MI lay off $MR = NQ$ (and in the same direction); through R draw $RP \perp IM$, thus determining a point P. [This construction is made in the figure for the series of points with subscript 7, and the points are indicated (without actually drawing the lines) for those having the subscripts 1, 2, 3 and 6.]



The point P is a point of the graph. The proof rests upon the theorem that if a perpendicular p is drawn from the vertex of a right triangle to the hypotenuse, dividing the hypotenuse into segments m and n, then $mn = p^2$. For if AK' (not shown in the figure) is drawn perpendicular to AK to meet IM at K', and

The construction is shown for the tangents at P_1, P_3, P_4, P_5. T_2 coincides with R_2.

At diametrically opposite points, corresponding to equal and opposite values of k, the slope is the same; at two such points the tangents are parallel.

Graph for a Cubic Equation.—If the graph is to be used in connection with the solution of a cubic equation, the value of $\alpha = a^2$ is arbitrary, and may be so chosen as to separate or crowd the roots. The original axes may conveniently be located after the graph is drawn—with reference to the axes IL, IM, their intersection is at $(\beta/3, (1/27\alpha)(9\beta\gamma - 2\beta^3 - 27\delta))$. If it is convenient to make $a = c$, *i. e.*, $\alpha = \beta^2/3 - \gamma$, the construction will be somewhat simplified, as then $MR = IK$, and the point Q becomes superfluous.

The figure illustrates the construction for Case (1), showing seven points, with tangents, on each side of the inflection: the three special points P_2, P_4, P_5, a point in each of the segments IP_2 and P_2P_4, and two points beyond P_5.

In Case (2) or Case (3), fewer points would give an equally good guide for sketching the graph.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Ind.

Dr. W. H. WILSON has been appointed instructor in mathematics at the Massachusetts Institute of Technology.

Associate Professor W. B. FORD, of the University of Michigan, has been promoted to a full professorship of mathematics.

Miss PAULINE SPERRY, of Smith College, has accepted a position as instructor in mathematics at the University of California.

Assistant Professor L. A. RICE, of Syracuse University, has resigned to become instructor in mathematics at Tufts College.

Dr. IDA BARNEY, formerly instructor in Smith College, has been appointed professor of mathematics in Lake Erie College.

Professor W. A. GARRISON, of Union College, has been appointed professor of mathematics at King College, Bristol, Tennessee.

At the University of Nebraska, Miss L. RUNGE and Mr. A. BABBITT have been promoted to assistant professorships of mathematics.

Dr. A. W. HOBBS, of Johns Hopkins University, has been appointed instructor in mathematics at the University of North Carolina.

At the Southern Methodist University, Dallas, Texas, Associate Professor E. H. JONES has been promoted to a professorship of mathematics.

At the University of Toronto, Dr. SAMUEL BEATTY has been promoted to an assistant professorship of mathematics.

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Dr. R. A. JOHNSON, of Adelbert College, Western Reserve University, has been appointed professor of mathematics at Hamline University, St. Paul, Minn.

At McPherson College, Kansas, Professor A. B. FRIZELL has resigned from the department of mathematics.

At Northwestern University, Dr. E. J. MOULTON has been promoted to an associate professorship of mathematics; and Mr. F. L. KERR and Mr. A. D. CAMPBELL have been appointed to instructorships in mathematics.

Professor S. F. NORRIS, of the department of mathematics in Baltimore City College, and a charter member of the Association, died on September 4, 1917.

Dr. M. O. TRIPP has resigned the professorship of mathematics at Olivet College, Olivet, Mich., and has accepted an assistant professorship at the University of Maine.

Professor A. D. BUTTERFIELD, head of the department of mathematics at Worcester Polytechnic Institute, has resigned to accept a commission in the aviation service of the U. S. government.

Dr. C. H. YEATON, assistant professor of mathematics at Northwestern University, and Mr. G. R. MIRICK, assistant in mathematics at the University of Michigan, have entered the national service, the former in connection with the Signal Corps at St. Louis, Mo.

At Louisiana State University, Assistant Professor S. T. SANDERS has been promoted to the headship of the department of mathematics, and Dr. I. C. NICHOLS has been appointed associate professor in the department.

Associate Professor R. P. BAKER has been made acting head of the department of mathematics at the University of Iowa, and Mr. R. E. GLEASON has been made instructor in mathematics.

Cornell University has granted leave of absence for the year 1917–18 to Dr. L. L. SILVERMAN, instructor in mathematics, for service upon the committee of public safety of the state of Massachusetts.

Mr. M. HEDLUND, for two years instructor in mathematics at Beloit College, was commissioned a second lieutenant in the Adjutant General's office at the close of the first officers' training school at Ft. Sheridan.

Professor H. E. HAWKES, of the department of mathematics of Columbia University, will be acting dean of Columbia College during the absence of Dean KEPPEL who is serving as confidential secretary to the Secretary of War.

At the University of Michigan, Mr. HERMAN BETZ, of Cornell University, has been appointed instructor in mathematics.

Mr. C. H. CLEVENGER, instructor in mechanics and mathematics at the University of Minnesota, has resigned to accept a position as assistant in an

investigation of stresses in chilled cast iron car wheels being conducted at the University of Illinois.

Dr. G. W. Smith, formerly instructor in mathematics at the University of Colorado, has been appointed instructor in mathematics at Beloit College, Beloit, Wisconsin. Dr. Smith did his graduate work at the University of Illinois, where he received the doctorate in June of this year.

The De Morgan medal, awarded every third year by the London Mathematical Society for eminent work in mathematics, has been awarded during the past ten years as follows: to Professor Glazier (1908), to Professor Lamb (1911), to Professor Larmor (1914) and to Professor W. H. Young (1917).

Professor J. F. Millis, who for a number of years has been head of the department of mathematics in the Parker School of Chicago, died at his home in Chicago on October 25, 1917, at the age of forty-two years. Professor Millis was the joint author of the well-known Stone-Millis series of elementary mathematical text-books.

Two charter members of the Association, Mr. O. S. Adams, and Mr. W. D. Lambert, of the Coast and Geodetic Survey, have been transferred to the War Department as first lieutenants and have entered the active service. Mr. P. C. Porter, also a charter member, is enrolled in the army aviation ground-school at the University of Texas.

At the College for Women, Western Reserve University, Professor Anna H. Palmié has been granted leave of absence, her place being supplied by Dr. Mary F. Curtis as instructor in mathematics. Mrs. W. E. Beckwith has been promoted to an assistant professorship of mathematics.

During the summer Professor W. R. Ransom, of Tufts College, had charge of Navigation Schools for the U. S. Shipping Board at Philadelphia. Professor D. T. Wilson, of Case School of Applied Science, gave instruction in the elements of navigation to men who are candidates for the merchant marine, and who have only elementary mathematical knowledge.

The feature of the opening number of Vol. XXIV of the *Bulletin* of the American Mathematical Society is a forty-six page paper by Professor G. A. Bliss, of the University of Chicago, on the "Integrals of Lebesgue," giving in detail the substance of Professor Bliss's symposium lecture presented at the April meeting of the Chicago Section of the Society.

At Purdue University, Professor T. G. Alford and Professor Jacob Westlund have retired from active service, the former on account of age, the latter on account of disability. Each has been granted a retiring allowance. Associate Professor William Marshall has been promoted to a full professorship of mathematics, and instructor Glenn James has been promoted to an assistant professorship.

Dr. Ruth E. Gentry, formerly instructor in mathematics at Vassar College and a member of the American Mathematical Society, died at Indianapolis, Indiana, on October 15, 1917. Miss Gentry was a graduate of the University of Michigan, received her doctorate at Byrn Mawr College, and as holder of a scholarship from the Association of Collegiate Alumnae she studied in Berlin and Paris.

The "*Guide to the Study and Use of Reference Books,*" recently issued by the American Library Association, lists the following mathematical works: Encyclopédie des Sciences Mathématiques; Historical Introduction to Mathematical Literature, by G. A. Miller; Mathematisches Vokabularium, by Felix Müller; Smithsonian Mathematical Tables; and Memorabilia Mathematica, by R. E. Moritz.

Dr. J. D. Maddrill, of the Travelers Insurance Company, and former director of the International Geodetic Observatory at Ukiah, Calif., and instructor in insurance mathematics at the University of California, has become actuary of the bureau of efficiency and economy at Washington. Among Dr. Maddrill's duties at Washington will be the preparation of a plan for pensioning the 300,000 civil employees of the government.

The Paris Academy of Sciences has awarded the following prizes in mechanics and mathematics: The Bordin prize of 3,000 francs to M. Gaston Julia, now an officer in the French army; the Francoeur prize of 1,000 francs to M. Henri Villat, for his publications on hydrodynamics; the Montyon prize of 700 francs to M. René de Sausseure, of Geneva, for his work in mechanics; and the Poncelet prize of 200 francs to M. Jules Andrade, professor at Besançon, for work in applied mechanics.

"A Sixth List of Writings on Determinants" by Sir Thomas Muir has appeared in Vol. XLVII, 1917, of *The Quarterly Journal of Pure and Applied Mathematics*. The five lists preceding the present one, also appearing in *The Quarterly,* cover the periods to 1880, 1885, 1900, 1905, 1910, respectively, and the sixth list covers the period from 1910–1915. The sixth list consists of titles and authors, date and place of publication, of four hundred twenty-four different publications.

At the University of Illinois, Dr. E. F. Simonds, of Columbia University, has been appointed instructor in mathematics; Mr. A. W. Larsen has been appointed assistant in mathematics; and instructors Dr. C. M. Hebbert and Dr. F. W. Reed are teaching in the school of aviation. Dr. L. M. Kells, instructor in mathematics, has entered the National Army at Camp Dodge.

At the University of Maine, Assistant Professor T. L. Hamlin has resigned to accept a position in the department of mathematics at Union College, Schenectady, N. Y., and Mr. R. Woods, instructor in mathematics, has resigned to do graduate work in the University of Illinois.

Baron KIKUCHI, the noted Japanese mathematician and scientist, died at his home in Japan on August 19, 1917, at the age of sixty-three years. His university training was obtained in England, where he was sent by the Japanese government. He entered Cambridge at the age of fifteen and was graduated among the wranglers at the age of twenty-two. Returning to Japan, he was immediately appointed professor of mathematics in the Imperial University, a position which he occupied for twelve years. Later he was dean of the College of Science, president of the Tokyo Imperial University, director of the Peer's School, president of the Kyoto Imperial University, and in 1901 he was appointed minister of education. In 1909–1910 Baron KIKUCHI visited England and the United States, giving lectures on Japanese education.

M. PAUL PAINLEVÉ, formerly professor of mathematics at the University of Paris and professor of mechanics at the Paris Polytechnic School, was chosen premier of the French Republic and served in that capacity until the recent overthrow of the French cabinet. While he was premier, the New York Evening Post made the following comment: "France has three of her most important posts in government and army filled by men especially renowned in mathematical proficiency. The Premier, M. PAINLEVÉ, knew enough at eleven and a half to have entitled him to his bachelor's degree, and later he attracted the attention of the mathematicians of Paris and was appointed to the professorships mentioned above. The commander-in-chief of the army, General PÉTAIN, now directing the gigantic struggle of the French armies, is a fine mathematician. And there is M. LOUCHEUR, the new minister of armaments, who gained especial mathematical distinction while a student at the Ecole Polytechnique.

The seventeenth annual meeting of the Central Association of Science and Mathematics Teachers will be held at Columbus, Ohio, on November 30 and December 1, 1917. At the opening general session Dr. W. O. THOMPSON, President of Ohio State University, will deliver an address on "Immediate and ultimate aims of science and mathematics teaching." At the mathematics section, Professor HARRIS HANCOCK, of the University of Cincinnati, will speak on "What course of study should be taken by a boy or girl in High School;" Mr. S. A. COURTIS, Supervisor of Educational Research, Detroit, Michigan, will speak on "Measurement of the products of teaching high school mathematics;" and Mr. J. A. FOBERG, of the Crane Junior College, Chicago, will present a report as chairman of the section committee on "mathematical requirements." Mr. Foberg is the representative of the Central Association delegated to act as a member of the National Committee on Mathematical Requirements now working under the auspices of the Mathematical Association of America.

The one hundred and ninety-third regular meeting of the American Mathematical Society was held in New York on October 27, 1917. Eleven papers appeared on the printed program. The eleventh regular meeting of the Southwestern Section will be held at the University of Oklahoma on December 1, 1917.

The seventieth meeting of the American Association for the Advancement of Science will be held at Pittsburgh during the week from December 28, 1917, to January 2, 1918. The chairman of Section A, Mathematics and Astronomy, is Professor H. N. Russell of Princeton University, and the secretary is Professor F. R. Moulton of the University of Chicago.

The following extract of a letter from Professor Paul Montel of Paris to Professor Hedrick will be of interest to many of our readers:

"I received your letter of May 18th, and I want to thank you cordially for your kind expressions regarding my country. Our confidence was strengthened both morally and materially by the entry of the United States into the war, but no one here had any idea at first that this support would assume the formidable character which it now seems likely to take. I can assure you that the powerful effort which your country is now making fills us all with admiration. Yesterday I met the members of the scientific mission which has just returned from the United States: Fabry, Abraham, etc. They are all elated over the warmth and the sympathy of the reception which they received.

"I desire to tell you that the Council of the French Mathematical Society has decided to accept articles written in either French, English or Italian. Hence whenever you send us an article, it will not be necessary to have it translated, unless you yourself desire."

NOTES ON THE ASSOCIATION.

The third annual meeting of the Mathematical Association of America will be held at the University of Chicago on Thursday and Friday, December 27–28, 1917, immediately preceding and in conjunction with the Winter meeting of the Chicago Section of the American Mathematical Society. There will be a joint session of the two organizations on Friday at which Professor W. B. Ford, of the University of Michigan, will deliver his retiring address as chairman of the Chicago Section on the subject: "A conspectus of the modern theory of divergent series." There will be a joint dinner on Thursday evening. The dinner was previously announced for Friday evening but was changed to Thursday in order to avoid conflict with the dinner of the Association of University Professors which occurs on Friday evening and which many of the mathematicians will desire to attend.

Arrangements for these meetings are in the hands of a joint committee consisting of W. D. Cairns of Oberlin College, Arnold Dresden of the University of Wisconsin, W. B. Ford of the University of Michigan, H. L. Rietz of the University of Illinois, J. A. Foberg of the Crane Junior College, Chicago, D. F. Campbell of Armour Institute, H. E. Cobb of Lewis Institute, E. J. Moulton of Northwestern University, and H. E. Slaught, chairman, of the University of Chicago.

The program of the Association meetings is in charge of a committee of three: Elizabeth B. Cowley of Vassar College, O. D. Kellogg of the University of Missouri, and E. J. Wilczynski, chairman, of the University of Chicago. Heretofore all places on the programs of the Association meetings have been filled by direct invitation of the Program Committee, but on this occasion it is proposed to devote one session to papers submitted by members on their own initiative. Abstracts of papers intended for this meeting should be submitted in form for publication to the Program Committee for approval and should be in the hands of the Chairman on or before Saturday, December 1, 1917. Some portion of the program will also be devoted to questions of interest to institutions, and all institutional members are invited to send delegates.

Programs and full announcements of arrangements for this meeting will be mailed to all members early in December.

In accordance with the action of the Council at Cleveland, President Cajori has appointed a committee of five to consider the feasibility of publishing a mathematical dictionary. The committee consists of R. C. Archibald of Brown University, H. L. Rietz of the University of Illinois, D. E. Smith of Columbia University, H. E. Slaught of the University of Chicago, and E. R. Hedrick, chairman, of the University of Missouri. The committee is expected to make at least a preliminary report at the annual meeting in Chicago.

Following up the suggestion of Professor Archibald in discussing undergraduate clubs at the Cleveland meeting (see October MONTHLY, page 357) a decision has been reached to form a special department of the MONTHLY to be devoted to undergraduate clubs. In the next January issue we should like to publish as complete a list of clubs as possible together with statistics as to membership and meetings. Will our readers, then, furnish us with such information as they have relative to existing clubs and their programs and membership for both this year and last year. Secretaries of clubs are particularly requested to furnish the desired information. Communications relative to this matter may be sent to Professor R. C. Archibald, Brown University, Providence, R. I.

President Cajori has announced that he proposes to adhere to the principle declared by Professor Hedrick last year, namely, that it is deemed to be for the best interests of the Association that the president should not be renominated for a second consecutive term. This announcement has already been made known to the members in connection with the preliminary ballot sent out about November first.

The Council has authorized the submission of certain amendments to the Constitution and By-Laws of the Association. These will be voted on at the annual meeting in Chicago, and in accordance with the constitution they are published herewith for comment or suggestion by members:

Article III, Section 2, to be modified to read as follows:

The President and Vice-Presidents shall be elected annually for a term of one year, and four members of the Council shall be elected annually for a term of three years. They shall be eligible for reëlection, but not for more than two consecutive terms. The Secretary-Treasurer, and the Committee on Publications, consisting of the Manager, the Editor, and one other member, shall be appointed by the Council.

Article VI, Section 2, to be modified to read as follows:

The Council shall have full control of the publication and sale of the official journal.

By-Law 1, last sentence, to be modified to read:

Those who were admitted to membership before April 1, 1916, constitute the list of charter members.

By-Law 3, first sentence, to be replaced by the following:

Committees. The official journal shall be under the general management of the Committee on Publications. There shall also be appointed by the Council a Board of Associate Editors who shall give assistance in connection with the official journal under the direction of the Committee on Publications.

The above modifications are substantially as passed upon by the Council, except in the case of By-Law 1, which involves merely a change in tense of the verbs so as to conform to the present date.

The important changes in Article III of the Constitution pertain (1) to the Committee on Publications and (2) to the manner of choosing the Secretary-Treasurer. The reason for the first change is that the burdens of editorial work need to be further divided and distributed, and the reason for the second change is that the strictly business affairs of the Association need to be in the hands of an officer selected by the Council rather than elected by popular vote. All the other changes are consequences of these two.

The attention of institutional members is especially called to the report of the Committee on Libraries published in the October MONTHLY. A great amount of work has been expended by the committee in compiling this classified list and it is hoped that it may prove to be of real use, not only to institutional members, but also to all institutions of collegiate rank in the country, both as a means of checking the books now in their libraries and also as a stimulus to securing the books of this list not already possessed. In particular, it is hoped that this report may prove a real benefit to the mathematical departments of these institutions. The chairman invites correspondence on any matters pertaining to the suggestions contained in this report, and it is likely that a place may be provided on the program of the meeting for further discussion of it.

Townsend's Functions of a Complex Variable

By E. J. TOWNSEND, Professor and Head of the Department of Mathematics in the University of Illinois. (AMERICAN MATHEMATICAL SERIES.) vii + 384 pp. 8vo. $4.00.

This book is based upon a course of lectures given by the author for a number of years at the University of Illinois. It is intended as an introduction to the subject, suitable for seniors and first year graduate students. It presupposes a knowledge of calculus such as is usually given in the undergraduate course. Such topics from real variables as are not ordinarily included in an elementary course in calculus, and made use of in the development of the theory of functions of a complex variable are discussed in the text.

Both the Cauchy-Riemann and the Weierstrass points of view are made use of in developing the general properties of analytic functions. In a chapter following the discussion of the general properties of single-valued functions, the author shows how those properties may be extended to the consideration of multiple-valued functions by the aid of Riemann surfaces.

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"On the whole the book is well coordinated with our under-graduate courses and covers just about the ground in function theory which the first year graduate student of mathematics should get well in hand."—H. B. PHILLIPS, in the Bulletin of the American Mathematical Society.

"A book as carefully and well written as the present volume, and which deals with such a fundamental and important field in mathematics, will undoubtedly be well received by both teachers and students."—HANS H. DALAKER, in the American Mathematical Monthly.

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ERRATA NOTED IN VOLUME XXIV.

Page 20, line 17 up, for "$x - y$" read "x-y."
Page 30, line 4 up, for "Pitisi" read "Pitisci."
Page 39, line 12 up, for "W. H. Thome" read "W. J. Thome."
Page 44, line 17 up, for "Montana" read "Minnesota"; for "F. L." read "H. L."
Line 12 up, for "H. W. Myers" read "H. S. Myers."
Line 11 up, for "H. W. Roever" read "W. H. Roever."
Page 70, line 6 down, for "Suma" read "Sūma."
Page 71, line 19 up, for "l_1" read "λ."
Page 85, last line, for "Hosley" read "Lasley."
Page 88, line 8 up, for "equation becomes" read "equations become."
Page 94, line 11 up, add "except as first noted."
Page 125, line 16 down, for "M. T. Reed," read "M. T. Peed."
Line 17 down, for "Whiteford" read "Whitford."
Page 132, lines 3 down and 4 up, for "Whiteford" read "Whitford."
Page 139, lines 4 and 5 down, for "Glazier" read "Glaisher."
Page 176, line 9 up, for "2 A D" read "2 A B."
Page 307, line 10 up, for "Left" read "Gauche."
Page 328, line 5 down, for "$^n2$" read "2^n."
Page 332, line 6 down, for "Yenn" read "Yen."
Line 4 down, for "Woods" read "Wood."
Page 336, line 8 down, for "Sir Arthur" read "Arthur."
Page 428, line 14 up, for "C. H." read "E. H."
Page 441, line 9 down, for "Professor Glazier" read "Dr. J. W. L. Glaisher."

VOLUME XXIV DECEMBER, 1917 NUMBER 10

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EDITORIAL CORRESPONDENCE should be addressed to the Managing Editor, H. E. Slaught, 5548 Kenwood Avenue, Chicago, Ill.

BUSINESS CORRESPONDENCE should be addressed to the Secretary-Treasurer of the Association, W. D. Cairns, 27 King Street, Oberlin, Ohio.

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THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME XXIV DECEMBER, 1917 NUMBER 10

CAVALIERI'S THEOREM IN HIS OWN WORDS.

By G. W. EVANS, Boston, Mass.

Introduction. Cavalieri's Theorem has recently come into some use in the teaching of elementary geometry as a means of unifying the ideas that lie at the basis of the mensuration of solids. The substance of the theorem is as follows:

If two solids have equivalent bases, and if sections parallel to the bases and equally distant from them in the two solids are also equivalent, then the solids are equivalent.

It is natural to think of the two solids referred to as being generated by two surfaces constantly equivalent to each other and constantly parallel to the respective bases, and we have an instinctive willingness to accept the statement that the two solids, so generated, are equivalent.

It is possible, however, to prove the theorem with as much rigor as is looked for in any demonstration involving the theory of limits; and it would be desirable, if it were possible, to prove this theorem at the beginning of the mensuration work of solid geometry. We then should be able to do away with the somewhat vexatious theorem about the equivalence of two parallelepipeds having equivalent bases and equal altitudes; and also to do away with the theorem about the equivalence of two pyramids having equivalent bases and equal altitudes. Moreover, the proof of the volume of a cylinder, or even of a cone, could be made, by means of Cavalieri's theorem, to depend directly upon the theorems in plane geometry about the area of a circle.

The more one examines the objects attainable by the use of this theorem as a basic proposition of mensuration, the more attractive it looks, especially in the matter of diminishing recourse to the theory of limits. On that account, the proof given by Cavalieri himself nearly three hundred years ago will doubtless be of interest to students and teachers of solid geometry. The diagram is a substantially exact copy of Cavalieri's own drawing, but the lettering has been

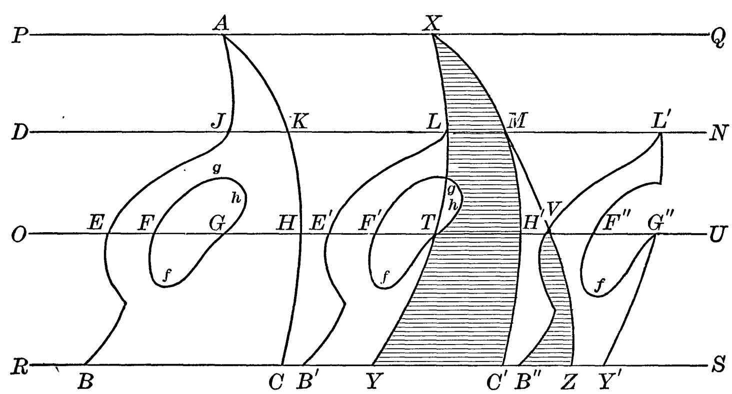
changed to take advantage of the modern habit of lettering a diagram so as to give some clue to the relations of the figure. The translation is intended to give, as faithfully as possible, the verbal meaning of the Latin, while not, of course, following the prolix idiom of the time in all its ramifications.

The book from which the following is taken is *Geometria Indivisibilibus Continuorum Nova quadam ratione promota. Authore P. Bonaventura Cavalerio.* . . . Bononiæ MDCLIII.

The Theorem.[1] *If between the same parallels any two plane figures are constructed, and if in them, any straight lines being drawn equidistant from the parallels, the included portions of any one of these lines are equal, the plane figures are also equal to one another; and, if between the same parallel planes any solid figures are constructed, and if in them, any planes being drawn equidistant from the parallel planes, the included plane figures out of any one of the planes so drawn are equal, the solid figures are likewise equal to one another.*

The figures so compared let us call analogues,[2] *the solid as well as the plane. . . .*

The Proof. Let any two plane figures ABC and XYZ be constructed be-



tween the same parallels PQ, RS; and let DN, OU, be drawn parallel to the aforesaid PQ, RS; and let the portions, *e. g.*, of DN, included in the figures, namely JK, LM, be equal to each other; and again, in the line OU, let the portions EF, GH, taken together (for the figure ABC, *e. g.*, may be hollow within, according to the contour of FgG), be likewise equal to TV; and let this happen in all the other lines equidistant from PQ. I say that the figures, ABC, XYZ, are equal to each other.

---

[1] FIGURÆ PLANÆ QUÆCUMQUE IN EISDEM PARALLELIS CONSTITUTÆ IN QUIBUS, DUCTIS QUIBUSCUMQUE EISDEM PARALLELIS ÆQUIDISTANTIBUS RECTIS LINEIS, CONCEPTÆ CUIUSCUMQUE RECTÆ LINEÆ PORTIONES SUNT ÆQUALES, ETIAM INTER SE ÆQUALES ERUNT; ET FIGURÆ SOLIDÆ QUÆCUMQUE IN EISDEM PLANIS PARALLELIS CONSTITUTÆ, IN QUIBUS, DUCTIS QUIBUSCUMQUÆ PLANIS EISDEM PLANIS PARALLELIS ÆQUIDISTANTIBUS, CONCEPTÆ CUIUSCUMQUE SIC DUCTI PLANI IN IPSIS SOLIDIS FIGURÆ PLANÆ SUNT ÆQUALES, PARITER INTER SE ÆQUALES ERUNT. . . . Op. cit., p. 484.

[2] Æqualiter analogæ.

Let either, then, of the two figures ABC, XYZ be taken, for example ABC itself, with the portions of the parallels PQ, RS coterminous with it, namely the portions PA, RB, and let it be superposed upon the other figure XYZ, but so that the lines PA, RB may fall upon AQ, CS; then either the whole figure ABC coincides with the whole figure XYZ (and thus, since they coincide with each other they are equal), or not; yet let there be some part which will coincide with some part, as $XMC'YThL$, part of the figure ABC, with $XMC'YThL$, part of the figure XYZ.

It is manifest, moreover, if the superposition of the figures is effected in such a way that portions of the parallels PQ, RS coterminous with our two figures are mutually superposed, that whatever straight lines (included in the figures) are in line, remain in line; as, for example, since EF, GH are in line with TV, when the aforesaid superposition is made they will remain in line (namely $E'F'TH'$ in line with TV), for the distance of those lines EF, GH from PQ is equal to the distance of TV from PQ; whence, no matter how many times PA is placed over AQ, at any place, EF, GH will always remain in line with TV, which is clearly apparent not only for this but for all other lines parallel to PQ in either figure.

In the case where part of one figure (as ABC) coincides of necessity with part of the figure XYZ, and not with the whole, granting that the superposition be made by such a rule as has been told, the demonstration will be as follows. For since when any parallels are drawn to PQ, the portions of them, included in the figures, which were in line, will still remain in line after superposition, and moreover since they were by hypothesis equal before superposition, therefore, after superposition the portions included in the figures will likewise be equal—as, *e. g.*, $E'F'$, TH' taken together will be equal to TV—therefore, if $E'F'$, TH' do not coincide with the whole of TV, then, one part [of one] coinciding with some part [of the other], as TH' with TH' itself, $E'F'$ will be equal to $H'V$, $E'H'$ being in the residuum of the figure ABC which is superposed, and $H'V$ in the figure XYZ upon which the other is superposed. In the same way we shall show that to any line whatever parallel to PQ, and included in the residuum of the superposed figure ABC (which may be $LB'YTF'$) corresponds an equal straight line, in line [with the former], which will be in the residuum of the figure XYZ on which ABC is superposed; therefore, the superposition being made by this rule, when anything of the superposed figure is left over and does not fall upon the figure, it must be that something of the other figure must also be left over, and have nothing superposed upon it.

Since, moreover, to each of the straight lines parallel to PQ and included in the residuum (or residua, for there may be several residual figures) of the superposed figure ABC (or $XB'C'$) there corresponds another straight line, in line [with the first] and included in the residuum (or residua) of the figure XYZ, it is manifest that these residual figures, or their aggregates, are between the same parallels; so since the residual figure $LB'YTF'$ is between the parallels DN, RS, likewise the residual figure (or aggregate of residual figures) of the figure XYZ

(because it has the frusta Thg, $MC'Z$) will be between the same parallels DN, RS. For if it did not extend both ways to the parallels DN, RS, as for example if it extended up to DN, but not down to RS, only as far as OU, then to the straight lines included in the frustum $E'B'YfF'$, and parallel to PQ, there would not be found in the residuum of the figure XYZ (or in the aggregate of the residua) other corresponding lines as has been proved to be unavoidable. Therefore these residua, or their aggregates, are between the same parallels; and the portions of the lines parallel to PQ, RS, included therein, are equal, as we have shown above; therefore the residua are subject to the same condition as has been assumed for ABC, XYZ; that is, they are analogues.

So let the residua be now superposed, but so that the parallels KL, CY may fall upon the parallels LN, YS, and the part $VB''Z$ of the frustum $LB'YTF'$ may coincide with the part $VB''Z$ of the frustum $MC'Z$; then we shall show, as above, that as long as there is found a residuum of one, there will be found also a residuum of the other, and these residua, or aggregates of residua, will be found within the same parallels. Let $L'VZY'G''F''$ be a residuum belonging to the figure ABC; and let $MC'B''V$, Thg, be residua belonging to the figure XYZ, whose aggregate is between the same parallels as the residuum $L'VZY'G''F''$, that is, between DN, RS. If now we superpose these residua again, but so that the parallels between which they lie be always superposed respectively, and this is supposed to be done continually, until the whole figure ABC shall have been superposed, I say the whole of it must coincide with XYZ; otherwise if there were any residuum of the figure XYZ, upon which nothing is superposed, there would be also some residuum of the figure ABC which would not have been superposed, as we have shown above to be unavoidable; but it is granted that the whole of ABC is superposed upon XYZ, therefore they are so superposed upon each other that there are no residua of either, therefore they are so superposed that they coincide, therefore the figures ABC, XYZ are equal to each other.

Now in the same diagram let ABC, XYZ be any two solid figures constructed between the same parallel planes PQ, RS; and let DN, OU be any planes drawn equidistant from the planes previously spoken of; and let the figures that lie in the same plane and that are included in the solids be equal to each other always; as JK equal to LM, and EF, GH, taken together (for a solid figure, for example ABC, may be hollow in any way within, according to the surface $FfGg$), equal to TV. I say that these solid figures are equal to each other.

For if we superpose the solid ABC, with the portions PA, RC of the planes PQ, RS, coterminous with it, upon the solid XYZ, in such a way that the plane PA be on the plane PQ, and the plane RC on the plane RS, we shall show (as we did above about the portions of the lines parallel to PQ included in the plane figures ABC, XYZ) that the figures included in the solids and lying in the same plane will also after superposition remain in the same plane; and therefore thus far the figures included in the superposed solids are equal,—and parallel to PQ, RS.

Then unless the entire solid coincides with the other solid entire in the first

superposition, residual solids will remain, or solids composed of residua, in either solid, which will not be superposed upon each other. Because since for example the figures $E'F'$, TH' are equal to the figure TV, then when the common figure TH' is taken away, the remaining figure $E'F'$ will be equal to the remaining figure $H'V$; and this will happen in any plane whatever parallel to PQ and meeting the solids ABC, XYZ. Therefore whenever we have a residuum of one solid, we shall always have a residuum of the other also; and it will be evident, according to the method applied in the former part of this Proposition in the case of plane figures, that the residua of the solids, or the aggregates of residua, will always be between the same parallel planes (as the residua $LB'YTF'$, $MC'Z$, Thg are between the same parallels DN, RS) and will be analogues.

Now if these residua be superposed again, so that the plane DL will be placed on the plane LN, and RY on YS, and this is understood to be done continually, until ABC, which is being superposed, is entirely taken, the entire solid ABC will finally coincide with the entire solid XYZ. For when the entire solid ABC is superposed upon XYZ, unless they coincided there would be some residuum of one, as of the solid XYZ, therefore also some residuum of the solid $XB'C'$, or ABC, and this residuum would not be superposed; which is absurd, for it is already assumed that the entire solid ABC is superposed on XYZ. Therefore there will not be any residuum in these solids; therefore they will coincide; therefore the solid figures spoken of, ABC, XYZ, will be equal to each other, which was to be proved of them.

MATHEMATICAL FORMS OF CERTAIN ERODED MOUNTAIN SIDES.

By T. M. PUTNAM, University of California.

In some of the desert regions of the west, visited by occasional heavy rainstorms, the formation is such that erosion takes place in a way that makes it possible to calculate approximately the form of the curve of intersection of the eroded slope and the alluvial fan of the plain.

This problem has been suggested by Professor A. C. Lawson of the department of geology of the University of California. He has formulated the hypotheses used below and the forms of the curves here obtained have been closely approximated by the actual geological conditions observed by him.

In the figure $S_0OR_0T_0$ is the contour of valley, hillside and plateau at some initial time. The table land R_0T_0 may be taken as inclined at a small angle C to the horizontal direction T_0Y. The valley OS_0 is assumed to be inclined at an angle B. The eroded material OR_0RH comes off in layers and is deposited in the valley in the position S_0SHO. The slope of erosion as well as the floor of the valley maintains, in the observed regions, a fairly constant inclination. Both inclinations are assumed constant in these calculations. The line S_0S, at the low

condition there is for every set of angles A, B, C one ratio $a : b$ for which the locus is a straight line.

The tangent to the curve at O is parallel to the line S_0R_0 and at any point H it is parallel to the line SR. The curve is concave downward

$$\text{if } \frac{a}{b} < \frac{1 + \sqrt{1 + PQ}}{P}, \qquad \text{upward if } \frac{a}{b} > \frac{1 + \sqrt{1 + PQ}}{P}.$$

The special case $B = C = 0$ is of interest. The equation now becomes $\cos Ay^2 + 2xy - 2bx + 2ay = 0$. Using rectangular coördinates, u and v, with the old origin and X-axis,

$$u = x + y \cos A, \qquad v = y \sin A,$$

the equation becomes

$$\cot A \cdot v^2 - 2uv + 2ku - 2hv = 0,$$

where $h = a + b \cos A$, $k = b \sin A$. The center is at the point $(-a, k)$ which is the point where S_0S meets the now horizontal line T_0R_0. One of the asymptotes is the line R_0T_0. The curve is concave downward in this case and the uneroded portion tends to become horizontal as erosion progresses. It might be expected that R_0T_0 is always an asymptote, but this is the case only when $\angle B = \not\angle C$, *i. e.*, $P = 0$.

THE OBSOLETE IN MATHEMATICS.

By G. A. MILLER, University of Illinois.

The present article was suggested by extravagant claims made in the advertisements of some general works of reference. As an extreme case we may refer to recent advertisements of the *Mathematical Dictionary and Cyclopedia of Mathematical Science* by Charles Davies and W. G. Peck, which contained the following statements: "A standard work for 60 years." "Definitions of all terms employed in mathematics, an analysis of each branch and the whole as forming a single science."

The claims represented by these quotations, which many readers of this MONTHLY have doubtless seen, present a serious affront to our intelligence, since any mathematical reader should at once recognize that these claims must be false even if he has never heard of the work in question. Hence it is not the object of the present article to establish their extravagance. Those who have only a slight knowledge of the mathematical advances during the last sixty years know that a work written at the beginning of this period could not contain definitions of all the important terms now employed in mathematics.

On the other hand, it will probably be of interest to those readers who have

not had an opportunity to examine the mathematical dictionary mentioned above to be furnished a few evidences from its pages of the great mathematical advances made during the last sixty years, especially in our own country. It may be desirable to emphasize the last phrase in the preceding sentence since American mathematics was not then in as close contact with the best mathematics of the world as it is to-day.

As evidence of this fact we may note that the dictionary under consideration does not contain the term *determinants,* although the subject of determinants had been extensively developed in Europe at a much earlier date and had soon found its way into many of the textbooks on algebra.[1] In fact, two separate books on determinants, viz., *Elementary Theorems Relating to Determinants,* 1851, by W. Spottiswoode, and *La Teorica dei Determinanti,* 1854, by F. Brioschi, had appeared before the date 1855, when the dictionary by Davies and Peck was "entered according to act of Congress."

The cited omission may serve to illustrate the fact that this dictionary was in many ways a half century behind the times when it was published, and hence it is simply ridiculous to claim that it has been "a standard work for 60 years." As evidence of the fact that in some respects it was actually more than a half century behind the times when it was published we need only cite the following definite statement, "The Calculus of Variations is the highest branch of mathematics," which appears under the term *mathematics* and also under *Calculus of Variations.* Another evidence of the same fact appears under the term *limit* in the following words: "A quantity towards which a varying quantity may approach to within less than any assignable quantity, but which it cannot pass."

The preceding remarks are not intended to convey the idea that the mathematical dictionary by Davies and Peck is of no value to the modern student of mathematics. They are, however, intended to convey the writer's conviction that as a work of reference for reliable information this dictionary cannot be recommended since it represents too many obsolete views and omits too many of the modern notions and developments. Some of these obsolete views are of much interest, partly because they represent approximately the stage of American mathematics a little more than half a century ago and also because they exhibit difficulties which the teacher is apt to overlook unless they are explicitly brought to his attention.

Among the obsolete mathematical methods which have received considerable attention in the history of elementary mathematics are those which relate to the four fundamental operations of arithmetic. Some of these methods are quite complicated from our present point of view but they often serve to give a deeper insight into the real meaning of these operations. This is true, in particular, as regards the operation of division.

Among the methods of division which were employed before the fifteenth century, when the present method began to be used, those included under the term complementary division were widely adopted during the Middle Ages.

[1] Cf. T. Muir, *The Theory of Determinants,* second edition, 1906, p. 14.

[the letter A] was used as a numeral denoting 500, or, with a dash over it, thus, $\bar{A}$, it stood for 500,000."

The question at once arises what ancients are meant. The number systems of the ancient Babylonians, the ancient Egyptians, the ancient Greeks, etc., do not reveal such a use of the letter A as a number symbol. From the fact that in medieval times the Romans sometimes used the horizontal bar above a number symbol to denote that the number represented by this symbol is to be multiplied by 1,000 leads one to suspect that the term "ancients" in this quotation probably refers to the Romans in the Middle Ages, but there is nothing in the article itself which would aid one in reaching this conclusion.

To teachers of mathematics the obsolete methods, viewpoints, and terms should be of peculiar interest since they involve elements which were once attractive, and were replaced by others which were still more attractive. The genesis of our modern methods and viewpoints may reveal possible further improvements, and a knowledge of the obsolete may enable us to assist more effectively in consigning to the obsolete those things which could now be replaced by the more useful.

A SUBSTITUTE FOR DUPIN'S INDICATRIX.

By C. L. E. MOORE, Massachusetts Institute of Technology.

1. **Dupin's Indicatrix.** Let the surface be given in the form $z = f(x, y)$ and let us take the origin at a non-singular point and take the tangent plane at the origin for the xy-plane. Then z can be expanded into an infinite series in x, y which will begin with the second-degree terms,

$$z = \tfrac{1}{2}(ax^2 + 2hxy + by^2) + \cdots, \tag{1}$$

where the terms omitted are of higher than the second order. One method of obtaining Dupin's indicatrix then is to write

$$ax^2 + 2hxy + by^2 = \pm 1. \tag{2}$$

If this conic is an ellipse then use the sign on the right, which will make it real and if it is an hyperbola use either sign. We note that in case this conic becomes a parabola it degenerates into two coincident lines and never takes the form of a general parabola. Points on the surface are then classified as elliptic, hyperbolic or parabolic according as the indicatrix is an ellipse, hyperbola or two coincident straight lines. The axes of the indicatrix correspond to the directions of the lines of curvature on the surface. The direction of the asymptotes of the indicatrix are the same as the direction of the asymptotic lines of the surface. Two directions on the surface are said to be conjugate if they coincide with conjugate diameters of the indicatrix. So we see that the indicatrix is quite intimately associated with the important directions on the surface.

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If this conic is an ellipse then use the sign on the right, which will make it real and if it is an hyperbola use either sign. We note that in case this conic becomes a parabola it degenerates into two coincident lines and never takes the form of a general parabola. Points on the surface are then classified as elliptic, hyperbolic or parabolic according as the indicatrix is an ellipse, hyperbola or two coincident straight lines. The axes of the indicatrix correspond to the directions of the lines of curvature on the surface. The direction of the asymptotes of the indicatrix are the same as the direction of the asymptotic lines of the surface. Two directions on the surface are said to be conjugate if they coincide with conjugate diameters of the indicatrix. So we see that the indicatrix is quite intimately associated with the important directions on the surface.

It always seemed to me however that there was something arbitrary in the

choice of this conic because we take the second-degree terms and set them equal to a constant and be careful that the constant is so chosen that the conic is real. It would seem more reasonable to me to set the second-degree terms equal to zero. If this were done the two lines into which it factors would give the asymptotic directions and the bisectors of the angle between them would give the direction of the lines of curvature. Conjugate diameters do not depend on the equation of a conic but only on the directions of the asymptotes so that conjugate directions could be defined if we should take as directrix the two lines obtained by setting the second-degree terms equal to zero. The method used by de la Vallee-Poussin in his *Cours d'analyse* to obtain the Dupin indicatrix is not open to the above objection. He measures on each tangent line to the surface at O a length OT equal to the square root of the radius of normal curvature for that direction. The locus of T is the Dupin indicatrix.

If one wishes to study the theory of surfaces in higher dimensions he will find that the indicatrix does not easily generalize and that the sort of things which it shows about the surface have very slight connection with the sort of things which the indicatrix shows for surfaces in three dimensions. For these reasons I venture to offer a substitute for the indicatrix.

2. **Substitute for Dupin's Indicatrix.** In what follows we shall be more interested in curvature than the radius of curvature of a curve and shall use the term normal curvature at a point to denote the curvature of the section of the surface made by a plane which contains a tangent line and normal to the surface at that point. The normal curvature depends on the particular tangent line taken. The following discussion will be given only to show properties of the surface in the neighborhood of a point and for simplicity we will take the point under discussion for the origin and will use equation (1) for the equation of the surface. The substitute which I propose for the indicatrix is then obtained as follows: Measure on the normal to the surface at O (the axis of z) a distance OC equal to the normal curvature in any direction. The locus of C, as we vary the direction, is a section of the normal. The position of this segment with respect to the surface point (origin in this case) tells us as much about the character of the surface as does the indicatrix of Dupin. We shall see that this segment may, with advantage, be considered as a degenerate ellipse. This locus generalizes quite readily for higher dimensions, but in that case the locus is not, in general, a degenerate ellipse.

Let the axes of the conic (2) be taken for the x- and y-axes, that is let the x- and y-axes be tangent to the lines of curvature. The equation of the surface then takes the form

$$z = Ax^2 + By^2 + \cdots. \tag{3}$$

Any plane passing through the normal (z-axis) is

$$y = kx, \tag{4}$$

and any normal section is obtained by taking (3) and (4) simultaneously. We

will now find the curvature of such a normal section. The curvature of a twisted curve given in the form

$$x = f(t), \qquad y = g(t), \qquad z = h(t)$$

is

$$(5) \quad c = \sqrt{\frac{d^2x}{ds^2} + \frac{d^2y}{ds^2} + \frac{d^2z}{ds^2}} = \frac{[(g'h'' - h'g'')^2 + (g'f'' - f'g'')^2 + (h'f'' - f'h'')^2]^{1/2}}{[f'^2 + g'^2 + h'^2]^{3/2}},$$

where s denotes the arc length and primes denote derivatives with respect to t.[1] To apply formula (5) to the curve in hand we first substitute (4) in (3) and then the parametric equations of the curve become

$$x = x, \qquad y = kx, \qquad z = (A + Bk^2)x^2 + \cdots,$$

where x is the parameter. Now applying (5) we have

$$(6) \qquad c = \frac{A + Bk^2}{1 + k^2}.$$

Solving for k^2

$$(7) \qquad k^2 = \frac{A - c}{c - B},$$

and from this last expression we see that for real normal sections, that is for real values of k, c must lie between A and B. If then we measure on the normal $OA = A$ and $OB = B$ we see that the point C (the end of the normal curvature) will trace out the segment AB twice, since for each value of c there are two values of k. To the same value of c there correspond two values of k equal but opposite in sign, hence the directions in which the normal curvatures are equal are equally inclined to the lines of curvature.

If we denote the normal curvature in the direction of the lines of curvature, that is along the x- and y-axis, by c_1 and c_2 and their radii of curvature by R_1 and R_2 we have

$$c_1 = \frac{1}{R_1} = A, \qquad c_2 = \frac{1}{R_2} = B.$$

The mean curvature is then

$$H = \tfrac{1}{2}(A + B).$$

If we substitute H for c in (7) we find that the directions on the surface for which the normal curvature is H are $k = \pm 1$ and that therefore these directions bisect the angles between the lines of curvature and are perpendicular to each other. Now take two perpendicular directions k and $-(1/k)$ on the surface. For the normal curvature in these directions we have

$$c' = \frac{A + Bk^2}{1 + k^2}, \qquad c'' = \frac{Ak^2 + B}{1 + k^2},$$

[1] See Wilson's *Advanced Calculus*, p. 85.

and hence

$$\tfrac{1}{2}(c' + c'') = \tfrac{1}{2}(A + B) = H.$$

That is the average of the normal curvatures in two perpendicular directions is equal to the mean curvature. This means that C' and C'' are equally distant from the middle of the segment AB. If now we look on AB as a degenerate ellipse A and B are its vertices and H its center and to each direction on the surface corresponds a point on this degenerate ellipse. The ends of a diameter here will be points equally distant from the center but the values of k to which they correspond must be taken with opposite signs. With this convention then we can say that the ends of a diameter correspond to perpendicular directions. As appears then the two halves of the ellipse can be distinguished by saying that one side corresponds to positive values of k and the other side corresponds to negative values.

For a surface whose equation has the form (1), conjugate directions coincide with the directions of the conjugate diameters of the conic

$$Ax^2 + By^2 = 1.$$

Then the conjugate to any direction k is $-\ (A/Bk)$ and we find for the radii of curvature for normal sections in two such directions

$$R' = \frac{1 + k^2}{A + Bk^2}, \qquad R'' = \frac{A^2 + B^2k^2}{AB(A + Bk^2)}.$$

Hence

$$\tfrac{1}{2}(R' + R'') = \frac{A + B}{2AB} = \tfrac{1}{2}(R_1 + R_2) = R_m,$$

where R_m simply denotes the mean of the radii of curvature in the directions of the lines of curvature. The normal radii of curvature in two conjugate directions have the same relation to the mean radius of curvature that the curvature in two perpendicular directions has to the mean curvature. This property can then be used to define conjugate directions on the surface, that is two directions are conjugate if the mean of their normal radii of curvature is the mean radius of curvature.

The asymptotic directions are those in which the normal curvature is zero. These directions are $k = \pm\sqrt{-\ A/B}$. If then $A = 0$ both these directions coincide in the x-axis and likewise if $B = 0$ they both coincide in the y-axis. But if either $A = 0$ or $B = 0$ the segment AB has one end at O. If A and B have like signs the above values of k are imaginary and therefore the asymptotic directions are imaginary. This means that O is not a point of the segment AB. If A and B have different signs the asymptotic directions are real. In this case O is a point of the segment. A minimal surface is one for which H is zero for each point of the surface. Expressed in terms of the segment we see that for a minimal surface O must be the middle point of the segment.

The segment AB will then indicate about as much as the Dupin indicatrix.

The only exception being conjugate directions which coincide with conjugate diameters of the indicatrix. The segment shows that if O is not a point of AB the curvature of each normal section through O has the same sign and hence the surface is dome shaped at this point. This is usually called an elliptic point. If O is a point of the segment, not coincident with A or B, then some directions have positive normal curvature and some have negative. At such a point the surface is saddle shaped. These points are usually called hyperbolic. If O coincides with either A or B then in all directions the normal curvature has the same sign but there is a single value for which it is zero. These are called parabolic points.

The advantage of using the locus of the end of the normal curvature OC for an indicatrix shows up best in higher dimensions for the locus is, in general, not a degenerate ellipse. In higher dimensions there is a different direction of the normal to the surface for each direction in the tangent plane. The character of the surface is indicated by the position of the ellipse with reference to the surface point O. For example if the surface is a general one there is no relation between the plane of the ellipse and the point O. If the plane of the ellipse passes through O then, at this point, the surface has the character of a surface in a four-dimensional space. If O lies on the ellipse the point is called parabolic. If the ellipse degenerates into a segment, but the line on which it lies does not pass through O, the surface then has a special four-dimensional character having on it lines which have many of the properties of lines of curvature on surfaces in three dimensions. If the line of the segment passes through O then the surface has the character of a surface in ordinary space, at that point, and all the different cases which we have discussed in the preceding paragraph have the same meaning that they had in three dimensions. For the proof of these results for higher dimensions see "Differential Geometry of Surfaces in Hyperspace," by E. B. Wilson and C. L. E. Moore, *Proceedings of the American Academy of Arts and Sciences*, Vol. 52, 1916.

3. **Expression in Parametric Form.** The equation of the surface is often given in parametric form and therefore I shall indicate briefly how these same results would appear in parametric form. The formulas used can be found in any book on surface theory such as Eisenhart's *Differential Geometry*. Let the surface be given by the equations

$$x = \varphi(u, v), \qquad y = \chi(u, v), \qquad z = \psi(u, v). \tag{8}$$

If u and v are functions of a single paramenter or if v is a function of u then equations (8) are the equations of a curve traced on the surface. From differential geometry it is well known that the differential of arc of a curve traced on the surface is

$$ds^2 = Edu^2 + 2Fdu\,dv + Gdv^2,$$

where

$$E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2,$$

$$F = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}, \qquad G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2.$$

If the parameter curves, that is the curves $u =$ const., $v =$ const., are orthogonal it is well known that $F = 0$. The curvature of any normal section of the surface is given by the formula

$$c = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2}, \tag{9}$$

where

$$L = \frac{1}{\sqrt{EG - F^2}}\begin{vmatrix} \frac{\partial^2 x}{\partial u^2} & \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{d^2 y}{\partial u^2} & \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial^2 z}{\partial u^2} & \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}, \qquad M = \frac{1}{\sqrt{EG - F^2}}\begin{vmatrix} \frac{\partial^2 x}{\partial u \partial v} & \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial^2 y}{\partial u \partial v} & \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial^2 z}{\partial u \partial v} & \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix},$$

$$N = \frac{1}{\sqrt{EG - F^2}}\begin{vmatrix} \frac{\partial^2 x}{\partial v^2} & \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial^2 y}{\partial v^2} & \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial^2 z}{\partial v^2} & \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}.$$

If the parameter curves are conjugate curves then $M = 0$. The directions of the lines of curvature are known to be conjugate and also orthogonal directions. Then if we take them for parameter curves both $M = 0$ and $F = 0$. The direction of a curve on the surface is defined by the ratio dv/du, that is, to each direction through a given point on the surface corresponds a single value of dv/du and vice versa. Then writing $k = dv/du$ and taking the lines of curvature for parameter lines equation (9) takes the simple form

$$(10)\quad c = \frac{L + Nk^2}{E + Gk^2}, \qquad \text{from which} \qquad (11)\quad k^2 = \frac{L - Ec}{cG - N}.$$

From the expressions for E and G we see that they are both positive and therefore from (11) for real values of k, that is for real directions on the surface, c must lie between L/E and N/G. Hence if we measure on the normal line a length equal to the normal curvature as was done in § 2 we see that the end of the curvature will trace out a segment from L/E to N/G. Since there are two values of k for a single value of c each point of this segment will be counted twice. The values of k corresponding to the same value of c differ only in sign, hence directions for which the normal curvatures are equal are equally inclined to the lines of curvature (the parameter lines in this case). We can see this by applying the formula for the angle between two directions k_1 and k_2

$$\cos\theta = \frac{E + Gk_1k_2}{\sqrt{E + Gk_1^2}\sqrt{E + Gk_2^2}}, \tag{12}$$

for the case in which the parameter curves are orthogonal. If we set $k_1 = k$, $k_2 = 0$ the above statement is verified.

If we denote the curvature of the lines of curvature by c_1 and c_2 and the corresponding radii of curvature by R_1, R_2 we have

$$c_1 = \frac{1}{R_1} = \frac{L}{E}, \qquad c_2 = \frac{1}{R_2} = \frac{N}{G}. \tag{13}$$

The mean curvature is then

$$H = \tfrac{1}{2}(c_1 + c_2) = \frac{LG + NE}{2EG}.$$

If H is substituted for c in (10) it is found that the two directions on the surface whose normal curvature is H are $k = \pm\sqrt{E/G}$. Substituting these values for k_1 and k_2 in (12) it is seen that these two directions are orthogonal. From (12) we see, in general, that two directions are orthogonal if

$$E + Gk_1k_2 = 0.$$

Now if we take a direction k and the direction perpendicular to it we find for the normal curvature in these two directions

$$c' = \frac{L + Nk^2}{E + Gk^2}, \qquad c'' = \frac{LG^2k^2 + E^2N}{EG(E + Gk^2)},$$

and hence

$$\tfrac{1}{2}(c' + c'') = \frac{LG + NE}{2EG} = H.$$

Hence the average of the curvatures in two perpendicular directions is constant and equal to the mean curvature H. We can now consider the segment as a degenerate ellipse just as in § 2.

From surface theory we also know that, if $M = 0$, conjugate directions are such that

$$L + Nk_1k_2 = 0. \tag{14}$$

For two such directions we have for the radii of curvature R', R''

$$R' = \frac{E + Gk^2}{L + Nk^2}, \qquad R'' = \frac{EN^2k^2 + GL^2}{NL(L + Nk^2)},$$

and hence

$$\tfrac{1}{2}(R' + R'') = \tfrac{1}{2}(R_1 + R_2) = R_m.$$

This last equation may then be used as a definition for conjugate directions. The remainder of the discussion would be the same as that given in § 2.

Topic A.—A preliminary report on this topic has been submitted to Professor Young by Mr. J. A. Foberg. This report was drawn up by a sub-committee of a committee of the Central Association of Science and Mathematics Teachers. A discussion of this topic has also been secured at the request of Professor Young from Professor Hedrick. Certain other preliminary material is expected soon and it is hoped that a report may be made to the committee within the next few months.

Topic B.—Professors Tyler and Smith are coöperating on a report which, it is expected, will be ready some time in December.

Topic C.—Miss Blair submitted a report on her topic last summer and is now engaged in a revision and expansion of it with a view to early publication.

Topic E.—Professor Crathorne submitted an extensive summary of the results of recent scientific, statistical investigations relating to mathematical requirements last spring and submitted a second report during the summer, suggesting that the committee carry out certain specific investigations of its own. At its meeting in Cleveland last September it was decided to carry out the work suggested by Professor Crathorne. It is hoped that a detailed account of the nature of the proposed investigation can be published in an early number of the MONTHLY.

Topic G.—Responsibility for this topic will be assigned in the near future.

This indicates in broad outline the larger phases of the committee's work. Many matters of more special interest have been before the committee during the past year but hardly seem to call for comment here. It is hoped that further information concerning the work of the committee may be published as occasion arises in future numbers of the MONTHLY.

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

Introduction to the Calculus of Variations. By WILLIAM ELWOOD BYERLY. Harvard University Press, Cambridge, 1917. 48 pages.

Having just finished using this little book as a basis of instruction in a short course in advanced calculus given during the summer session, we venture to give expression to our sentiments regarding it.

In the first place, we believe that any book on an advanced mathematical topic may be called good for one or the other of two reasons: either it is good because of the depth and rigor of its scholarship or it is good from the fact that, though written quite regardless of the "fine points," it somehow meets very squarely the actual needs of the great majority of our advanced students as we find them in our ordinary American university classes to-day. It might seem that these two types should be the same, but they are not. It is of course highly desirable that we cultivate and produce books of the first or primarily scholarly

Topic A.—A preliminary report on this topic has been submitted to Professor Young by Mr. J. A. Foberg. This report was drawn up by a sub-committee of a committee of the Central Association of Science and Mathematics Teachers. A discussion of this topic has also been secured at the request of Professor Young from Professor Hedrick. Certain other preliminary material is expected soon and it is hoped that a report may be made to the committee within the next few months.

Topic B.—Professors Tyler and Smith are coöperating on a report which, it is expected, will be ready some time in December.

Topic C.—Miss Blair submitted a report on her topic last summer and is now engaged in a revision and expansion of it with a view to early publication.

Topic E.—Professor Crathorne submitted an extensive summary of the results of recent scientific, statistical investigations relating to mathematical requirements last spring and submitted a second report during the summer, suggesting that the committee carry out certain specific investigations of its own. At its meeting in Cleveland last September it was decided to carry out the work suggested by Professor Crathorne. It is hoped that a detailed account of the nature of the proposed investigation can be published in an early number of the MONTHLY.

Topic G.—Responsibility for this topic will be assigned in the near future.

This indicates in broad outline the larger phases of the committee's work. Many matters of more special interest have been before the committee during the past year but hardly seem to call for comment here. It is hoped that further information concerning the work of the committee may be published as occasion arises in future numbers of the MONTHLY.

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

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A number of minor corrections follow:
Page 24, line 8, change "order" to "degree."
Page 30, last line, numerator of first integral, change ∂ to δ.
Page 38, next to last line, change ∂ to δ.

WALTER B. FORD.

UNIVERSITY OF MICHIGAN,
August, 1917.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

ALGEBRA.

492. Proposed by ARTEMUS MARTIN, LL.D., Washington, D. C.

If two numbers, A and B, $B > A$, be selected at random, what is the probability that they have no common divisor?

493. Proposed by ALBERT BABBITT, University of Nebraska.

Determine the coefficients b, c, d of the equation $x^3 + bx^2 + cx + d = 0$ so that they should be roots of the same equation. [From *Supplemento a Periodico di Matematica.*]

494. Proposed by N. P. PANDYA, Sojitra, India.

Solve algebraically and also graphically, $\log \sin x = \sin \log x$.

GEOMETRY.

524. Proposed by NORMAN ANNING, Somewhere in France.

Many railways use as "easement curve" the cubic parabola. If points on such a curve are named by their distances measured along the curve from the point of inflection ("flat end") show that, within the limits of ordinary practice, *i. e.*, for angles so small that the difference between arc and sine is inappreciable, the deflection from tangent at m to set n is $(n - m)(n + 2m)$ times the deflection from tangent at 0 to 1.

525. Proposed by C. N. SCHMALL, New York City.

Given a quadrant of a circle AOB, where OA and OB are bounding radii, and a semicircle ACO having OA as a diameter and lying on the same side as the quadrant. Describe a circle which shall touch the two arcs and the radius OB.

CALCULUS.

440. Proposed by JOSEPH B. REYNOLDS, Lehigh University.

If t is the differential vector joining two consecutive points on a space curve and R is the radius of curvature of the curve at that point, prove that

$$R^2 = \frac{(t \cdot t)^3}{(t \times \dot{t}) \cdot (t \times \dot{t})}.$$

441. Proposed by J. L. RILEY, Stephenville, Texas.

Find the minimum value of

$$\int \left\{ \left(\frac{dy}{dx} \right)^2 \sin x + (y + x - \sin x)^2 / \sin x \right\} dx.$$

MECHANICS.

356. Proposed by FLORENCE P. LEWIS, Goucher College, Baltimore.

A ray of light enters a prism having vertex angle α. If the angle between the incoming and the outgoing directions is defined as the angle of deviation, at what angle must the ray enter the prism in order that the angle of deviation may be a minimum?

357. Proposed by JOSEPH B. REYNOLDS, Lehigh University.

Two beads each of mass m connected by a string of length $2l$ and carrying a mass m' at its middle point are threaded symmetrically with respect to the major axis which is vertical on a

smooth ellipse of eccentricity e and latus rectum $2l$. The string is held taut and horizontal, then released; find the velocities of the beads when the end ones impinge.

NUMBER THEORY.

274. Proposed by J. L. RILEY, Stephenville, Texas.

Solve in positive integers the equation

$$x^4 + x^3 + x^2 + x + 1 = y^2.$$

275. Proposed by V. M. SPUNAR, Chicago, Illinois.

A square, side $2a$, is represented by the equation $x^n + y^n = a$ $(n = \infty)$. Find a like formula for an equilateral triangle.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

480. Proposed by FRANK IRWIN, University of California.

Solve the following equation

$$(x-1) - 2\left(1-\frac{1}{x}\right) - 3\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right) - 4\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)\left(1-\frac{3}{x}\right) - \cdots - n\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)\cdots\left(1-\frac{n-1}{x}\right) = 0.$$

Also the equation

$$(x-a_1) - a_2\left(1-\frac{a_1}{x}\right) - a_3\left(1-\frac{a_1}{x}\right)\left(1-\frac{a_2}{x}\right) - \cdots - a_n\left(1-\frac{a_1}{x}\right)\left(1-\frac{a_2}{x}\right)\cdots\left(1-\frac{a_{n-1}}{x}\right) = 0.$$

[Adapted from a formula of Tait's.]

SOLUTION BY A. M. HARDING, University of Arkansas.

It is evident that

$$(x-1)(x-2) \equiv x(x-1) - 2(x-1)$$

and

$$(x-1)(x-2)(x-3) \equiv x^2(x-1) - 2x(x-1) - 3(x-1)(x-2).$$

Let us assume that

$$(x-1)(x-2)\cdots(x-n+1) \equiv x^{n-2}(x-1) - 2x^{n-3}(x-1) - 3x^{n-4}(x-1)(x-2) - \cdots - (n-1)(x-1)(x-2)(x-3)\cdots(x-n+2). \quad (1)$$

If we multiply both members of this equation by $x - n$ and rearrange the terms we obtain

$$(x-1)(x-2)\cdots(x-n+1)(x-n) \equiv x^{n-1}(x-1) - 2x^{n-2}(x-1) - 3(x-1)(x-2) - \cdots - n(x-1)(x-2)(x-3)\cdots(x-n+1).$$

Hence, equation (1) is true for every value of n.

After dividing both members of the last equation by x^{n-1}, we find

$$\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)\left(1-\frac{3}{x}\right)\cdots\left(1-\frac{n}{x}\right) \equiv (x-1) - 2\left(1-\frac{1}{x}\right) - 3\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right) - \cdots - n\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)\cdots\left(1-\frac{n-1}{x}\right).$$

Hence, the roots of the given equation are $x = 1, 2, 3, \cdots, n$.

By a method similar to the one used above, it may be shown that the roots of the second equation are $x = a_1, a_2, a_3, \cdots, a_n$.

Also solved by HORACE OLSON, V. M. SPUNAR, E. F. CANADAY, and C. P. SOUSLEY.

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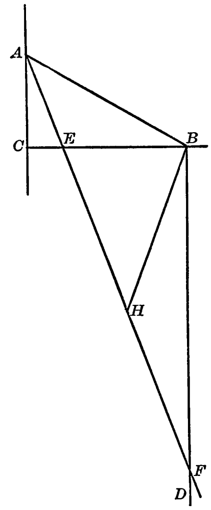
By a method similar to the one used above, it may be shown that the roots of the second equation are $x = a_1, a_2, a_3, \cdots, a_n$.

Also solved by HORACE OLSON, V. M. SPUNAR, E. F. CANADAY, and C. P. SOUSLEY.

Also solved by E. H. VANCE, J. B. REYNOLDS, and WILLIAM HOOVER.

513. Proposed by ALBERT A. BENNETT, University of Texas.

The following construction for angle trisection was given some years ago in a non-mathematical journal. Let ABC be a right triangle with AB as hypotenuse. Let BD be a ray drawn parallel to AC and extending in the same direction. Let AEF be a variable ray meeting the segment BC in E and the ray BD in F. Show, by elementary methods, that when the variable ray is so adjusted that $EF = 2AD$ then $\angle EAC = \frac{1}{3} \angle BAC$.



SOLUTION BY JOS. B. REYNOLDS, Lehigh University.

I take it that the reading should be $EF = 2AB$. Let the figure be constructed according to the given directions. Join B to H, the middle point of EF. Now $BH = HE = HF = AB$ so that the triangles ABH and BHF are isosceles and, therefore,

$$\angle BAE = \angle BHA = 2 \angle BFH = 2 \angle EAC.$$

Hence,

$$\angle EAC = \tfrac{1}{3} \angle BAC.$$

Also solved by S. W. REAVES and HORACE OLSON.

MECHANICS.

337. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Assuming that a train may be accelerated by the application of a force equal to 1/40 of its gross weight and be braked with a force equal to 1/10 of its gross weight, show that the least time in which it may run from one to another of two stopping stations 5,000 feet apart is 2 minutes and 5 seconds. Also find the greatest speed during the run to be 54-6/11 miles per hour.

SOLUTION BY CAPTAIN STUART C. GODFREY, West Point, N. Y.

A force applied to a body equal to one fortieth of its weight will produce an acceleration of $g/40 = 32/40 = 4/5$ ft. per sec^2.

Then $\alpha_1 = 4/5$ = increasing acceleration during first period of train's motion.

$\alpha_2 = 32/10 = 16/5$ = decreasing acceleration during final period of train's motion.

Let t_1 and t_2 represent the corresponding time intervals, and let $V_1 = \alpha_1 t_1 = \alpha_2 t_2$ = maximum speed attained.

Since $\alpha_2 = 4\alpha_1$, $t_2 = 1/4t_1$. Also, since $s = 1/2\alpha t^2$, the distance corresponding to the final period is

$$\tfrac{1}{2}\alpha_2 t_2^2 = \frac{1}{2}\left(4\alpha_1 \cdot \frac{t_1^2}{4^2}\right) = \tfrac{1}{4}(\tfrac{1}{2}\alpha_1 t_1^2) = \tfrac{1}{4}, \text{ of the distance for the first period.}$$

Now if the reasonable assumption be made that the speed of the train continues to accelerate until the moment when the brakes must be applied to bring it to a stop at the final station, the problem is readily solved without using the maxima and minima method.

For the distances are obviously 4,000 feet and 1,000 feet.

$$t = \sqrt{\frac{2s_1}{\alpha_1}} = \sqrt{\frac{8000 \cdot 5}{4}} = 100 \text{ secs.}$$

$$t_2 = \frac{t_1}{4} = 25 \text{ secs.} \quad \textit{Total time} = \textit{2 min. 5 secs.}$$

Also $V_1 = \alpha_1 t_1 = 4/5 \cdot 100 = 80$ ft. per sec. = 54-6/11 miles per hour.

More rigorously, it is conceivable that the train might retain its maximum speed V_1 for a period t_3 before the brakes were applied. Then, letting $4s$ and s represent the distances corresponding to the periods t_1 and t_2, the time function in terms of s becomes (since $t = \sqrt{2s/\alpha}$)

$$t = \sqrt{\frac{8s}{\alpha_1}} + \frac{1}{4}\sqrt{\frac{8s}{\alpha_1}} + \frac{5000 - 5s}{V_1} = \frac{5}{4}\sqrt{10s} + \frac{5000 - 5s}{\frac{4}{5}\sqrt{10s}} = \frac{5}{4} \cdot \frac{10s + 5000 - 5s}{\sqrt{10s}}.$$

$$\frac{dt}{ds} = \frac{5}{4} \cdot \frac{25s - 25000}{(10s)^{3/2}} = 0.$$

Hence, $s = 1{,}000$ feet, $4s = 4{,}000$ feet, and the distance travelled at uniform speed is zero.

In this case, t is a decreasing function of s until s reaches the critical value 1,000; but it can only become an increasing function thereafter by 5,000—$5s$ becoming negative, which makes the third term of t, and consequently t itself, no longer represent possible physical conditions. The function itself, then, has a true minimum state, but this is an end point of the curve, so far as its physical interpretation is concerned.

Also solved by O. S. ADAMS, CHRISTIAN HORNUNG, FRANK IRWIN, HORACE OLSON, and G. PAASWELL.

338. Proposed by J. B. REYNOLDS, Lehigh University.

A comet in a parabolic orbit crosses the earth's orbit (assumed circular) so that it remains a maximum time within it; find the comet's maximum velocity in miles per second and its time within the earth's orbit in years.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

Let ρ be the distance at any moment from the sun, p the perpendicular from the sun upon the tangent to the comet's path when the comet is at the distance ρ, then, $4a$ being by the problem the latus rectum of the parabolic path of the comet, we have

$$p^2 = a\rho. \tag{1}$$

Also let F be the attractive force of the sun, h the double area generated by ρ in a unit of time, and θ the angular coördinate corresponding to ρ.

From the theory of central forces,

$$F = \frac{h^2}{p^3}\frac{dp}{d\rho}. \tag{2}$$

From (1),

$$\frac{2}{p^3}\frac{dp}{d\rho} = \frac{1}{a\rho^2}\cdots, \tag{3}$$

and this in (2) gives

$$F = \frac{h^2}{2a\rho^2}. \tag{4}$$

Let $F = \mu$ when $\rho = 1$; then (4) gives

$$h = \sqrt{2a\mu}, \tag{5}$$

(4) then becoming

$$F = \frac{\mu}{\rho^2}. \tag{6}$$

For the velocity v of the earth,

$$v^2 = rF = \frac{\mu}{r}, \tag{7}$$

ρ in (6) being now the earth's distance from the sun, *i. e.*, r. Then $\theta r \div v = \theta r^{3/2}/\sqrt{\mu}$ = the time required by the earth to describe the arc of its circular orbit subtending the angle θ at the sun.

For the comet,

$$dt = \frac{\rho^2 d\theta}{h}, \tag{8}$$

and the polar equation of its orbit is

$$\rho = \frac{2a}{1 + \cos\theta} = \frac{a}{\cos^2\frac{\theta}{2}}. \text{ Hence, } \rho^2 d\theta = \frac{a^2 d\theta}{\cos^4\frac{\theta}{2}}, \tag{9}$$

and (8) becomes

$$t = \frac{2a^2}{h}\int\frac{d\theta}{\cos^4\frac{\theta}{2}} = \frac{2a^2}{\sqrt{2a\mu}}\left(\tan\frac{\theta}{2} + \tfrac{1}{3}\tan^3\frac{\theta}{2}\right). \tag{10}$$

At the point of intersection of the two orbits, $\rho = r$, and so by (9),

$$\cos\frac{\theta}{2} = \sqrt{\frac{a}{r}}, \qquad \sin\frac{\theta}{2} = \sqrt{\frac{r-a}{r}}, \qquad \text{and} \qquad \tan\frac{\theta}{2} = \sqrt{\frac{r-a}{a}},$$

(10) then becoming

$$2t = 2a^{3/2}\sqrt{\frac{2}{\mu}}\left(\sqrt{\frac{r-a}{a}} + \frac{1}{3}\sqrt{\frac{(r-a)^3}{a^3}}\right) = \frac{2}{3}\sqrt{\frac{2}{\mu}}(r+2a)\sqrt{r-a}, \tag{11}$$

which by the problem is to be a maximum. For this, the first derivative of both members of (11) with respect to α must vanish, giving $r = 2a$.

$$\text{Hence, } \theta = \frac{\pi}{2}, \text{ and the time, } \frac{\theta r}{v} = \frac{2\pi a\sqrt{2a}}{\sqrt{\mu}};$$

and (11) becomes

$$t = \frac{4a}{3}\sqrt{\frac{2a}{\mu}},$$

and the greatest part of the earth's year during which a parabolic comet can remain in the earth's orbit is given by

$$\frac{4a}{3}\sqrt{\frac{2a}{\mu}} \div \frac{2\pi a\sqrt{2a}}{\sqrt{\mu}} = \frac{2}{3\pi}.$$

Again by the theory of central forces, in the notation above defined, the velocity v_1 of the comet is

$$v_1 = \frac{h}{p}, \text{ or, by (1) and (5), } v_1^2 = \frac{2a\mu}{a\rho} = \frac{2\mu}{\rho}.$$

This value of v_1 is greatest when ρ is least, which is at perihelion when $\rho = a$; so that the required maximum velocity is

$$v_1 = \sqrt{\frac{2\mu}{a}}.$$

Also solved by C. F. GUMMER and O. S. ADAMS.

NUMBER THEORY.

248. Proposed by E. T. BELL, Seattle, Washington.

NOTE BY THE PROPOSER.

The solution in the June MONTHLY (p. 295) is obviously incomplete. It is not shown that: (1) for general n the expression under the radical is a perfect square; (2) the stated form of Δ_n is both necessary and sufficient. The completion of (1) will prove only the sufficiency; the necessity is also asked for in the problem.

256. Proposed by FRANK IRWIN, University of California.

Let p be an odd prime, and let the notation $1/k$ stand for the solution of $kx \equiv 1 \pmod{p}$. Then show that if the sum of the numbers $1, \frac{1}{2}, \frac{1}{3}, \cdots \frac{1}{(p-1)/2}$ be congruent to zero (mod p), should that be possible, the same is true for the sum of their products two at a time, as well as four at a time.

SOLUTION BY E. B. ESCOTT, Kansas City, Mo.

Since the congruence $(x-1)(2x-1)(3x-1)\cdots[(p-1)x-1] \equiv 0 \pmod{p}$ has $p-1$ roots, namely, $1, 2, 3, \cdots (p-1)$, and since the same is true of the congruence $1 - x^{p-1} \equiv 0 \pmod{p}$, by Fermat's Theorem it follows that

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$$(x-1)(2x-1)(3x-1)\cdots[(p-1)x-1] \equiv 1 - x^{p-1} \tag{1}$$

is an identical congruence. It is also satisfied by $x \equiv 0$ so that it has p roots while it is only of the $(p-1)$th degree.

The last may be written

$$(x-1)(2x-1)\cdots[\tfrac{1}{2}(p-1)x-1][-\tfrac{1}{2}(p-1)x-1]\cdots(-2x-1)(-x-1) \equiv 0,$$

or

$$(x-1)(2x-1)\cdots[\tfrac{1}{2}(p-1)x-1]\cdot(-1)^{1/2(p-1)}(x+1)(2x+1)\cdots[\tfrac{1}{2}(p-1)x+1] \equiv 0;$$

whence

$$(x-1)(x-\tfrac{1}{2})\cdots\left(x-\frac{1}{\frac{1}{2}(p-1)}\right)\cdot(x+1)(x+\tfrac{1}{2})\cdots\left(x+\frac{1}{\frac{1}{2}(p-1)}\right) \equiv 0.$$

Let

$$S_1 = 1 + \tfrac{1}{2} + \cdots + \frac{1}{\frac{1}{2}(p-1)},$$

$$S_2 = 1\cdot\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{3} + \cdots, \quad \text{etc.}$$

The congruence may be written

$$(x^{1/2(p-1)} - S_1x^{1/2(p-1)-1} + S_2x^{1/2(p-1)-2} - \cdots)\cdot(x^{1/2(p-1)} + S_1x^{1/2(p-1)-1} + S_2x^{1/2(p-1)-2} + \cdots) \equiv 0,$$

i. e.,

$$(x^{1/2(p-1)} + S_2x^{1/2(p-1)-2} + \cdots)^2 - (S_1x^{1/2(p-1)-1} + \cdots)^2 \equiv 0.$$

Expanding, we find the coefficients of the first few powers of x to be 1, $2S_2 - S_1^2$, $S_2^2 + 2S_4 - 2S_1S_3$, $\cdots$. Since by (1) this congruence is identical with $1 - x^{p-1}$, it follows that

$$2S_2 - S_1^2 \equiv 0 \pmod{p}, \tag{2}$$

and

$$S_2^2 + 2S_4 - 2S_1S_3 \equiv 0 \pmod{p}. \tag{3}$$

Therefore, by (2) if $S_1 \equiv 0$, it follows that $S_2 \equiv 0$ and by (3) that $S_4 \equiv 0$, which was to be proved.

NOTE: The same property is true of any $\frac{1}{2}(p-1)$ numbers, where the sum of no two is congruent to zero (mod p). The proof is the same.

Examples: (1) If $p = 7$;

$$1 + 2 + 4 = 7 \equiv 0 \pmod{7},$$

$$1\cdot2 + 1\cdot4 + 2\cdot4 = 14 \equiv 0 \pmod{7}.$$

(2) If $p = 11$;

$$1 + 3 + 4 + 5 + 9 = 22 \equiv 0 \pmod{11},$$

$$1\cdot3 + 1\cdot4 + \cdots = 176 \equiv 0 \pmod{11},$$

$$1\cdot3\cdot4\cdot5 + 1\cdot3\cdot4\cdot9 + \cdots = 1{,}023 \equiv 0 \pmod{11}.$$

Also solved by C. F. GUMMER.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence.

DISCUSSIONS.

I. A REPLY TO PROFESSORS ZIWET AND JACKSON ON MECHANICS.[1]

BY EDWARD V. HUNTINGTON, Harvard University.

I cannot help being highly gratified at the favorable reception accorded my "Logical Skeleton of Elementary Dynamics"[2] by Professor ZIWET of Michigan and Professor JACKSON of Harvard, who say they regard it as "a really serviceable concrete scheme" by which "the whole approach to dynamical theory is extraordinarily simplified."

There are certain points, however, on which they confess they are still "unconvinced."

I. Professor JACKSON's criticisms concern chiefly my treatment of "*mass*," which he regards, not as erroneous, but as inadequate. After some witty remarks about the proper draperies for skeletons like $F/F' = a/a'$ to wear, he contends that the "good old idea of mass" should be "searched for all there is in it." But, one asks at once, *which idea of mass does he mean?* For, as I have pointed out in my "Bibliographical Note on the Use of the Word Mass in Current Textbooks,"[3] there are at least four different senses in which the word mass is commonly used. Professor Jackson's preference, apparently, is to use the term in two of these senses at once, so that it shall convey simultaneously the idea of "inertia" (as measured by force over acceleration), and the idea of what he calls "gravitational power" (as measured by a beam balance). I do not believe that this double use of a hard-worked term like mass conduces to clearness; my reasons are set forth at length in the "Bibliographical Note" already cited (see especially Propositions F and G), and a reference to this note must here suffice.

II. Professor ZIWET's criticisms cover a wider range.

For one thing, he points out an error on page 14, where I inadvertently stated a proposition about the resultant of a set of forces in three dimensions in a form applicable only to two dimensions; this I must acknowledge as an unfortunate blunder, which I beg the reader to correct.

Among Professor Ziwet's other criticisms, there are several which seem to indicate that certain passages in my "Skeleton" were not as clear as I had hoped

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[1] A. ZIWET, "Some Reflections on the Teaching of Mechanics, suggested by Professor E. V. Huntington's Article 'A Logical Skeleton of Elementary Dynamics'"; D. JACKSON, "Mass and Force in Elementary Dynamics," AMERICAN MATHEMATICAL MONTHLY, Vol. 24, pp. 296–297 and 298–300, June, 1917.

[2] AMERICAN MATHEMATICAL MONTHLY, Vol. 24, pp. 1–16, January, 1917.

[3] [This "Bibliographical Note" will be published in an early number of the MONTHLY. *Editors.*]

to make them; and in a spirit of clarification rather than of controversy, I should like to set myself right in regard to some of these misunderstandings.

1. For example, referring to § 2 of Professor Ziwet's paper, I certainly did not mean to contend that "*force* as the active, aggressive principle" has any "logical priority" over "inert, passive *matter*."

In my opening section I meant, on the contrary, to set forth the fundamental concepts, "active forces" and "inert material bodies," as of absolutely coördinate importance, neither of them having any sort of priority over the other.

Professor Ziwet's misconception in regard to my intention is apparently due to a confusion on his part between the idea of "material body" on the one hand and the idea of the "mass of a body" on the other hand. Thus, in § 2, he identifies "mass" with "particle"; again, in § 5, he expressly uses "mass" as a synonym for "matter," "lump of matter," "body," etc. This particular confusion was one which I had especially hoped to guard against, by carefully avoiding, throughout the body of my paper, all use of the ambiguous word "mass"; the reappearance of the term "mass" in Professor Ziwet's discussion only goes to show how persistently intrusive a trouble-making little word may be.

It need hardly be said that a confusion of this sort absolutely vitiates all of Professor Ziwet's arguments in which the idea of the "mass of a body" occurs.

2. Again, judging from Professor Ziwet's remarks in § 4, I appear to have given him the impression that both of the "*lists of names of derived units*," tabulated on pp. 15–16, were regarded by me as of equal importance; whereas I had intended to make the contrast between these two tables as striking as possible.

Table II, which was placed in an appendix and carefully labelled "A useless complication still found in many text-books," contains names which are avowedly given as "horrible examples," like the *Lb.-ft*^{2} *per sec.*^{2} (!) as the unit of work, or the *Gm. per sec.*^{2}*-cm.* (!) as the unit of pressure.

Table I, on the other hand, was printed in the body of the text, and contains, as its heading indicates, *only names which are "actually used by engineers and pure scientists."* This Table I, which is so simple and obvious that there is nothing to "learn" about it, is the one that I advocate.

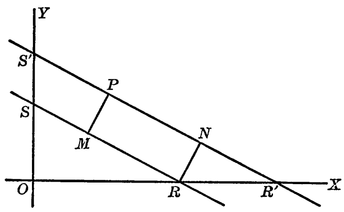
Professor Ziwet groups these two lists together quite indiscriminately,—sometimes condemning them equally as "a nuisance anyhow," which "the student should not be encouraged to consult," sometimes endorsing them equally, by saying that the student must be told that "both systems are in use and both are equally justifiable." Only as a sort of afterthought, at the end of his paper, does he really distinguish between them, and then he makes a statement which I confess I am at a loss to understand. He says, namely, that "the system of units here advocated is not used by a single writer on higher mechanics." This assertion seems to me to be quite unfounded, if "system of units" means the actual list of names of the units that I give in Table I; for all these names are the commonplace of all writers on the subject. For example, if you ask any physicist "What is an erg?" he will immediately reply: "an erg is a dyne-centimeter";

Other points in the papers of Professor Ziwet and Professor Jackson are met, I believe, by my discussion of "mass" in the "Bibliographical Note" already cited. I beg to repeat, however, that I do not put forward this discussion with any idea that it would be profitable for use in the class room. The chief end of a course in dynamics is not a definition of "mass," but the development of ability to master fundamental dynamical problems. *Any teacher who desires to avoid all these controversial matters, and proceed as rapidly as possible to the important principles,* has only to follow the program outlined in my "Logical Skeleton," from which all polemical matter has been intentionally excluded.[1]

II. Relating to the Derivation of a Distance Formula.

By R. M. Mathews, Junior College, Riverside, Calif.

In the usual course in analytic geometry the formula for the distance from a point to a line is derived from Hesse's "normal form" for the equation of a straight line. It is not necessary, however, to introduce the "normal form"



merely for this purpose. The distance formula may easily be derived directly in the manner given below.

In the accompanying figure let (x_1, y_1) be the coördinates of the given point P and $ax + by + c = 0$ the equation of the given line SR. Also let $S'R'$ be a line through P parallel to SR and let RN be perpendicular to SR at R.

Then

$$d = PM = RN.$$

From similar triangles

$$\frac{RN}{RR'} = \frac{OS}{SR} \qquad \text{and} \qquad d = \frac{RR' \cdot OS}{SR}.$$

The equation of $S'R'$ is

$$ax + by + k = 0, \qquad \text{where} \qquad k = -ax_1 - by_1.$$

Also,

$$RR' = OR' - OR = -\frac{k}{a} + \frac{c}{a} = \frac{c-k}{a},$$

[1] [Reprints of this "Skeleton," which a number of teachers have found useful in the class room, may be obtained from the Secretary, Professor W. D. Cairns, 27 King Street, Oberlin, O., at ten cents a copy. *Editors.*]

$$OS = -\frac{c}{b}, \quad SR = \pm\frac{c}{ab}\sqrt{a^2+b^2}.$$

Hence,

$$d = \frac{c-k}{a}\left(-\frac{c}{b}\right)\frac{ab}{\pm c\sqrt{a^2+b^2}} = \frac{ax_1+by_1+c}{\mp\sqrt{a^2+b^2}}.$$

As $-c$ was a factor, we have the usual convention of taking the sign of the radical opposite to that of c.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Indiana.

Mr. H. R. DOUGHERTY, head of the department and professor of mathematics in New York Military Academy, Cornwall-on-Hudson, has been commissioned first lieutenant of infantry in the officers' reserve corps of the national army.

Professor M. E. GRABER, of the department of physics, Heidelberg University, Tiffin, O., and a charter member of the Association, is on leave of absence studying at the University of Chicago.

Miss B. M. TURNER, a charter member of the Association, and formerly principal of the Moundsville (W. Va.) High School, is assistant director of The Phebe Anna Thorne Model School, of Bryn Mawr College.

Mr. RALPH BEATLEY, formerly instructor in mathematics in the Horace Mann High School, New York City, has been commissioned second lieutenant in the Coast artillery service of the regular army.

Professor L. P. EISENHART, of Princeton University, as vice-president, section A, American Association for the Advancement of Science, delivered his retiring address at the Pittsburgh meeting, on the subject "The kinematical generation of surfaces."

In Vol. 11, No. 4, of *The Tôhoku Mathematical Journal,* Sir THOMAS MUIR has a paper on "A theorem including Cayley's on zero axial skew determinants of even order"; and in the *Transactions of the Royal Society of South Africa,* Vol. VI, Part I, a paper entitled "A note on Pfaffians connected with the difference-product."

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According to *Science,* Professor C. W. COBB, of Amherst College, has been granted leave of absence to enter the aviation service of the government as an instructor in connection with the eight ground schools for aviators.

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Hence,

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Professor L. C. KARPINSKI, of the University of Michigan, addressed the high school teachers of mathematics at the Rural Life Conference held at Kirksville (Mo.) Normal School, November 2, on "Practical applications of high school mathematics"; he also addressed the general session of the Conference on the "Causes of the war."

Professor G. VERONESE, of the University of Padua, died on July 17, 1917, at the age of sixty-three years. He was the author of works on hypergeometry, was a member of many learned societies including the Accademia dei Lincei, and the National Italian Academy, and had been professor of mathematics at Padua during the past thirty-four years.

The eighth regular meeting of the Southwestern Section of the American Mathematical Society was held at the University of Oklahoma, Norman, Oklahoma, on December 1, 1917. The members present were guests at a luncheon served by the University on Saturday at noon. Eleven papers appeared upon the printed program, presented by representatives from the following educational institutions: University of Arkansas, University of Illinois, University of Missouri, University of Kansas, University of Nebraska, University of Oklahoma, Washington University and William Jewell College.

The address of Sir JOSEPH LARMOR, retiring president of the London Mathematical Society, appears in Part I, Vol. 16, of the *Proceedings* of the Society, published in April, 1917. The address of the retiring president was devoted to the history of the Society, tracing briefly its growth and usefulness from its origin at University College, London, on January 16, 1866, with twenty-seven members, under the leadership of Professor De Morgan, to the present time. The early membership included such noted mathematicians as De Morgan, Sylvester, Cayley, Harley, Tucker, H. J. S. Smith, Archibald Smith, Clerk Maxwell, Clifford, Stirling, and Lord Rayleigh.

The spring meeting of the Swiss Mathematical Society was held at Zürich on May 10, under the presidency of MARCEL GROSSMANN. At this meeting Professor J. HADAMARD, of Paris, delivered an address on "The notion of analytic function and partial differential equations." The regular meeting of the Society was held at Zürich on September 11. At this meeting, besides the papers by local members of the Society, addresses were made by Professor D. HILBERT, of Göttingen, on "Axiomatic thought"; by Professor C. CARATHEODORY, of Göttingen, "Concerning the geometric treatment of extrema in double integrals"; and by Professor A. EMCH, of the University of Illinois, "Concerning plane curves with certain properties."

The Mathematics Club of Chicago High School Teachers, in coöperation with a national committee of the Mathematical Association of America, is investigating the reasons for the teaching of mathematics in secondary schools. The results of a part of the study appear in *School and Society*, November 17, under the title The Status of Mathematics in Secondary Schools. Six questions were sub-

mitted to a large number of business and professional men of Chicago; fifty-five replies were received. The following is the substance of the questions, with the answers: (1) Was the study of high school mathematics worth while to you? Answers: Yes, 43; very important, 7; negative, 6. (2) Did its study contribute anything which could not have been obtained in equal degree by some other subject? Answers: Yes, 36; uncertain, 8; negative, 6. (3) In employing or advising a young person in line of your profession what importance would you attach to a good high school record in mathematics? Answers: Great importance, 19; considerable importance, 20; little importance, 2; no importance, 3. (4) In planning the secondary education of your son or daughter would you include algebra and geometry? Answers: Both, 42; algebra, 3; neither, 2. (5) Is a knowledge of algebra and geometry of any practical value in your profession? Answers: No, 26; yes, 23. (6) Do you think algebra and geometry should be retained in our secondary schools? Answers: Yes, 46; optional, 2; negative 2. The study also contains quotations from a number of the replies received, showing interesting comments upon the importance of mathematical training afforded by the secondary schools.

There are numerous calls for back numbers of the MONTHLY which cannot be supplied from any files now in our possession, and yet it is of great importance, especially in the case of libraries, that missing copies should be found in order to complete their files. For example, the library of the University of Wisconsin, with our assistance, has been able to complete its set with the exception of the June-July number of volume IV, 1897, and the March number of volume V, 1898. A liberal price will be paid for these two numbers. Please report to the Secretary, W. D. Cairns, who has charge of all the files.

The Missouri Section of the Mathematical Association of America met at Kansas City on Saturday, November 16, 1917. The chairman, Professor L. D. AMES, of the University of Missouri, presided, and nine papers were presented during the two sessions. A full report of this meeting will appear in an early issue of the MONTHLY.

Mr. S. A. JOFFE, assistant actuary of the Mutual Life Insurance Company of New York, has an article entitled "Interpolation Formulae and Central Difference Notation" in the *Transactions of the Actuarial Society of America*, Vol. XVIII, Part I, No. 57, in which a review of the subject is made from the time of Newton down to the present.

Professor OSWALD VEBLEN, of Princeton University, has been commissioned a captain of ordnance in the reserve corps.

Lieutenant WALTER L. HART, formerly of Harvard University, who was commissioned with the field artillery of the regular army and stationed at Fort Russell, Wyoming, has been transferred to the coast artillery and is now at Fort Monroe. Professor A. A. BENNETT, of the University of Texas, and Dr. E. A. T. KIRCHER, of Harvard University, are captains in the coast artillery.

One of our members wishes to secure a complete set of Forsyth's *Theory of Differential Equations,* the first volume, at least, of which is out of print, and our Bureau of Information is unable to give any assistance. Any one who may know where this set may be obtained will confer a favor by reporting to the Secretary.

The library of the Association is now being organized in a room provided by the Oberlin College library. Members are again reminded that it is their privilege to contribute to this library copies of their own published books, or of other books in their possession or control. A special label has been provided for all contributed books showing the donor's name and date of gift, and a personal acknowledgment will be made of all books received. As soon as practicable a report will be made showing a list of the books already in hand.

THE following 34 doctorates in American Universities were conferred in the year 1916–1917 with mathematics as major subject; the subject of the thesis is added in each case:

J. D. BARTER (California), "A contribution to the theory of vector functions";

C. C. BRAMBLE (Johns Hopkins), "A complete system for a collineation group isomorphic with the group of the double tangents of a plane quartic";

C. C. CAMP (Cornell), "An extension of the Sturm-Liouville expansion";

SARAH E. CRONIN (Columbia), "Geometric properties completely characterizing all the curves in a plane along which the constrained motion is such that the pressure is proportional to the normal component of the acting force";

W. L. CRUM (Yale), "Perturbations caused by close approach of two asteroids";

MARY F. CURTIS (Harvard), "Curves invariant under point transformations of special type";

T. DANTZIG (Indiana), "Contributions to the theory of plane transformations";

L. R. FORD (Harvard), "On rational approximations to an irrational complex number";

SISTER M. GERVAS (Catholic University of America), "On the cardioids fulfilling certain assigned conditions";

E. D. GRANT (Chicago), "The motion of a flexible cable in a vertical plane";

C. M. HEBBERT (Illinois), "Projective pencils of stelloids and the curve generated by them";

A. W. HOBBS (Johns Hopkins), "On a problem of projectiles";

T. R. HOLLCROFT (Cornell), "On the types of plane (2, 3) correspondences";

GOLDIE P. HORTON (Texas), "Lebesgue integrals";

M. T. HU (Harvard), "Linear integro-differential equations with a boundary condition";

E. P. HUBBLE (Chicago), "Photographic investigations of faint nebulae";

W. G. HUBERT (New York University), "Sextic curves with two triple points";

K. W. LAMSON (Chicago), "A general implicit function theorem with an application to problems of relative minimum";

GILLIE A. LAREW (Chicago), "Necessary conditions for the problem of Mayer in the calculus of variations";

FLORA E. LESTOURGEON (Chicago), "Minima of functions of lines";

A. S. MERRILL (Chicago), "An isoperimetric problem with variable end-points";

B. E. MITCHELL (Columbia), "Complex conics and their real representation";

H. C. M. MORSE (Harvard), "Certain types of geodesic motion on a surface of negative curvature";

G. W. MULLINS (Columbia), "Differential invariants under the inversion group";

I. C. NICHOLS (Michigan), "A comparative study of fractions in the early treatises on the Hindu art of reckoning";

J. N. RICE (Catholic University of America), "On the in- and circumscribed triangles of the plane rational quartic curve";

J. F. RITT (Columbia), "On a general class of linear homogeneous differential equations of infinite order with constant coefficients";

E. F. SIMONDS (Columbia), "Differential invariants in the plane";

T. MCN. SIMPSON (Chicago), "Relations between the metric and projective theories of space curves";

D. M. SMITH (Chicago), "Jacobi's condition for problems of Lagrange in the calculus of variations";

G. W. SMITH (Illinois), "On nilpotent algebras generated by two units such that i is not an independent unit";

E. A. WHITE (Cornell), "Mechanical theory of the plow";

W. H. WILSON (Illinois), "On a certain class of functional equations."

NOTES ON THE THIRD ANNUAL MEETING OF THE ASSOCIATION.

By the time this issue[1] reaches the members, the third annual meeting of the Association will have taken place at Chicago. The program was in charge of a committee of three consisting of Professors ELIZABETH B. COWLEY of Vassar College, O. D. KELLOGG of the University of Missouri, and E. J. WILCZYNSKI of the University of Chicago.

The meetings were arranged in conjunction with the Chicago Section of the American Mathematical Society by a joint committee including representatives of all the local institutions as follows: Professors W. D. CAIRNS of Oberlin College, D. F. CAMPBELL of Armour Institute, ARNOLD DRESDEN of the University of Wisconsin, J. A. FOBERG of Crane Junior College, W. B. FORD of the University of Michigan, E. J. MOULTON of Northwestern University, H. L. RIETZ of the University of Illinois, and H. E. SLAUGHT of the University of Chicago.

The meetings of the Association opened on Thursday, December 27, with a number of papers on miscellaneous topics presented by members. Thursday

[1] Delayed on account of miscarriage of proof in the holiday congestion of the mails.

afternoon was devoted to a consideration of the teaching of Descriptive Geometry as a collegiate subject. An address was given by Professor Roever of Washington University, St. Louis, Missouri, who has made a prolonged study of this question, and this was followed by several short discussions from teachers of mathematics and teachers of drawing in both colleges and high schools. A special invitation was sent out to teachers of drawing, art, and architecture in all the states contiguous to Chicago to attend this session.

Following the afternoon session, the annual meeting of the Council was held. Important matters of policy and procedure were among the items of business.

On Friday morning the time was divided between the Association and the Society. At nine o'clock, reports were made by three standing Committees of the Association, namely, (1) the Committee on Mathematical Requirements, (2) the Committee on Libraries, and (3) the Committee on a proposed Mathematical Dictionary. At this session matters of vital importance to Institutional Representatives were discussed. At half-past ten the opening session of the Chicago Section of the American Mathematical Society was held and members of the Association were invited to attend this session.

On Friday afternoon there was a joint session of the two organizations at which Professor W. B. Ford of the University of Michigan gave his retiring address as Chairman of the Chicago Section on "A Conspectus of the Modern Theory of Divergent Series" and Professor L. D. Ames of the University of Missouri gave an address on behalf of the Association on "A Definition of the Real Number System by means of Infinite Decimals."

At the close of this session came the annual business meeting of the Association at which certain amendments to the Constitution and By-Laws were voted upon and the officers for 1918 were elected.

Both sessions on Saturday were devoted to the reading of papers before the Chicago Section of the American Mathematical Society.

On Thursday evening occurred the joint dinner of the two organizations, and on Friday evening all were invited to attend the dinner of the American Association of University Professors whose meetings were in session at this time.

A full report of the third annual meeting of the Association will appear in the February issue of the MONTHLY.

SPECIAL NOTICE TO ALL MEMBERS.

In accordance with the prevailing spirit of economy with respect to things not absolutely necessary, it is proposed to greatly condense the Register of the Association for 1918, and to issue, early in January, a mailing list of officers and members containing all promotions, appointments and changes of address which members have reported or may report immediately to the Secretary. His mailing-list is already fairly complete, but in order to insure greater accuracy and completeness, will each member "do his bit" by reporting to the Secretary at once any item which should be included?

This is the only notice that will be given.

ERRATA NOTED IN VOLUME XXIV.

Page 20, line 17 up, for "$x - y$" read "x-y."
Page 30, line 4 up, for "Pitisi" read "Pitisci."
Page 39, line 12 up, for "W. H. Thome" read "W. J. Thome."
Page 44, line 17 up, for "Montana" read "Minnesota"; for "F. L." read "H. L."
Line 12 up, for "H. W. Myers" read "H. S. Myers."
Line 11 up, for "H. W. Roever" read "W. H. Roever."
Page 70, line 6 down, for "Suma" read "Sūma."
Page 71, line 19 up, for "l_1" read "λ."
Page 85, last line, for "Hosley" read "Lasley."
Page 88, line 8 up, for "equation becomes" read "equations become."
Page 94, line 11 up, add "except as first noted."
Page 125, line 16 down, for "M. T. Reed," read "M. T. Peed."
Line 17 down, for "Whiteford" read "Whitford."
Page 132, lines 3 down and 4 up, for "Whiteford" read "Whitford."
Page 139, lines 4 and 5 down, for "Glazier" read "Glaisher."
Page 176, line 9 up, for "2 A D" read "2 A B."
Page 307, line 10 up, for "Left" read "Gauche."
Page 328, line 5 down, for "$^n2$" read "2^n."
Page 332, line 6 down, for "Yenn" read "Yen."
Line 4 down, for "Woods" read "Wood."
Page 336, line 8 down, for "Sir Arthur" read "Arthur."
Page 428, line 14 up, for "C. H." read "E. H."
Page 441, line 9 down, for "Professor Glazier" read "Dr. J. W. L. GLAISHER."